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HOMOMORPHISMS BETWEEN C^*-ALGEBRAS ASSOCIATED WITH THE TRIF FUNCTIONAL EQUATION AND LINEAR DERIVATIONS ON C^*-ALGEBRAS

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ABSTRACT. It is shown that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ of a unital C^*-algebra $\mathcal{A}$ to a unital C^*-algebra $\mathcal{B}$ is a homomorphism under some condition on multiplication, and that every almost linear continuous mapping $h : \mathcal{A} \to \mathcal{B}$ of a unital C^*-algebra $\mathcal{A}$ of real rank zero to a unital C^*-algebra $\mathcal{B}$ is a homomorphism under some condition on multiplication.

Furthermore, we are going to prove the generalized Hyers-Ulam-Rassias stability of $*$-homomorphisms between unital C^*-algebras, and of $\mathbb{C}$-linear $*$-derivations on unital C^*-algebras.

1. Introduction

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Rassias [7] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

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for all $x \in X$. Găvruta [2] generalized the Rassias' result.

Recently, Trif [8] proved the following: let $q := \frac{l(d-1)}{d-l}$, $r := -\frac{l}{d-l}$. Denote by $\varphi : X^d \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x_1, \ldots, x_d) = \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \ldots, q^j x_d) < \infty$$

for all $x_1, \ldots, x_d \in X$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\begin{aligned} &\| d \ {}_{d-2}C_{l-2} f\Big(\frac{x_1 + \cdots + x_d}{d}\Big) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} f(x_j) \\ &- l \sum_{1 \le j_1 < \cdots < j_l \le d} f\Big(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\Big) \| \\ &\le \varphi(x_1, \ldots, x_d) \end{aligned}$$

for all $x_1, \ldots, x_d \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{1}{l \cdot {}_{d-1}C_{l-1}} \widetilde{\varphi}(qx, \underbrace{rx, \ldots, rx}_{d-1 \text{ times}})$$

for all $x \in X$. And Park [6] applied the Trif's result to the Trif functional equation in Banach modules over a C^*-algebra.

B. E. Johnson [3, Theorem 7.2] also investigated almost algebra $*$-homomorphisms between Banach $*$-algebras : Suppose that $\mathcal{U}$ and $\mathcal{B}$ are Banach $*$-algebras which satisfy the conditions of [3, Theorem 3.1]. Then for each positive ϵ and K there is a positive δ such that if $T \in L(\mathcal{U}, \mathcal{B})$ with $\|T\| < K$, $\|T^\vee\| < \delta$ and $\|T(x^*)^* - T(x)\| \le \delta \|x\| \quad (x \in \mathcal{U})$ then there is a $*$-homomorphism $T' : \mathcal{U} \to \mathcal{B}$ with $\|T - T'\| < \epsilon$. Here $L(\mathcal{U}, \mathcal{B})$ is the space of bounded linear maps from $\mathcal{U}$ into $\mathcal{B}$, and $T^\vee(x, y) = T(xy) - T(x)T(y) \quad (x, y \in \mathcal{U})$. See [3] for details.

Throughout this paper, let $\mathcal{A}$ be a unital C^*-algebra with norm $|| \cdot ||$ and unit e, and $\mathcal{B}$ a unital C^*-algebra with norm $\| \cdot \|$. Let $\mathcal{U}(\mathcal{A})$ be the set of unitary elements in $\mathcal{A}$, $\mathcal{A}_{sa} = \{x \in \mathcal{A} \mid x = x^*\}$, and $I_1(\mathcal{A}_{sa}) = \{v \in \mathcal{A}_{sa} \mid ||v|| = 1, v \text{ is invertible}\}$. Let $q = \frac{l(d-1)}{d-l}$ and $r = -\frac{l}{d-l}$ for integers l, d with $2 \le l \le d-1$.

In this paper, we prove that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism when $h(q^n uy) = h(q^n u)h(y)$ holds for all $u \in \mathcal{U}(\mathcal{A})$, all

$y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, and that for a unital C^*-algebra $\mathcal{A}$ of real rank zero (see [1]), every almost linear continuous mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism when $h(q^n uy) = h(q^n u)h(y)$ holds for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$.

