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ALGORITHMS FOR SYSTEMS OF NONLINEAR VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce and study a new system of nonlinear variational inequalities. The existence and uniqueness of solution for this problem are proved and an iterative algorithm for approximating the solution of system of nonlinear variational inequalities is constructed.

1. Introduction

Variational inequalities not only have stimulated the new results dealing with partial differential equations, but also have been used in a large variety of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc. For these reasons, variational inequality and its generalizations have been well studied in recent years. For details, we refer to [1]-[26] and the references therein.

Recently, some problems consisting of two variational inequalities or two complementarity problems have been introduced and studied. Huang and Fang [15] introduced a system of order complementarity problems and established some existence results by fixed point theory. Kassay and Kolumbán [16] introduced a system of variational inequalities and proved an existence theorem by Ky Fan's lemma. In [22]-[24], Verma introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of system of variational inequalities.

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Motivated by these works, in this paper, we introduce and study a new system of variational inequalities and construct an iterative algorithm for finding the solution of our problem.

Throughout this paper, let H_1, H_2 be two real Hilbert spaces, $A \subset H_1$ and $B \subset H_2$ be two nonempty closed convex sets, $\varphi : H_1 \to R \cup \{+\infty\}$ and $\phi : H_2 \to R \cup \{+\infty\}$ be two proper convex lower semicontinuous functionals. Let $F : H_1 \times H_2 \to H_1$ and $G : H_1 \times H_2 \to H_2$ be two nonlinear mappings.

Consider the following problem: find $(a, b) \in H_1 \times H_2$ such that

$$(1.1) \qquad \begin{cases} \langle F(a,b), x-a\rangle + \varphi(x) - \varphi(a) \geq 0, & x \in H_1, \\ \langle G(a,b), y-b\rangle + \phi(y) - \phi(b) \geq 0, & y \in H_2, \end{cases}$$

which is called the *system of nonlinear variational inequalities.*

If $\varphi = \delta_A$ (the indicator function of A) and $\phi = \delta_B$, then the problem (1.1) reduces to the following system of variational inequalities: find $(a, b) \in A \times B$ such that

$$(1.2) \qquad \begin{cases} \langle F(a,b), x-a\rangle \geq 0, & x \in A, \\ \langle G(a,b), y-b\rangle \geq 0, & y \in B, \end{cases}$$

which is just the problem considered in [16] with F, G being single-valued mappings.

The purpose of this paper is to construct an iterative algorithm for approximating the solution of the problem (1.1) and discuss the convergence analysis of our algorithm. As consequences, the corresponding algorithm and convergence analysis of algorithm for the problem (1.2) are presented too.

The rest of this paper is organized as follows: in Section 2, some concepts and lemmas are given. Section 3 is devoted to the existence of solutions of the problems (1.1) and (1.2). In the final section, some algorithms for the problems (1.1) and (1.2) are constructed and the corresponding convergence analysis of algorithms is discussed.

2. Preliminaries

In this section, we give some concepts and lemmas which will be used in next sections. In the sequel, we always suppose that H is a Hilbert space.

DEFINITION 2.1. Let $T : H \to H$ be a nonlinear mapping. T is said to be *strongly monotone* if there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq r\|x - y\|^2, \quad x, y \in H.$$

DEFINITION 2.2. A nonlinear mapping $T : H \to H$ is said to be *Lipschitz continuous* if there exists a constant $s > 0$ such that

$$\|Tx - Ty\| \leq s\|x - y\|, \quad x, y \in H.$$

DEFINITION 2.3. Let $f : H \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous functional. The *resolvent operator* $J_f^\beta : H \to H$ is defined by

$$J_f^\beta(x) = (I + \beta\partial f)^{-1}(x), \quad x \in H,$$

where I is the identity mapping on H, ∂f is the subdifferential of f and $\beta > 0$ is a constant. Remark that $J_f^\beta = P_K$ if $f = \delta_K$, where K is a nonempty closed convex subset of H and P_K denotes the projection operator.

For our main results in this paper, we need the following lemmas.

