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CLOSURE OPERATORS ON BL-ALGEBRAS

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ABSTRACT. We study relationships between closure operators and BL-algebras. We investigate the properties of closure operators and BL-homomorphisms on BL-algebras. We show that the image of a closure operator on a BL-algebra is isomorphic to a quotient BL-algebra.

1. Introduction and preliminaries

Hájeck [10] introduced a BL-algebra as the foundation of the algebraic structures of fuzzy logic. Closure operators play an important role in (fuzzy) topological spaces [14, 16, 18], lattices [2-4, 6, 9, 11, 19, 20], Boolean algebras [2, 5, 6, 9, 11, 17] and deductive systems [1-3, 5, 6, 13, 20]. Recently, they have been developed in many view points [1, 3, 4-8, 10-20].

In this paper, we will study relationships between BL-algebras and closure operators in a sense Chakraborty [4]. We investigate the properties of closure operators and BL-homomorphisms on BL-algebras. Moreover, we study the first isomorphism theorem induced by closure operators on BL-algebras. We study the properties of quotient induced by closure operators on BL-algebras. We give the examples of them.

DEFINITION 1.1 ([5, 10, 20]). A lattice $(L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ is called a *BL-algebra* if it satisfies the following conditions: for each $x, y, z \in L$,

(B1) $(L, \odot, 1)$ is a commutative monoid,
(B2) if $x \leq y$, then $x \odot z \leq y \odot z$ ($\odot$ is an isotone operation),
(B3) (Galois correspondence) $x \odot y \leq z$ if and only if $x \leq y \to z$,
(B4) $x \wedge y = x \odot (x \to y)$,

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(B5) $x \vee y = \Big((x \to y) \to y\Big) \wedge \Big((y \to x) \to x\Big)$,
(B6) $(x \to y) \vee (y \to x) = 1$.

In a BL-algebra L, $x^* = (x \to 0)$ is called *complement* of $x \in L$.

LEMMA 1.2 ([10, 20]). *In a BL-algebra $(L, \leq, \wedge, \vee, \odot, \to, 0, 1)$, we have the following properties: for $x, y, z \in L$,*
(1) $x = 1 \to x$,
(2) $1 = x \to x$,
(3) $x \odot y \leq x, y$,
(4) $x \odot y \leq x \wedge y$,
(5) $y \leq x \to y$,
(6) $x \odot y \leq x \to y$,
(7) $x \leq y$ *if and only if* $1 = x \to y$,
(8) $x = y$ *if and only if* $1 = x \to y = y \to x$,
(9) $x \odot (x \to y) \leq y$,
(10) $x \to (y \to z) = y \to (x \to z)$.

DEFINITION 1.3 ([5, 20]). Let L be a BL-algebra. A subset D of L is called a *deductive system* of L if it satisfies the following conditions:
(1) $1 \in D$,
(2) if $x, x \to y \in D$, then $y \in D$.

DEFINITION 1.4 ([5, 20]). Let $\sim$ be an equivalence relation on A. Let $f : A^m \to A$ be an m-ary operation on A. We say that $\sim$ is a *congruence* with respect to f if $a_i \sim b_i$ for each $i = 1, \ldots, m$, then $f(a_1, \ldots, a_m) \sim f(b_1, \ldots, b_m)$.

THEOREM 1.5 ([5, 20]). *If $\sim$ is a congruence relation on a BL-algebra L. then $D = \{a \in L \mid a \sim 1\}$ is a deductive system.*

THEOREM 1.6 ([5, 20]). *Let L be a BL-algebra and D a deductive system of L. Define $a \sim b$ if and only if $(a \to b) \odot (b \to a) \in D$. Then $\sim$ is a congruence relation with respect to $\to, \odot, ^*, \vee, \wedge$.*

THEOREM 1.7 ([5, 20]). *Let D be a deductive system of a BL-algebra L. Define on L/D which is the set of equivalence classes $\{|a| \mid a \in L\}$, for all $a, b \in L$,*

$$|a| \leq |b| \text{ if and only if } a \to b \in D.$$

Then

$$(L/D, \leq, \wedge, \vee, \odot, \to, |0|, |1|)$$

is a BL-algebra where $|a| \wedge |b| = |a \wedge b|$, $|a| \vee |b| = |a \vee b|$, $|a| \odot |b| = |a \odot b|$ $|a| \to |b| = |a \to b|$.