Furthermore, we are going to prove the generalized Hyers-Ulam-Rassias stability of $*$-homomorphisms between unital C^*-algebras, and of $\mathbb{C}$-linear $*$-derivations on unital C^*-algebras.

2. $*$-homomorphisms between unital C^*-algebras

We are going to investigate $*$-homomorphisms between unital C^*-algebras.

THEOREM 1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(q^n uy) = h(q^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, for which there exists a function $\varphi : \mathcal{A}^d \to [0, \infty)$ such that*

$$\text{(i)} \qquad \widetilde{\varphi}(x_1, \ldots, x_d) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \ldots, q^j x_d) < \infty,$$

$$\text{(ii)} \qquad \begin{aligned} &\| d \cdot {}_{d-2}C_{l-2} h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d}\right) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j) \\ &- l \sum_{1 \le j_1 < \cdots < j_l \le d} \mu h\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) \| \le \varphi(x_1, \ldots, x_d), \end{aligned}$$

$$\text{(iii)} \qquad \| h(q^n u^*) - h(q^n u)^* \| \le \varphi(\underbrace{q^n u, \ldots, q^n u}_{d \text{ times}})$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, all $u \in \mathcal{U}(\mathcal{A})$, all $n = 0, 1, 2, \ldots$, and all $x_1, \ldots, x_d \in \mathcal{A}$. Assume that (iv) $\lim_{n\to\infty} \frac{h(q^n e)}{q^n}$ *is invertible. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism.*

Proof. Put $\mu = 1 \in \mathbb{T}^1$. It follows from the Trif theorem [8, Theorem 3.1] that there exists a unique additive mapping $\Theta : \mathcal{A} \to \mathcal{B}$ such that

$$(\dagger) \qquad \| h(x) - \Theta(x) \| \le \frac{1}{l \cdot {}_{d-1}C_{l-1}} \widetilde{\varphi}(qx, \underbrace{rx, \ldots, rx}_{d-1 \text{ times}})$$

for all $x \in \mathcal{A}$. The additive mapping $\Theta : \mathcal{A} \to \mathcal{B}$ is given by

$$\Theta(x) = \lim_{n\to\infty} \frac{1}{q^n} h(q^n x)$$

for all $x \in \mathcal{A}$.

Put $x_1 = \cdots = x_d = x$ in (ii). For each $\mu \in \mathbb{T}^1$,

$$\|d \cdot {}_{d-2}C_{l-2}(h(\mu x) - \mu h(x))\| \le \varphi(\underbrace{x, \ldots, x}_{d \text{ times}})$$

for all $x \in \mathcal{A}$. So

$$q^{-n}\|d \cdot {}_{d-2}C_{l-2}(h(\mu q^n x) - \mu h(q^n x))\| \le q^{-n}\varphi(\underbrace{q^n x, \ldots, q^n x}_{d \text{ times}})$$

for all $x \in \mathcal{A}$. By (i),

$$q^{-n}\|d \cdot {}_{d-2}C_{l-2}(h(\mu q^n x) - \mu h(q^n x))\| \to 0$$

as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Thus

$$q^{-n}\|h(\mu q^n x) - \mu h(q^n x)\| \to 0$$

as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Hence

$$\Theta(\mu x) = \lim_{n\to\infty} \frac{h(q^n \mu x)}{q^n} = \lim_{n\to\infty} \frac{\mu h(q^n x)}{q^n} = \mu\Theta(x) \tag{1}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \ne 0$) and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [4, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. So by (1)

$$\begin{aligned}\Theta(\lambda x) &= \Theta(\frac{M}{3} \cdot 3\frac{\lambda}{M}x) = M \cdot \Theta(\frac{1}{3} \cdot 3\frac{\lambda}{M}x) = \frac{M}{3}\Theta(3\frac{\lambda}{M}x) \\ &= \frac{M}{3}\Theta(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(\Theta(\mu_1 x) + \Theta(\mu_2 x) + \Theta(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)\Theta(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}\Theta(x) = \lambda\Theta(x)\end{aligned}$$

for all $x \in \mathcal{A}$. Hence

$$\Theta(\zeta x + \eta y) = \Theta(\zeta x) + \Theta(\eta y) = \zeta\Theta(x) + \eta\Theta(y)$$

for all $\zeta, \eta \in \mathbb{C}(\zeta, \eta \ne 0)$ and all $x, y \in \mathcal{A}$. And $\Theta(0x) = 0 = 0\Theta(x)$ for all $x \in \mathcal{A}$. So the unique additive mapping $\Theta : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

