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PRIME IDEALS OF SUBRINGS OF MATRIX RINGS

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ABSTRACT. In a ring $R_n(K,J)$ where K is a commutative ring with identity and J is an ideal of K, all prime ideals of $R_n(K,J)$ are of the form either $M_n(P)$ or $R_n(P, P \cap J)$. Therefore there is a one to one correspondence between prime ideals of K not containing J and prime ideals of $R_n(K,J)$.

1. Introduction

Let $NT_n(K)$ be the ring of all lower triangular $n \times n$ matrices over an associative ring K with zeros on and above the main diagonal.

It is well known that all ideals(resp. prime ideals) of a matrix ring $M_n(R)$ where R is an associative ring with identity are $M_n(I)$(resp. $M_n(P)$) where I(resp. P) is an ideal(resp. a prime ideal) of R. Dubish and Perlis [1, Thm.9] gave a uniform construction of all ideals of the ring $NT_n(K)$ over a field K. Levchuk [4, sec.2] similarly constructed ideals of the ring $NT_n(K)$ over a division ring K. It is impossible to give a similar description of the ideals of $NT_n(K)$ for the case $K = Z$ (see [4]).

Now we assume that K is a commutative ring with identity and J is an ideal of K and denote the ring $NT_n(K) + M_n(J)$ by $R_n(K,J)$.

Kuzucuoglu and Levchuk [3] described all ideals of $R_n(K,J)$ when J is a strongly maximal ideal of K. In this paper we will characterize all prime ideals of $R_n(K,J)$ when J is an ideal (not necessarily strongly maximal ideal) of K. The results are followings;

(1) If $P \subsetneq J$ is a J-submodule of K. Then P is a prime ideal of K if and only if $M_n(P)$ is a prime ideal of $R_n(K,J)$.

(2) If P is J-submodule of K such that $P \not\subseteq J$ and $P \not\supseteq J$. Then P is a prime ideal of K if and only if $R_n(P, P \cap J)$ is a prime ideal of $R_n(K,J)$.

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For the convenience we denote following notations:
(1) e_{ij} ; matrix unit on $M_n(K)$.
(2) π_{km} ; the canonical projection on $M_n(K)$.
That means $\pi_{km}(e_{ij}) = 1$ if $(k,m) = (i,j)$ and $\pi_{km}(e_{ij}) = 0$ if $(k,m) \neq (i,j)$.
(3) $(k,m) < (i,j)$ means that $k \leq i, j \leq m$ and $(k,m) \neq (i,j)$.

2. Prime ideals of $R_n(K,J)$

If H is an ideal of $R_n(K,J)$, $\pi_{n1}(H)$ is a J-submodule (not necessarily an ideal) of K. The following example shows that $\pi_{n1}(H)$ is J-submodule but not an ideal of K.

EXAMPLE 2.1. For $K = Z \oplus Z \oplus Z$, $J = 0 \oplus 0 \oplus 2Z$ and $\tilde{T} = \{(n,n,k)e_{21} + (0,0,2k)e_{12} \mid n,k \in Z\}$, let

$$H = Je_{11} + \tilde{T} + Je_{22} + M_2(J^2).$$

Then H is an ideal of $R_2(K,J)$ and $H \neq \sum \pi_{ij}(H)e_{ij}$. Also, $\pi_{21}(H) = \{(n,n,k) \mid n,k \in Z\}$ is a J-submodule of K but not an ideal of K.

But we can get the following lemma.

LEMMA 2.2. *Let H be a subset of $R_n(K,J)$. If H is an ideal of $R_n(K,J)$ Then the followings hold;*
(1) $\pi_{ij}(H)$ is a J-submodule of K.
(2) For all $(k,m) < (i,j)$, $K\pi_{km}(H) \subseteq \pi_{ij}(H)$.
(3) For all $s > i$, $J\pi_{sj}(H) \subseteq \pi_{ij}(H)$.
(4) For all $t < j$, $J\pi_{it}(H) \subseteq \pi_{ij}(H)$.
Moreover, If $\sum \pi_{ij}(H)e_{ij} = H$, the converse is true.