Definition 1.8 ([5, 20]). Let L, K be two BL-algebras. A map $h : L \to K$ is called a *BL-homomorphism* if for all $x, y \in L$, it satisfies the following conditions:

(1) $h(x \to y) = h(x) \to h(y)$,

(2) $h(x \odot y) = h(x) \odot h(y)$, $h(0) = 0$.

A BL-homomorphism $h : L \to K$ is called a *BL-isomorphism* if h is bijective.

Theorem 1.9 ([5, 20]). *Let L, K be two BL-algebras. Let $h : L \to K$ be a BL-homomorphism. Then, for all $x, y \in L$,*

(1) $h(x^*) = h(x)^*, h(1) = 1$,

(2) if $x \leq y$, then $h(x) \leq h(y)$,

(3) $h(x \wedge y) = h(x) \wedge h(y)$, $h(x \vee y) = h(x) \vee h(y)$,

(4) if D is a deductive system of L, then $h(D)$ is a deductive system of K.

2. Closure operators and BL-algebras

Definition 2.1 ([4]). Let $(L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ be a BL-algebra. A map $c : L \to L$ is called a *closure operator* if for all $x, y \in L$, it satisfies the following conditions:

(c1) $x \leq c(x)$.

(c2) If $x \leq y$, then $c(x) \leq c(y)$.

(c3) $c(c(x)) = c(x)$.

(c4) $c(x) \odot c(y) \leq c(x \odot y)$.

Theorem 2.2. *Let $(L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ be a BL-algebra. If $c : L \to L$ is a closure operator, then $D = \{a \in L \mid c(a) = 1\}$ is a deductive system.*

Proof. By (c1), $1 \in D$. Let $a, a \to b \in D$. Then $c(a) = 1, c(a \to b) = 1$. Since $b \geq a \odot (a \to b)$ from Lemma 1.2(9), by (c2) and (c4), $c(b) \geq c(a \odot (a \to b)) \geq c(a) \odot c(a \to b) = 1$. Thus $c(b) = 1$. So, $b \in D$.□

Theorem 2.3. *Let $(L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ be a BL-algebra, $c : L \to L$ a closure operator and $c(L) = \{x \in L \mid x = c(x)\}$. Then we have the following properties.*

(1) $c(L) = \{c(x) \mid x \in L\}$ and it is closed under $\wedge$ and $\to$.

(2) $(c(L), \leq_c, \wedge_c, \vee_c, \odot_c, \rightarrow_c, c(0), 1)$ *is a BL-algebra defined as follows: for each* $x, y, z \in c(L)$,

$$x \wedge_c y = c(x \wedge y), \quad x \vee_c y = c(x \vee y),$$
$$x \odot_c y = c(x \odot y), \quad x \rightarrow_c y = c(x \rightarrow y).$$

Furthermore,

$$\wedge_c = \wedge, \vee_c = \vee, \rightarrow_c = \rightarrow .$$

(3) *If* L *satisfies* $x \odot x = x$ *for each* $x \in L$, *then*

$$c(x \odot y) = c(x) \odot c(y).$$

PROOF. (1) We easily prove $c(L) = \{c(x) \mid x \in L\}$ from (c3). For each $x, y \in c(L)$, $x \wedge y \in c(L)$ from:

$$c(x) \wedge c(y) = x \wedge y \leq c(x \wedge y) \leq c(x) \wedge c(y) = x \wedge y.$$

We show that, for all $x, y \in c(L)$, $c(x \rightarrow y) = x \rightarrow y$. By (c1), $(x \rightarrow y) \leq c(x \rightarrow y)$. Since $c(x) \odot c(x \rightarrow y) \leq c\Big(x \odot (x \rightarrow y)\Big) \leq c(y)$ from (c4) and Lemma 1.2(9), then, by (B3), $c(x \rightarrow y) \leq c(x) \rightarrow c(y) = x \rightarrow y$.