By (i) and (iii), we get

$$\Theta(u^*) = \lim_{n\to\infty} \frac{h(q^n u^*)}{q^n} = \lim_{n\to\infty} \frac{h(q^n u)^*}{q^n} = (\lim_{n\to\infty} \frac{h(q^n u)}{q^n})^* = \Theta(u)^*$$

for all $u \in \mathcal{U}(\mathcal{A})$. Since Θ is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [5, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$,

$$\begin{aligned}\Theta(x^*) &= \Theta(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j}\Theta(u_j^*) = \sum_{j=1}^m \overline{\lambda_j}\Theta(u_j)^* \\ &= (\sum_{j=1}^m \lambda_j \Theta(u_j))^* = \Theta(\sum_{j=1}^m \lambda_j u_j)^* = \Theta(x)^*\end{aligned}$$

for all $x \in \mathcal{A}$.

Since $h(q^n uy) = h(q^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \dots,$

$$(2) \quad \Theta(uy) = \lim_{n\to\infty} \frac{1}{q^n} h(q^n uy) = \lim_{n\to\infty} \frac{1}{q^n} h(q^n u)h(y) = \Theta(u)h(y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of Θ and (2),

$$q^n\Theta(uy) = \Theta(q^n uy) = \Theta(u(q^n y)) = \Theta(u)h(q^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$(3) \qquad \Theta(uy) = \frac{1}{q^n}\Theta(u)h(q^n y) = \Theta(u)\frac{1}{q^n}h(q^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (3) as $n \to \infty$, we obtain

$$(4) \qquad \Theta(uy) = \Theta(u)\Theta(y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since Θ is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$, it follows from (4) that

$$\begin{aligned}\Theta(xy) &= \Theta(\sum_{j=1}^m \lambda_j u_j y) = \sum_{j=1}^m \lambda_j \Theta(u_j y) \\ &= \sum_{j=1}^m \lambda_j \Theta(u_j)\Theta(y) = \Theta(\sum_{j=1}^m \lambda_j u_j)\Theta(y) \\ &= \Theta(x)\Theta(y)\end{aligned}$$

for all $x, y \in \mathcal{A}$.

By (2) and (4),

$$\Theta(e)\Theta(y) = \Theta(ey) = \Theta(e)h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n\to\infty} \frac{h(q^n e)}{q^n} = \Theta(e)$ is invertible,

$$\Theta(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, as desired.□

COROLLARY 2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(q^n uy) = h(q^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|d \cdot {}_{d-2}C_{l-2} h(\frac{\mu x_1 + \cdots + \mu x_d}{d}) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j)$$

$$- l \sum_{1\leq j_1<\cdots<j_l\leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l})\| \leq \theta(\sum_{j=1}^{d} ||x_j||^p),$$

$$\|h(q^n u^*) - h(q^n u)^*\| \leq d q^{np}\theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, all $n = 0, 1, 2, \ldots$, and all $x_1, \ldots, x_d \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{h(q^n e)}{q^n}$ is invertible. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism.

Proof. Define $\varphi(x_1, \ldots, x_d) = \theta(\sum_{j=1}^{d} ||x_j||^p)$, and apply Theorem 1. □

THEOREM 3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(q^n uy) = h(q^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, for which there exists a function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying* (i), (iii), *and* (iv) *such that*

$$\|d \cdot {}_{d-2}C_{l-2} h(\frac{\mu x_1 + \cdots + \mu x_d}{d}) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j)$$

$$- l \sum_{1\leq j_1<\cdots<j_l\leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l})\|$$

$$\text{(v)} \qquad \leq \varphi(x_1, \ldots, x_d)$$

for $\mu = 1, i$, and all $x_1, \ldots, x_d \in \mathcal{A}$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism.