PROOF. (1) is clear.
(2) It is enough to show

$$K\pi_{i-1,j}(H) \subseteq \pi_{ij}(H)$$

for $i > 1$ and

$$K\pi_{i,j+1}(H) \subseteq \pi_{ij}(H)$$

for $j < n$.
Since H is an ideal, $Ke_{i,i-1}H \subseteq H$. So,

$$K\pi_{i-1,j}(H) = \pi_{ij}(Ke_{i,i-1}H) \subseteq \pi_{ij}(H).$$

On the other hand, since H is an ideal, $HKe_{j+1,j} \subseteq H$. so,

$$K\pi_{i,j+1}(H) = \pi_{ij}(HKe_{j+1,j}) \subseteq \pi_{ij}(H).$$

(3) For $s > i$ the fact $Je_{is} \subseteq R_n(K,J)$ implies

$$J\pi_{sj}(H) = \pi_{ij}(Je_{is}H) \subseteq \pi_{ij}(H).$$

(4) For $t < j$ the fact $Je_{tj} \subseteq R_n(K,J)$ implies

$$J\pi_{it}(H) = \pi_{it}(H)J = \pi_{ij}(HJe_{tj}) \subseteq \pi_{ij}(H).$$

Moreover, by hypothesis and by (1), H is a subgroup of $R_n(K,J)$. So,

$$\begin{aligned}
&\pi_{ij}(R_n(K,J)H) \\
&= \sum_{\lambda=1}^{n} \pi_{i\lambda}(R_n(K,J))\pi_{\lambda j}(H) \\
&= \sum_{\alpha=1}^{i-1} \pi_{i\alpha}(R_n(K,J))\pi_{\alpha j}(H) + \sum_{\beta=i}^{n} \pi_{i\beta}(R_n(K,J))\pi_{\beta j}(H) \\
&= \sum_{\alpha=1}^{i-1} K\pi_{\alpha j}(H) + \sum_{\beta=i}^{n} J\pi_{\beta j}(H) \subseteq \pi_{ij}(H)
\end{aligned}$$

by (2) and (3).

Thus, $R_n(K,J)H \subseteq H$. Similarly, by (2) and (4) $HR_n(K,J) \subseteq H$. Therefore, H is an ideal of $R_n(K,J)$. □

REMARK. If H is an ideal of $R_n(K,J)$, $\pi_{n1}(H) \supseteq \pi_{ij}(H)$ by Lemma 2.2 (2). This means that if H is a nonzero ideal, then $\pi_{n1}(H)$ is a nonzero J-submodule of K.

PROPOSITION 2.3. *If H is an ideal of $R_n(K,J)$ such that $\pi_{n1}(H) \equiv T$. Then $H \supseteq M_n(TJ^2)$.*

PROOF. Since H is an ideal, $\bar{H} \equiv (Je_{1n})H(Je_{1n}) \subseteq H$ and

$$\pi_{ij}(\bar{H}) = \begin{cases} TJ^2, & \text{if } (i,j) = (1,n) \\ 0, & \text{if } (i,j) \neq (1,n). \end{cases}$$

That is, $\bar{H} = TJ^2 e_{1n}$. So, for $s \neq 1$

$$(Ke_{s1})\bar{H} = TJ^2 e_{sn},$$

for $t \neq n$

$$\bar{H}(Ke_{nt}) = TJ^2 e_{1t},$$

and for $s \neq 1$ and $t \neq n$

$$(Ke_{s1})\bar{H}(Ke_{nt}) = TJ^2 e_{st}.$$

Therefore,

$$\begin{aligned} M_n(TJ^2) = & \sum_{s=2}^{n}(Ke_{s1})\bar{H} + \sum_{t=1}^{n-1}\bar{H}(Ke_{nt}) \\ & + \sum_{s=2}^{n}\sum_{t=1}^{n-1}(Ke_{s1})\bar{H}(Ke_{nt}) + \bar{H} \subseteq H. \end{aligned}$$

□

If J is a zero ideal of K, then $R_n(K, J) = NT_n(K)$ is a nilpotent ring and there is no prime ideals of $NT_n(K)$. So, in this paper we assume J is a nonzero ideal of K.