(2) (B4-B5) We show that, for all $x, y, z \in c(L)$,

$$\begin{aligned} x \wedge_c y &= c(x \wedge y) = c\Big(x \odot (x \rightarrow y)\Big) = x \odot_c (x \rightarrow_c y) \\ x \vee_c y &= c(x \vee y) = c\Big(\big((x \rightarrow y) \rightarrow y\big) \wedge \big((y \rightarrow x) \rightarrow x\big)\Big) \\ &= \Big((x \rightarrow_c y) \rightarrow_c y\Big) \wedge_c \Big((y \rightarrow_c x) \rightarrow_c x\Big). \end{aligned}$$

Furthermore, by (1),

$$\begin{aligned} x \wedge_c y &= c(x \wedge y) = x \wedge y, \\ x \vee_c y &= c(x \vee y) \\ &= c\Big(\big((x \rightarrow y) \rightarrow y\big) \wedge \big((y \rightarrow x) \rightarrow x\big)\Big) \\ &= \Big((c(x) \rightarrow c(y)) \rightarrow c(y)\Big) \wedge \Big((c(y) \rightarrow c(x)) \rightarrow c(x)\Big) \\ &= c(x) \vee c(y) = x \vee y. \end{aligned}$$

Since $x \wedge_c y = x \wedge y$ and $x \vee_c y = x \vee y$ for all $x, y, z \in c(L)$, we have $\wedge_c = \wedge, \vee_c = \vee$ and $\leq_c = \leq$.

We substitute $\leq$ for $\leq_c$ in following statements.

(B3) We show that, for all $x, y, z \in c(L)$,

$$x \odot_c y \leq z \quad \text{if and only if} \quad x \leq y \rightarrow_c z.$$

($\Rightarrow$) Since $c(x) \odot c(y) \leq c(x \odot y) = x \odot_c y \leq z$, we have $x = c(x) \leq c(y) \rightarrow c(z) = c(y \rightarrow z) = y \rightarrow_c z$.

($\Leftarrow$) Since $x \odot_c y \leq (y \rightarrow_c z) \odot_c y = c(y \odot (y \rightarrow z)) \leq c(z)$, we have $x \odot_c y \leq z$.

(B1) $(c(L), \odot_c, 1)$ is a commutative monoid because, for all $x, y, z \in c(L)$,

$$x \odot_c y = c(x \odot y) = c(y \odot x) = y \odot_c x$$
$$x \odot_c 1 = c(x \odot 1) = c(x) = x.$$

Suppose that there exist $x, y, z \in c(L)$ such that

$$x \odot_c (y \odot_c z) \not\geq (x \odot_c y) \odot_c z.$$

Then there exist $t \in c(L)$ such that

$$x \odot_c (y \odot_c z) \leq t < (x \odot_c y) \odot_c z.$$

By (B2),

$$\begin{aligned}
&x \odot_c (y \odot_c z) \leq t \\
&\quad \text{if and only if} \quad y \odot_c z \leq x \rightarrow_c t = c(x \rightarrow t) \\
&\quad \text{if and only if} \quad y \leq z \rightarrow_c c(x \rightarrow t) \\
&\quad \text{if and only if} \quad y \leq c\Big(z \rightarrow c(x \rightarrow t)\Big) \\
&\quad \text{if and only if} \quad y \leq c(z) \rightarrow c(x \rightarrow t) \\
&\quad \text{if and only if} \quad y \leq c(z) \rightarrow \Big(c(x) \rightarrow c(t)\Big) \\
&\quad \text{if and only if} \quad y \leq c(x) \rightarrow \Big(c(z) \rightarrow c(t)\Big) \quad \text{(by Lemma 1.2(10))} \\
&\quad \text{if and only if} \quad y \leq x \rightarrow_c (z \rightarrow_c t) \\
&\quad \text{if and only if} \quad x \odot_c y \leq z \rightarrow_c t \\
&\quad \text{if and only if} \quad (x \odot_c y) \odot_c z \leq t.
\end{aligned}$$

It is a contradiction. Hence $x \odot_c (y \odot_c z) \geq (x \odot_c y) \odot_c z$ for all $x, y, z \in c(L)$. Similarly, we have $x \odot_c (y \odot_c z) \leq (x \odot_c y) \odot_c z$ for all $x, y, z \in c(L)$.