LEMMA 2.1. ([4], [5]) *Let $f : H \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous functional and J_f^β be the resolvent operator for f. Then J_f^β is nonexpensive, that is,*

$$\|J_f^\beta(x) - J_f^\beta(y)\| \leq \|x - y\|, \quad x, y \in H.$$

LEMMA 2.2. ([5]) *Let $\varphi : H \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous functional. For any given $u \in H$, the point $z \in H$ satisfies the following inequality*

$$\langle u - z, x - u\rangle + \rho\varphi(x) - \rho\varphi(u) \geq 0, \quad x \in H,$$

if and only if $u = J_\varphi^\rho(z)$, where $\rho > 0$ is a constant.

LEMMA 2.3. *For any given $(a, b) \in H_1 \times H_2$, (a, b) is a solution of the problem (1.1) if and only if*

$$\begin{cases} a = J_\varphi^\rho[a - \rho F(a, b)], \\ b = J_\phi^\beta[b - \beta G(a, b)], \end{cases}$$

where $\rho > 0$ and $\beta > 0$ are two constant.

Proof. The fact directly follows from Lemma 2.2. □

LEMMA 2.4. *Let $\{c_n\}$ and $\{k_n\}$ be two real sequences of nonnegative numbers and satisfy the following conditions:*

(i) $0 \leq k_n < 1$ *for* $n = 0, 1, 2 \ldots$ *and* $\limsup_{n\to\infty} k_n < 1$,
(ii) $c_{n+1} \leq k_n c_n$ *for* $n = 0, 1, 2 \ldots$

Then the sequence $\{c_n\}$ converges to 0.

Proof. The condition (ii) implies that the sequence $\{c_n\}$ is decreasing and so $\{c_n\}$ has a limit c. Suppose by contradiction that $c \neq 0$. Choose a subsequence $\{k_{n_j}\}$ of $\{k_n\}$ such that $\{k_{n_j}\}$ converges to $\limsup_{n\to\infty} k_n$ as $j \to \infty$. By the condition (ii), $c_{n_j} \leq k_{n_j} c_{n_j}$ and so $c \leq (\limsup_{n\to\infty} k_n)c$, which contradicts the condition (i). Hence the sequence $\{c_n\}$ converges to 0. This completes the proof. □

3. The existence and uniqueness theorems

In this section, we shall establish the existence and uniqueness of solution for the problems (1.1) and (1.2), respectively.

THEOREM 3.1. *Let $\varphi : H_1 \to R \cup \{+\infty\}$ and $\varphi : H_1 \to R \cup \{+\infty\}$ be two proper lower semicontinuous functionals. Let $F : H_1 \times H_2 \to H_1$ be a nonlinear mapping such that, for any given $(a, b) \in H_1 \times H_2$, $F(\cdot, b)$ is strongly monotone and Lipschitz continuous with constants r_1 and s_1, respectively, and $F(a, \cdot)$ is Lipschitz continuous with constant τ. Let $G : H_1 \times H_2 \to H_2$ be a nonlinear mapping such that, for any given $(x, y) \in H_1 \times H_2$, $G(x, \cdot)$ is strongly monotone and Lipschitz continuous with constant r_2 and s_2 and $G(\cdot, y)$ is Lipschitz continuous with constant ξ. If there exist constants $\rho > 0$ and $\beta > 0$ such that*

$$\left\{ \begin{array}{l} \sqrt{1 - 2\rho r_1 + \rho^2 s_1^2} + \beta\xi < 1, \\ \sqrt{1 - 2\beta r_1 + \beta^2 s_2^2} + \rho\tau < 1. \end{array} \right. \tag{3.1}$$

Then the problem (1.1) *admits a unique solution.*

Proof. For any given $(u, v) \in H_1 \times H_2$, we define a mapping $Q : H_1 \times H_2 \to H_1 \times H_2$ by

$$Q(u, v) = (T(u, v), S(u, v)) = (J_\varphi^\rho[u - \rho F(u, v)], J_\phi^\beta[v - \beta G(u, v)])$$

for all $(u, v) \in H_1 \times H_2$. For any $(u_1, v_1), (u_2, v_2) \in H_1 \times H_2$, it follows from (3.2) and Lemma 2.1 that