THEOREM 2.4. *K is a prime ring if and only if $R_n(K, J)$ is a prime ring.*

PROOF. ($\Rightarrow$) Let H_1 and H_2 be nonzero ideals of $R_n(K, J)$. Then $\pi_{n1}(H_1) \equiv T_1 \neq 0$ and $\pi_{n1}(H_2) \equiv T_2 \neq 0$. Then $H_1 \supseteq M_n(T_1J^2) \neq 0$ and $H_2 \supseteq M_n(T_2J^2) \neq 0$ by Proposition 2.3 and primeness of K. So, since $T_1T_2J^4 \neq 0$, $H_1H_2 \supseteq M_n(T_1J^2)M_n(T_2J^2) \supseteq M_n(T_1T_2J^4) \neq 0$.
Therefore $R_n(K, J)$ is a prime ring.

($\Leftarrow$) Let I_1 and I_2 be nonzero ideals of K. Then $R_n(I_1, I_1 \cap J)$ and $R_n(I_2, I_2 \cap J)$ are nonzero ideals of $R_n(K, J)$. Since $R_n(K, J)$ is a prime ring, $R_n(I_1, I_1 \cap J)R_n(I_2, I_2 \cap J) \neq 0$. So, $0 \neq \pi_{n1}\{R_n(I_1, I_1 \cap J)R_n(I_2, I_2 \cap J)\} \subseteq I_1I_2$. This implies $I_1I_2 \neq 0$.
Therefore, K is a prime ring. □

LEMMA 2.5. *Let H be an ideal of $R_n(K, J)$ such that $\pi_{n1}(H) \equiv T$.*
(1) If $T \supsetneq J$, then H is not a prime ideal.
(2) If $T \subseteq J$ and $H \subsetneq M_n(T)$, then H is not a prime ideal.
(3) If $T \nsubseteq J, T \nsupseteq J$ and $H \subsetneq R_n(T, T \cap J)$, then H is not a prime ideal.

PROOF. (1) Since $\{R_n(K,J)\}^n \subseteq M_n(J)$ and by Proposition 2.3,

$$\{R_n(K,J)\}^{3n} \subseteq \{M_n(J)\}^3 \subseteq M_n(TJ^2) \subsetneq H.$$

So, H is not a prime ideal.

(2) Suppose H is a prime ideal and let $L \equiv M_n(KT)$. Then L is an ideal of $R_n(K,J)$ such that $L \supsetneq H$. Since $\{R_n(K,J)\}^n \subseteq M_n(J)$

$$L\ \{R_n(K,J)\}^{2n} \subseteq M_n(KTJ^2) = M_n(TJ^2) \subseteq H$$

by Proposition 2.3. Since $L \supsetneq H$, $\{R_n(K,J)\}^{2n} \subset H$. So, $R_n(K,J) \subseteq H$.

This is a contradiction.

(3) Suppose H is a prime ideal and let $L \equiv R_n(KT, KT\cap J)$. Then L is an ideal of $R_n(K,J)$ such that $L \supsetneq H$. Since $\{R_n(K,J)\}^n \subseteq M_n(J)$

$$L\ \{R_n(K,J)\}^{2n} \subseteq M_n(KTJ^2) = M_n(TJ^2) \subseteq H$$

by Proposition 2.3. Since $L \supsetneq H$, $\{R_n(K,J)\}^{2n} \subset H$. So, $R_n(K,J) \subseteq H$.

This is a contradiction. □

REMARK. By Lemma 2.5, the prime ideals of $R_n(K,J)$ are of the form $M_n(T)$ or $R_n(T, T\cap J)$ for some J-submodule T of K. Next theorem shows that T is actually an ideal of K.