Proof. Put $\mu = 1$ in (v). By the same reasoning as the proof of Theorem 1, there exists a unique additive mapping $\Theta : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (†). By the same reasoning as the proof of [7, Theorem], the additive mapping $\Theta : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (v). By the same method as the proof of Theorem 1, one can obtain that

$$\Theta(ix) = \lim_{n\to\infty} \frac{h(q^n ix)}{q^n} = \lim_{n\to\infty} \frac{ih(q^n x)}{q^n} = i\Theta(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned}\Theta(\lambda x) &= \Theta(sx + itx) = s\Theta(x) + t\Theta(ix)\\ &= s\Theta(x) + it\Theta(x) = (s+it)\Theta(x)\\ &= \lambda\Theta(x)\end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. So

$$\Theta(\zeta x + \eta y) = \Theta(\zeta x) + \Theta(\eta y) = \zeta\Theta(x) + \eta\Theta(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $\Theta : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as the proof of Theorem 1. □

From now on, assume that $\mathcal{A}$ is a unital C^*-algebra of real rank zero, where "real rank zero" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [1]).

Now we are going to investigate continuous $*$-homomorphisms between unital C^*-algebras.

THEOREM 4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(q^n uy) = h(q^n u)h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, for which there exists a function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying* (i), (ii), (iii), *and* (iv). *Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism.*

Proof. By the same reasoning as the proof of Theorem 1, there exists a unique $\mathbb{C}$-linear involution $\Theta : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (†).

Since $h(q^n uy) = h(q^n u)h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots,$

$$\Theta(uy) = \lim_{n\to\infty} \frac{1}{q^n} h(q^n uy) = \lim_{n\to\infty} \frac{1}{q^n} h(q^n u)h(y) = \Theta(u)h(y) \tag{5}$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. By the additivity of Θ and (5),

$$q^n\Theta(uy) = \Theta(q^n uy) = \Theta(u(q^n y)) = \Theta(u)h(q^n y)$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. Hence

$$\Theta(uy) = \frac{1}{q^n}\Theta(u)h(q^n y) = \Theta(u)\frac{1}{q^n}h(q^n y) \tag{6}$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. Taking the limit in (6) as $n \to \infty$, we obtain

$$\Theta(uy) = \Theta(u)\Theta(y) \tag{7}$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$.

By (5) and (7),

$$\Theta(e)\Theta(y) = \Theta(ey) = \Theta(e)h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n\to\infty} \frac{h(q^n e)}{q^n} = \Theta(e)$ is invertible,

$$\Theta(y) = h(y)$$

for all $y \in \mathcal{A}$. So $\Theta : \mathcal{A} \to \mathcal{B}$ is continuous. But by the assumption that $\mathcal{A}$ has real rank zero, it is easy to show that $I_1(\mathcal{A}_{sa})$ is dense in $\{x \in \mathcal{A}_{sa} \mid ||x|| = 1\}$. So for each $w \in \{z \in \mathcal{A}_{sa} \mid ||z|| = 1\}$, there is a sequence $\{\kappa_j\}$ such that $\kappa_j \to w$ as $j \to \infty$ and $\kappa_j \in I_1(\mathcal{A}_{sa})$. Since $\Theta : \mathcal{A} \to \mathcal{B}$ is continuous, it follows from (7) that

$$\begin{aligned}\Theta(wy) &= \Theta(\lim_{j\to\infty} \kappa_j y) = \lim_{j\to\infty} \Theta(\kappa_j y)\\ &= \lim_{j\to\infty} \Theta(\kappa_j)\Theta(y) = \Theta(\lim_{j\to\infty} \kappa_j)\Theta(y)\\ &= \Theta(w)\Theta(y) \qquad (8)\end{aligned}$$

for all $w \in \{z \in \mathcal{A}_{sa} \mid ||z|| = 1\}$ and all $y \in \mathcal{A}$.