$$\begin{aligned} &\|T(u_1, v_1) - T(u_2, v_2)\| \\ &\le \|u_1 - u_2 - \rho(F(u_1, v_1) - F(u_2, v_2))\| \\ &\le \|u_1 - u_2 - \rho(F(u_1, v_1) - F(u_2, v_1))\| \\ &\quad + \rho\|F(u_2, v_1) - F(u_2, v_2)\| \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\|S(u_1, v_1) - S(u_2, v_2)\| \\ &\le \|v_1 - v_2 - \beta(G(u_1, v_1) - G(u_2, v_2))\| \\ &\le \|v_1 - v_2 - \beta(G(u_1, v_1) - G(u_1, v_2))\| \\ &\quad + \beta\|G(u_1, v_2) - G(u_2, v_2)\|. \end{aligned} \tag{3.4}$$

By the assumptions, we have

$$\begin{aligned} &\|u_1 - u_2 - \rho(F(u_1, v_1) - F(u_2, v_1))\|^2 \\ &= \|u_1 - u_2\|^2 - 2\rho\langle F(u_1, v_1) - F(u_2, v_1), u_1 - u_2\rangle \\ &\quad + \rho^2\|F(u_1, v_1) - F(u_2, v_1)\|^2 \\ &\le (1 - 2\rho r_1 + \rho^2 s_1^2)\|u_1 - u_2\|, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\|v_1 - v_2 - \beta(G(u_1, v_1) - G(u_1, v_2))\|^2 \\ &= \|v_1 - v_2\|^2 - 2\beta\langle G(u_1, v_1) - G(u_1, v_2), v_1 - v_2\rangle \\ &\quad + \beta^2\|G(u_1, v_1) - G(u_1, v_2)\|^2 \\ &\le (1 - 2\beta r_2 + \beta^2 s_2^2)\|v_1 - v_2\|, \end{aligned} \tag{3.6}$$

$$\|F(u_2, v_1) - F(u_2, v_2)\| \le \tau\|v_1 - v_2\| \tag{3.7}$$

and

$$\|G(u_1, v_2) - G(u_2, v_2)\| \le \xi\|u_1 - u_2\|. \tag{3.8}$$

It follows from (3.4)$\sim$(3.8) that

$$\begin{cases} \|T(u_1, v_1) - T(u_2, v_2)\| \\ \le \sqrt{1 - 2\rho r_1 + \rho^2 s_1^2}\|u_1 - u_2\| + \rho\tau\|v_1 - v_2\|, \\ \|S(u_1, v_1) - S(u_2, v_2)\| \\ \le \sqrt{1 - 2\beta r_2 + \beta^2 s_2^2}\|v_1 - v_2\| + \beta\xi\|u_1 - u_2\|. \end{cases} \tag{3.9}$$

By (3.9), we have

$$
\begin{aligned}
&\|T(u_1, v_1) - T(u_2, v_2)\| + \|S(u_1, v_1) - S(u_2, v_2)\| \\
&\leq \left(\sqrt{1 - 2\rho r_1 + \rho^2 s_1^2} + \beta\xi\right)\|u_1 - u_2\| \\
&\quad + \left(\sqrt{1 - 2\beta r_2 + \beta^2 s_2^2} + \rho\tau\right)\|v_1 - v_2\| \\
&\leq \max\{\sqrt{1 - 2\rho r_1 + \rho^2 s_1^2} + \beta\xi, \sqrt{1 - 2\beta r_2 + \beta^2 s_2^2} + \rho\tau\} \\
&\quad \times (\|u_1 - u_2\| + \|v_1 - v_2\|).
\end{aligned} \tag{3.10}
$$

Now, define the norm $\|\cdot\|_1$ on $H_1 \times H_2$ by

$$\|(u, v)\|_1 = \|u\| + \|v\|, \quad (u, v) \in H_1 \times H_2.$$

It is easy to see that $(H_1 \times H_2, \|\cdot\|_1)$ is a Banach space. Let

$$k = \max\{\sqrt{1 - 2\rho r_1 + \rho^2 s_1^2} + \beta\xi, \sqrt{1 - 2\beta r_2 + \beta^2 s_2^2} + \rho\tau\}.$$