THEOREM 2.6. *If $M_n(T)$ or $R_n(T, T\cap J)$ is a prime ideal of $R_n(K,J)$ where T is a J-submodule of K. Then T is an ideal of K*

PROOF. Suppose $T \not\subseteq J, T \not\supseteq J$ and $R_n(T, T\cap J)$ is a prime ideal. Then

$$R_n(KT, KT\cap J)M_n(J) \subseteq M_n(KTJ) = M_n(TJ) \subseteq R_n(T, T\cap J).$$

Since $M_n(J) \not\subseteq R_n(T, T\cap J)$, $R_n(KT, KT\cap J) \subseteq R_n(T, T\cap J)$. So, $KT = T$, that is, T is an ideal of K.

Similarly, we have that T is an ideal of K if $M_n(T)$ is a prime ideal of $R_n(K,J)$. □

THEOREM 2.7. *If $P \subsetneq J$ is a J-submodule of K. Then P is a prime ideal of K if and only if $M_n(P)$ is a prime ideal of $R_n(K,J)$.*

PROOF. By first isomorphism theorem, we can easily show that

$$R_n(K,J)/M_n(P) \simeq R_n(K/P, J/P).$$

By Theorem 2.4, P is a prime ideal of K if and only if $R_n(K/P, J/P)$ is a prime ring if and only if $M_n(P)$ is a prime ideal of $R_n(K,J)$. □

THEOREM 2.8. *If P is a J-submodule of K such that $P \not\subseteq J$ and $P \not\supseteq J$. Then P is a prime ideal of K if and only if $R_n(P, P \cap J)$ is a prime ideal of $R_n(K,J)$.*

PROOF. ($\Rightarrow$) Suppose $R_n(P, P \cap J)$ is not prime. Then there exist ideals H_1,H_2($\supsetneq R_n(P, P\cap J)$) of $R_n(K,J)$ such that $H_1H_2 \subseteq R_n(P, P\cap J)$. Let $\pi_{n1}(H_1) = T_1$ and $\pi_{n1}(H_2) = T_2$. We may assume T_1 and T_2 are ideals. Thus,

$$P \supseteq \pi_{n1}(H_1H_2) \supseteq T_1J^2T_2J^2 = T_1T_2J^4.$$

Now $H_1 \supsetneq R_n(P, P \cap J)$. So, for some $i_1 > j_1, \pi_{i_1j_1}(H_1) \supsetneq P$ or for some $i_2 \leq j_2, \pi_{i_2j_2}(H_1) \supsetneq P \cap J$. In both cases $\pi_{n1}(H_1) = T_1 \supsetneq P$. Similarly, $\pi_{n1}(H_2) = T_2 \supsetneq P$. That is, $T_1, T_2, J \not\subseteq P$.

Therefore, P is not prime.

($\Leftarrow$) Suppose P is not prime. Then there exist ideals A,B($\supsetneq P$) of K such that $AB \subseteq P$.

Since$(A\cap J)(B\cap J) \subseteq AB \cap J \subseteq P \cap J$, $R_n(A, A\cap J)R_n(B, B\cap J) \subseteq R_n(P, P \cap J)$. But $R_n(A, A \cap J)$,$R_n(B, B \cap J)$ are ideals of $R_n(K,J)$ such that $R_n(A, A\cap J)$,$R_n(B, B\cap J) \supsetneq R_n(P, P\cap J)$.

Therefore, $R_n(P, P \cap J)$ is not prime. □

COROLLARY 2.9. *There is a one-to-one correspondence between the set of all prime ideals of K not containing J and the set of all prime ideals of $R_n(K,J)$.*

COROLLARY 2.10. *A prime ideal of $M_n(K)$ is $M_n(P)$ where P is a prime ideal of K.*

PROOF. Since $R_n(K,K) = M_n(K)$, we can easily prove by Theorem 2.7. □

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