(B2) If $x \leq y$ for each $x, y \in c(L)$,

$$x \odot_c z = c(x \odot z) \leq c(y \odot z) = y \odot_c z.$$

(B6) For each $x, y, z \in c(L)$,

$$\begin{aligned}(x \to_c y) \vee_c (y \to_c x) &= c\Big(c(x \to y) \vee c(y \to x)\Big) \\ &= c\Big(\big(c(x) \to c(y)\big) \vee \big(c(y) \to c(x)\big)\Big) \\ &= c(1) = 1.\end{aligned}$$

Thus $(c(L), \leq_c, \wedge_c, \vee_c, \odot_c, \to_c, c(0), 1)$ is a BL-algebra such that $\wedge_c = \wedge, \vee_c = \vee$ and $\to_c = \to$.

(3) Since $x \odot y \leq x \odot 1 = x$ and $x \odot y \leq y$, $c(x \odot y) \leq c(x)$ and $c(x \odot y) \leq c(y)$ imply

$$c(x \odot y) = c(x \odot y) \odot c(x \odot y) \leq c(x) \odot c(y).$$

□

EXAMPLE 2.4. Let $I = [0, 1]$ be the unit interval and $(I, \leq, \wedge, \vee, 0, 1)$ a lattice. We define $c : I \to I$ as follows:

$$c(x) = \begin{cases} \frac{1}{3}, & 0 \leq x \leq \frac{1}{3} \\ \frac{1}{2}, & \frac{1}{3} < x \leq \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1. \end{cases}$$

(1) Define binary operations $\odot$ and $\to$ on I by

$$x \odot y = x \wedge y,$$

$$x \to y = \begin{cases} 1, & x \leq y \\ y, & \text{otherwise.} \end{cases}$$

Then $(I, \leq, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra (*Gödel logic*, ref. [10]). Since $c(x \odot y) = c(x) \odot c(y)$, c is a closure operator. From Theorem 2.3, $(c(I), \leq, \wedge, \vee, \odot, \to, \frac{1}{3}, 1)$ is a BL-algebra.

(2) Define binary operations $\otimes$ and $\Rightarrow$ on I by

$$x \otimes y = max\{0, x + y - 1\},$$

$$x \Rightarrow y = min\{1, 1 - x + y\}.$$

Then $(I, \leq, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$ is a BL-algebra (*Lukasiewicz logic*, ref.[10]). But c is not a closure operator because

$$\frac{1}{3} = c(\frac{3}{7} \otimes \frac{4}{7}) \not\geq (c(\frac{3}{7}) \otimes c(\frac{4}{7})) = \frac{1}{2} \otimes 1 = \frac{1}{2}.$$

Moreover, $c(I) = \{\frac{1}{3}, \frac{1}{2}, 1\}$ is not closed under $\Rightarrow$ from

$$\frac{5}{6} = (c(\frac{1}{2}) \Rightarrow c(\frac{1}{3})) \neq c(\frac{1}{2} \Rightarrow \frac{1}{3}) = c(\frac{5}{6}) = 1.$$

The converse of Theorem 2.3 cannot be true from the following example.

EXAMPLE 2.5. Let $I = [0, 1]$ be the unit interval and $(I, \leq, \wedge, \vee, 0, 1)$ a lattice. We define $c : I \to I$ as follows:

$$c(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{4}, & 0 < x \leq \frac{1}{4} \\ \frac{1}{2}, & \frac{1}{4} < x \leq \frac{1}{2} \\ \frac{3}{4}, & \frac{1}{2} < x \leq \frac{3}{4} \\ 1, & \frac{3}{4} < x \leq 1. \end{cases}$$

Define binary operations $\otimes$ and $\Rightarrow$ on I by

$$x \otimes y = max\{0, x + y - 1\},$$

$$x \Rightarrow y = min\{1, 1 - x + y\}.$$

Then $(I, \leq, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$ is a BL-algebra. We define $\wedge_c, \vee_c, \odot_c, \to_c$ as in Theorem 2.3(2). Also, $(c(L), \wedge_c = \wedge, \vee_c = \vee, \odot_c = \otimes, \to_c = \Rightarrow, 0, 1)$ is a BL-algebra where $c(L) = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ from the following statement.