By (3.1), we know that $0 < k < 1$. It follows from (3.2) and (3.10) that

$$\|Q(u_1, v_1) - Q(u_2, v_2)\|_1 \leq k\|(u_1, v_1) - (u_2, v_2)\|_1.$$

This prove that $Q : H_1 \times H_2 \to H_1 \times H_2$ is a contractive mapping. Hence there exists a unique $(a, b) \in H_1 \times H_2$ such that

$$Q(a, b) = (a, b),$$

which implies that

$$
\begin{cases}
a = J_\varphi^\rho[a - \rho F(a, b)], \\
b = J_\phi^\beta[b - \beta G(a, b)].
\end{cases}
$$

Therefore, by Lemma 2.3, (a, b) is the unique solution of the problem (1.1). This completes the proof. □

By Theorem 3.1, we obtain the following existence result for the problem (1.2).

THEOREM 3.2. *Let $F : A \times B \to H_1$ be a nonlinear mapping such that, for any given $(a, b) \in A \times B$, $F(\cdot, b)$ is strongly monotone and Lipschitz continuous with constants r_1 and s_1, respectively, and $F(a, \cdot)$ is Lipschitz continuous with constant τ. Let $G : A \times B \to H_2$ be* *a nonlinear mapping such that, for any given* $(x, y) \in A \times B$, $G(x, \cdot)$ *is strongly monotone and Lipschitz continuous with constant* r_2 *and* s_2, *and* $G(\cdot, y)$ *is Lipschitz continuous with constant* ξ. *If there exist constants* $\rho > 0$ *and* $\beta > 0$ *such that*

$$\begin{cases} \sqrt{1 - 2\rho r_1 + \rho^2 s_1^2} + \beta\xi < 1, \\ \sqrt{1 - 2\beta r_1 + \beta^2 s_2^2} + \rho\tau < 1. \end{cases}$$

Then the problem (1.2) *admits a unique solution.*

4. Iterative algorithm and convergence theorems

In this section, we construct the Mann iterative algorithm for approximating the unique solution of the system of nonlinear variational inequalities and discuss the convergence analysis of the algorithm. As a consequence of our discussion, the Mann iterative algorithm for the system of variational inequalities is defined and the convergence of the iterative sequence is proved too.

THEOREM 4.1. *Let* $F : H_1 \times H_2 \to H_1$ *and* $G : H_1 \times H_2 \to H_2$ *be two nonlinear mappings,* $\varphi : H_1 \to R \cup \{+\infty\}$ *and* $\varphi : H_1 \to R \cup \{+\infty\}$ *be two proper lower semicontinuous functionals. Assume that all the conditions of Theorem 3.1 hold. For any given* $(a_0, b_0) \in H_1 \times H_2$, *define the Mann iterative sequences* $\{(a_n, b_n)\}$ *by*

$$\begin{cases} a_{n+1} = \alpha_n a_n + (1 - \alpha_n) J_\varphi^\rho [a_n - \rho F(a_n, b_n)], \\ b_{n+1} = \alpha_n b_n + (1 - \alpha_n)) J_\phi^\beta [b_n - \beta G(a_n, b_n)], \end{cases} \tag{4.1}$$

where

$$0 \le \alpha_n < 1, \quad \limsup_{n \to \infty} \alpha_n < 1. \tag{4.2}$$

Then the sequence $\{(a_n, b_n)\}$ *converges strongly to the unique solution* (a, b) *of the problem* (1.1).

Proof. By Theorem 3.1, the problem (1.1) admits a unique solution (a, b). It follows from Lemma 2.3 that

$$\begin{cases} a = \alpha_n a + (1 - \alpha_n) J_\varphi^\rho [a - \rho F(a, b)], \\ b = \alpha_n b + (1 - \alpha_n) J_\phi^\beta [b - \beta G(a, b)]. \end{cases} \tag{4.3}$$