If $\frac{k}{4} + \frac{l}{4} + \frac{m}{4} \leq 2$, then

$$\frac{k}{4} \otimes (\frac{l}{4} \otimes \frac{m}{4}) = (\frac{k}{4} \otimes \frac{l}{4}) \otimes \frac{m}{4} = 0.$$

If $\frac{k}{4}+\frac{l}{4}+\frac{m}{4}>2$, then

$$\frac{k}{4}\otimes(\frac{l}{4}\otimes\frac{m}{4})=(\frac{k}{4}\otimes\frac{l}{4})\otimes\frac{m}{4}=\frac{k}{4}+\frac{l}{4}+\frac{m}{4}-2.$$

Hence $x\otimes(y\otimes z)=(x\otimes y)\otimes z$ for all $x,y,z\in c(L)$. For all $x,y,z\in c(L)$.

$$\begin{aligned} x\otimes y\leq z &\text{ if and only if } x+y-1\leq z \text{ if and only if } x\leq 1-y+z \\ &\text{ if and only if } x\leq \min\{1,1-y+z\} \text{ if and only if } x\leq y\Rightarrow z. \end{aligned}$$

If $x\leq y$, then

$$x=x\wedge y=x\otimes 1=x\otimes(x\to y),$$
$$\big((x\to y)\to y\big)\wedge\big((y\to x)\to x\big)=(1\to y)\wedge y=y=x\vee y,$$
$$(x\to y)\vee(y\to x)=1\vee(1-y+x)=1.$$

If $x>y$, it is similarly proved. Thus, it satisfies the conditions (B4-B6). Other conditions easily proved. But c is not a closure operator from

$$0=c(\frac{3}{7}\otimes\frac{4}{7})\not\geq c(\frac{3}{7})\otimes c(\frac{4}{7})=\frac{1}{2}\otimes\frac{3}{4}=\frac{1}{4}.$$

THEOREM 2.6. *Let* $(L,\leq,\wedge,\vee,\odot,\to,0,1)$ *and* $(c(L),\leq_c,\wedge_c,\vee_c,\odot_c,\to_c,c(0),1)$ *be BL-algebras and let* $c:L\to L$ *be a closure operator. Then we have the following properties.*

(1) $c:L\to c(L)$ *is a BL-homomorphism.*

(2) *If* $D=\{a\in L\mid c(a)=1\}$, *then the map* $\bar{c}:L/D\to c(L)$ *defined by* $\bar{c}(|a|)=c(a)$ *is a BL-isomorphism.*

PROOF. (1) It is easy from $c(x\odot y)=c(x)\odot_c c(y)$ and $c(x\to y)=c(x)\to_c c(y)$.

(2) From Theorem 2.2, D is a deductive system. Let $a\sim_D b$. Then $(a\to b)\odot(b\to a)\in D$. It implies

$$c\Big((a\to b)\odot(b\to a)\Big)=\Big(c(a)\to c(b)\Big)\odot_c\Big(c(b)\to c(a)\Big)=1.$$

By Lemma 1.2(3,8), $c(a)=c(b)$. Thus, $\bar{c}$ is well defined.

Since $c:L\to c(L)$ is a BL-homomorphism, $\bar{c}:L/D\to c(L)$ is a BL-homomorphism from the following statements:

$$\bar{c}(|x|\to|y|)=\bar{c}(|x\to y|)=c(x\to y)=c(x)\to_c c(y)=\bar{c}(|x|)\to_c\bar{c}(|y|),$$

$$\bar{c}(|x| \odot |y|) = \bar{c}(|x \odot y|) = c(x \odot y) = c(x) \odot_c c(y) = \bar{c}(|x|) \odot_c \bar{c}(|y|),$$
$$\bar{c}(|0|) = c(0).$$

We only show that $\bar{c}$ is bijective. Let $c(a) = c(b)$. From Lemma 1.2(8),

$$c(a) \to_c c(b) = c(b) \to_c c(a) = 1.$$

It implies

$$\Big(c(a) \to_c c(b)\Big) \odot_c \Big(c(b) \to_c c(a)\Big) = c\Big((a \to b) \odot (b \to a)\Big) = 1.$$

Then $(a \to b) \odot (b \to a) \in D$. Thus, $a \sim_D b$, that is, $|a| = |b|$. Hence $\bar{c}$ is injective. Since c is surjective, $\bar{c}$ is surjective. □

We easily prove the following corollary.

COROLLARY 2.7. *Let X be a nonempty set and $P(X)$ a family of all subsets of X. Then $(P(X), \subset, \cap, \cup, \emptyset, X)$ is a lattice. For each $A, B \in P(X)$, we define the operations $\odot$ and $\to$ by*

$$A \odot B = A \cap B, \quad A \to B = A^c \cup B.$$

Then $(P(X), \subset, \cap, \cup, \odot, \to, \emptyset, X)$ is a BL-algebra.

EXAMPLE 2.8. Let $X = \{x_1, x_2, x_3\}$ be a set.
(1) Define $c : P(X) \to P(X)$ as follows:

$$c(A) = \begin{cases} \{x_1, x_2\}, & A \subset \{x_1, x_2\}, \\ X, & \text{otherwise.} \end{cases}$$

It satisfies the following conditions: for each $A, B \in P(X)$,

$$c(A \cap B) = c(A) \cap c(B), \ c(A \cup B) = c(A) \cup c(B),$$
$$c(A^c) = c(A)^c \cup c(\emptyset).$$

Since $A \to B = A^c \cup B$,

$$c(A \to B) = c(A^c \cup B) = c(A)^c \cup c(\emptyset) \cup c(B) = c(A) \to c(B).$$

From Theorem 2.3(2), $(c(P(X)), \subset, \cap, \cup, \odot = \cap, \to, c(\emptyset), X)$ is a BL-algebra. Hence $c : P(X) \to c(P(X))$ is a BL-homomorphism. Note that,

$$D = \{A \in P(X) \mid c(A) = X\} = \{\{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}, X\}.$$

From Theorem 1.6, since

$$\begin{aligned} A \sim B \ &\text{if and only if } (A \to B) \odot (B \to A) \in D \\ &\text{if and only if } (A^c \cup B) \cap (B^c \cup A) \in D \\ &\text{if and only if } (A \cap B) \cup (A \cup B)^c \in D. \end{aligned}$$

We can obtain:

$$\begin{aligned} &\emptyset \sim \{x_1\} \sim \{x_2\} \sim \{x_1, x_2\}, \\ &X \sim \{x_3\} \sim \{x_1, x_3\} \sim \{x_2, x_3\}. \end{aligned}$$

We obtain $P(X)/D = \{|\{x_1, x_2\}|, |X|\}$ and so $\overline{c} : P(X)/D \to c(P(X))$, $|A| \mapsto c(A)$, is a BL-isomorphism by Theorem 2.6(2).

(2) Define $c_1 : P(X) \to P(X)$ as follows:

$$c_1(A) = \begin{cases} \{x_1\}, & A \subset \{x_1\}, \\ X, & \text{otherwise}. \end{cases}$$

Then c_1 is not a closure operator because

$$X = c_1(\{x_1, x_2\}) \cap c_1(\{x_1, x_3\}) \not\leq c_1(\{x_1, x_2\} \cap \{x_1, x_3\}) = \{x_1\}.$$

(3) We define $(I, \leq, \wedge, \vee, \odot, \to, 0, 1)$ and a closure operator c as same as in Example 2.4(1). Then $c : I \to c(I)$ is a BL-homomorphism and $D = \{\frac{1}{2} < x \leq 1 \mid c(x) = 1\}$. From Theorem 1.6, we can obtain:

$$\begin{aligned} \frac{1}{3} \sim x, &\quad \forall x \in [0, \frac{1}{3}], \\ \frac{1}{2} \sim y, &\quad \forall y \in (\frac{1}{3}, \frac{1}{2}], \\ 1 \sim z, &\quad \forall z \in (\frac{1}{2}, 1]. \end{aligned}$$

We obtain $I/D = \{|\frac{1}{3}|, |\frac{1}{2}|, |1|\}$. Define $\overline{c} : I/D \to c(I)$ by $\overline{c}(|x|) = c(x)$. By Theorem 2.6, $\overline{c}$ is a BL-isomorphism.