By (4.1) and (4.3), we have

$$
\begin{aligned}
&\|a_{n+1} - a\| \\
&\leq \alpha_n \|a_n - a\| \\
&\quad + (1-\alpha_n)\|J_{\varphi}^{\rho}[a_n - \rho F(a_n, b_n)] - J_{\varphi}^{\rho}[a - \rho F(a,b)]\| \\
&\leq \alpha_n \|a_n - a\| + (1-\alpha_n)\|a_n - a - \rho(F(a_n,b_n) - F(a,b))\| \\
&\leq \alpha_n \|a_n - a\| + (1-\alpha_n)\|a_n - a - \rho(F(a_n,b_n) - F(a,b_n))\| \\
&\quad + (1-\alpha_n)\rho\|F(a,b_n) - F(a,b)\| \\
&\leq \alpha_n \|a_n - a\| + (1-\alpha_n)\sqrt{1 - 2\rho r_1 + \rho^2 s_1^2}\|a_n - a\| \\
&\quad + (1-\alpha_n)\rho\tau\|b_n - b\|
\end{aligned}
\tag{4.4}
$$

and

$$
\begin{aligned}
&\|b_{n+1} - b\| \\
&\leq \alpha_n \|b_n - b\| \\
&\quad + (1-\alpha_n)\|J_{\phi}^{\beta}[b_n - \beta G(a_n, b_n)] - J_{\phi}^{\beta}[b - \beta G(a,b)]\| \\
&\leq \alpha_n \|b_n - b\| + (1-\alpha_n)\|b_n - b - \beta(G(a_n,b_n) - G(a,b))\| \\
&\leq \alpha_n \|b_n - b\| + (1-\alpha_n)\|b_n - b - \beta(G(a_n,b_n) - G(a_n,b))\| \\
&\quad + (1-\alpha_n)\beta\|G(a_n,b) - G(a,b)\| \\
&\leq \alpha_n \|b_n - b\| + (1-\alpha_n)\sqrt{1 - 2\beta r_2 + \beta^2 s_2^2}\|b_n - b\| \\
&\quad + (1-\alpha_n)\beta\xi\|a_n - a\|.
\end{aligned}
\tag{4.5}
$$

It follows from (4.4) and (4.5) that

$$
\begin{aligned}
&\|a_{n+1} - a\| + \|b_{n+1} - b\| \\
&\leq \alpha_n(\|a_n - a\| + \|b_n - b\|) + (1-\alpha_n)k(\|a_n - a\| + \|b_n - b\|) \\
&= (k + (1-k)\alpha_n)(\|a_n - a\| + \|b_n - b\|),
\end{aligned}
\tag{4.6}
$$

where $0 < k < 1$ is defined by

$$
k = \max\{\sqrt{1 - 2\rho r_1 + \rho^2 s_1^2} + \beta\xi, \sqrt{1 - 2\beta r_2 + \beta^2 s_2^2} + \rho\tau\}.
$$

Let

$$
c_n = \|a_n - a\| + \|b_n - b\|, \quad k_n = k + (1-k)\alpha_n.
$$

Then (4.6) can be rewritten as follows:

$$c_{n+1} \leq k_n c_n, \quad n = 0, 1, 2 \ldots$$

By (4.2), we know that $\limsup_n k_n < 1$. Therefore, it follows from Lemma 2.4 that

$$\|a_n - a\| + \|b_n - b\| \to 0 \quad (n \to \infty)$$

and so the sequence $\{(a_n, b_n)\}$ converges strongly to the unique solution (a, b) of the problem (1.1). This completes the proof. □

By Theorem 4.1, we have the following result for the problem (1.2):

THEOREM 4.2. *Let $F : A \times B \to H_1$ and $G : A \times B \to H_2$ be two nonlinear mappings. Assume that all the conditions of Theorem 3.2 hold. For any given $(a_0, b_0) \in A \times B$, define the Mann iterative sequence $\{(a_n, b_n)\}$ by*

$$\begin{cases} a_{n+1} = \alpha_n a_n + (1 - \alpha_n) P_A[a_n - \rho F(a_n, b_n)], \\ b_{n+1} = \alpha_n b_n + (1 - \alpha_n)) P_B[b_n - \beta G(a_n, b_n)], \end{cases}$$

where

$$0 \leq \alpha_n < 1, \quad \limsup_{n \to \infty} \alpha_n < 1.$$

Then the sequence $\{(a_n, b_n)\}$ converges strongly to the unique solution (a, b) of the problem (1.2).

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