THEOREM 2.9. *Let L, K be two BL-algebras and $h : L \to K$ a BL-homomorphism. Let $c_1 : L \to c(L)$ and $c_2 : K \to c(K)$ be closure operators. Let $D_1 = \{x \in L \mid c_1(x) = 1\}$ and $D_2 = \{y \in K \mid c_2(y) = 1\}$. Then we have the following properties.*

(1) If $h(c_1(x)) \leq c_2(h(x))$ for all $x \in L$, which is called a closed map, a map $\overline{h} : L/D_1 \to K/D_2$ defined by $\overline{h}(|x|) = |h(x)|$ is a BL-homomorphism.

(2) If h is surjective such that $h(c_1(x)) = c_2(h(x))$ for all $x \in L$ and $h(c_1(x)) = 1$ implies $c_1(x) = 1$, then a map $\overline{h} : L/D_1 \to K/D_2$ is a BL-isomorphism.

PROOF. (1) Let $a \sim_{D_1} b$. From Theorem 1.6, $(a \to b) \odot (b \to a) \in D_1$. Since $c\Big((a \to b) \odot (b \to a)\Big) = 1$, by Theorem 1.9(1), we have $h\Big(c((a \to b) \odot (b \to a))\Big) = 1$. Since $h(c_1(x)) \leq c_2(h(x))$, $c_2\Big(h((a \to b) \odot (b \to a))\Big) = 1$. Then $h(a) \sim_{D_2} h(b)$. So, the map $\overline{h}$ is well defined. Since $|x \to y| = |x| \to |y|$ and $|h(x) \to h(y)| = |h(x)| \to |h(y)|$ from Theorem 1.7, we have

$$\begin{aligned}\overline{h}(|x| \to |y|) &= \overline{h}(|x \to y|) = |h(x) \to h(y)| \\ &= |h(x)| \to |h(y)| = \overline{h}(|x|) \to \overline{h}(|y|).\end{aligned}$$

Similarly, $\overline{h}(|x| \odot |y|) = \overline{h}(|x|) \odot \overline{h}(|y|)$, $\overline{h}(|0|) = |0|$. Thus, $\overline{h}$ is BL-homomorphism.

(2) Since h is surjective, $\overline{h}$ is surjective. We only show that $\overline{h}$ is injective. Let $h(x) \sim_{D_2} h(y)$. Then $\Big(h(x) \to h(y)\Big) \odot \Big(h(y) \to h(x)\Big) \in D_2$, that is,

$$\begin{aligned}1 &= c_2\Big(\big(h(x) \to h(y)\big) \odot \big(h(y) \to h(x)\big)\Big) \\ &= \Big(c_2(h(x)) \to_{c_2} c_2(h(y))\Big) \odot_{c_2} \Big(c_2(h(y)) \to_{c_2} c_2(h(x))\Big).\end{aligned}$$

By Lemma 1.2 (3,8), $c_2(h(x)) = c_2(h(y))$. Since $c_2(h(x)) = h(c_1(x))$ for all $x \in X$, $h(c_1(x)) = h(c_1(y))$. It implies

$$\begin{aligned}&h\Big(c_1\big((x \to y) \odot (y \to x)\big)\Big) \\ &= \Big(h(c_1(x)) \to h(c_1(y))\Big) \odot \Big(h(c_1(y)) \to h(c_1(x))\Big) \\ &= 1 \odot 1 = 1.\end{aligned}$$

Hence $c_1\big((x \to y) \odot (y \to x)\big) = 1$. Thus, $(x \to y) \odot (y \to x) \in D_1$. Hence $x \sim_{D_1} y$. □

EXAMPLE 2.10. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ be two sets. Define $h : P(X) \to P(Y)$ as follows:

$$\begin{gathered}h(\emptyset) = \emptyset, \quad h(X) = Y, \\ h(\{x_1\}) = \{y_1\}, \ h(\{x_2\}) = \{y_2\}, \ h(\{x_3\}) = \emptyset, \\ h(\{x_1, x_2\}) = \{y_1, y_2\}, h(\{x_1, x_3\}) = \{y_1\}, \ h(\{x_2, x_3\}) = \{y_2\}.\end{gathered}$$

It satisfies the following conditions: for each $A, B \in P(X)$,

$$h(A \cap B) = h(A) \cap h(B),\ h(A \cup B) = h(A) \cup h(B),\ \ h(A^c) = h(A)^c.$$

Since $A \to B = A^c \cup B$,

$$h(A \to B) = h(A^c \cup B) = h(A)^c \cup h(B) = h(A) \to h(B).$$

Hence $h : P(X) \to P(Y)$ is a BL-homomorphism.

Define $c_1 : P(X) \to P(X)$ as follows:

$$c_1(A) = \begin{cases} \{x_1, x_3\}, & A \subset \{x_1, x_3\}, \\ X, & \text{otherwise.} \end{cases}$$

Define $c_2 : P(Y) \to P(Y)$ as follows:

$$c_2(B) = \begin{cases} \{y_1\}, & B \subset \{y_1\}, \\ Y, & \text{otherwise.} \end{cases}$$

Then c_1 and c_2 are closure operators. It implies

$$h(c_1(A)) = \begin{cases} h(\{x_1, x_3\}) = \{y_1\} = c_2(h(A)), & A \subset \{x_1, x_3\}, \\ h(X) = Y = c_2(h(A)), & \text{otherwise.} \end{cases}$$

Moreover, $D_1 = \{\{x_2\}\{x_1, x_2\}\{x_2, x_3\}, X\}$, $D_2 = \{\{y_2\}, Y\}$. We define $\overline{h} : P(X)/D_1 \to P(Y)/D_2$ as $\overline{h}(|\{x_1, x_3\}|) = |\{y_1\}|$ and $\overline{h}(|X|) = |Y|$. Since $h(c_1(A)) = c_2(h(A))$ for each $A \subset X$ and $h(c_1(A)) = Y$ implies $c_1(A) = X$ by Theorem 2.9(2), $\overline{h}$ is a BL-isomorphism.

EXAMPLE 2.11. Let $I = [0, 1]$ be the unit interval and $(I, \leq, \wedge, \vee, 0, 1)$ be a lattice. We define $c_1, c_2 : I \to I$ as follows:

$$c_1(x) = \begin{cases} \frac{1}{3}, & 0 \leq x \leq \frac{1}{3} \\ \frac{1}{2}, & \frac{1}{3} < x \leq \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1, \end{cases} \quad c_2(x) = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{1}{2} \\ \frac{3}{4}, & \frac{1}{2} < x \leq \frac{3}{4} \\ 1, & \frac{3}{4} < x \leq 1. \end{cases}$$

Define on I binary operations $\odot$ and $\to$ by

$$x \odot y = x \wedge y,$$

$$x \to y = \begin{cases} 1, & x \leq y \\ y, & \text{otherwise.} \end{cases}$$

Then $(I, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Furthermore, for each $i = 1, 2$, c_i is a closure operator satisfying $c_i(x \odot y) = c_i(x) \odot c_i(y)$. By Theorem 2.3, $(c_1(I), \leq, \wedge, \vee, \odot, \rightarrow, \frac{1}{3}, 1)$ and $(c_2(I), \leq, \wedge, \vee, \odot, \rightarrow, \frac{1}{2}, 1)$ are BL-algebras. Define a function $h : I \rightarrow I$ by

$$h(x) = \begin{cases} \frac{3}{2}x, & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{2}, & \frac{1}{2} < x \leq 1. \end{cases}$$

We obtain $\overline{h} : I/D_1 \rightarrow I/D_2$ as

$$\overline{h}(|\frac{1}{3}|) = |\frac{1}{2}|, \quad \overline{h}(|\frac{1}{2}|) = |\frac{3}{4}|, \quad \overline{h}(|1|) = |1|.$$

Since $h(c_1(x)) = c_2(h(x))$ for each $x \in I$ and $h(c_1(x)) = 1$ implies $c_1(x) = 1$ by Theorem 2.9, $\overline{h}$ is a BL-isomorphism.

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