For each $x \in \mathcal{A}$, $x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i}$, where $x_1 := \frac{x+x^*}{2}$ and $x_2 := \frac{x-x^*}{2i}$ are self-adjoint.

First, consider the case that $x_1 \neq 0, x_2 \neq 0$. Since $\Theta : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (8) that

$$\begin{aligned}\Theta(xy) &= \Theta(x_1y + ix_2y) = \Theta(||x_1||\frac{x_1}{||x_1||}y + i||x_2||\frac{x_2}{||x_2||}y)\\ &= ||x_1||\Theta(\frac{x_1}{||x_1||}y) + i||x_2||\Theta(\frac{x_2}{||x_2||}y)\\ &= ||x_1||\Theta(\frac{x_1}{||x_1||})\Theta(y) + i||x_2||\Theta(\frac{x_2}{||x_2||})\Theta(y)\\ &= \{\Theta(||x_1||\frac{x_1}{||x_1||}) + i\Theta(||x_2||\frac{x_2}{||x_2||})\}\Theta(y) = \Theta(x_1 + ix_2)\Theta(y)\\ &= \Theta(x)\Theta(y)\end{aligned}$$

for all $y \in \mathcal{A}$.

Next, consider the case that $x_1 \neq 0, x_2 = 0$. Since $\Theta : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (8) that

$$\begin{aligned}\Theta(xy) &= \Theta(x_1y) = \Theta(||x_1||\frac{x_1}{||x_1||}y) = ||x_1||\Theta(\frac{x_1}{||x_1||}y)\\ &= ||x_1||\Theta(\frac{x_1}{||x_1||})\Theta(y) = \Theta(||x_1||\frac{x_1}{||x_1||})\Theta(y) = \Theta(x_1)\Theta(y)\\ &= \Theta(x)\Theta(y)\end{aligned}$$

for all $y \in \mathcal{A}$.

Finally, consider the case that $x_1 = 0, x_2 \neq 0$. Since $\Theta : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (8) that

$$\begin{aligned}\Theta(xy) &= \Theta(ix_2y) = \Theta(i||x_2||\frac{x_2}{||x_2||}y) = i||x_2||\Theta(\frac{x_2}{||x_2||}y)\\ &= i||x_2||\Theta(\frac{x_2}{||x_2||})\Theta(y) = \Theta(i||x_2||\frac{x_2}{||x_2||})\Theta(y) = \Theta(ix_2)\Theta(y)\\ &= \Theta(x)\Theta(y)\end{aligned}$$

for all $y \in \mathcal{A}$. Hence

$$\Theta(xy) = \Theta(x)\Theta(y)$$

for all $x, y \in \mathcal{A}$.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, as desired.□

Corollary 5. *Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(q^n uy) = h(q^n u)h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|d \cdot {}_{d-2}C_{l-2} h(\frac{\mu x_1 + \cdots + \mu x_d}{d}) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j)$$

$$- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l})\| \leq \theta(\sum_{j=1}^{d} ||x_j||^p),$$

$$\|h(q^n u^*) - h(q^n u)^*\| \leq d q^{np} \theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in I_1(\mathcal{A}_{sa})$, all $n = 0, 1, 2, \ldots$, and all $x_1, \ldots, x_d \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{h(q^n e)}{q^n}$ is invertible. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism.

Proof. Define $\varphi(x_1, \ldots, x_d) = \theta(\sum_{j=1}^{d} ||x_j||^p)$, and apply Theorem 4. □

Theorem 6. *Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(q^n uy) = h(q^n u)h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, for which there exists a function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying* (i), (iii), (iv), *and* (v). *Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism.*

Proof. By the same reasoning as the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $\Theta : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (†).

The rest of the proof is the same as the proofs of Theorems 1 and 4.□

3. Stability of $*$-homomorphisms between unital C^*-algebras

We are going to show the generalized Hyers-Ulam-Rassias stability of $*$-homomorphisms between unital C^*-algebras.

Theorem 7. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which* *there exists a function* $\varphi : \mathcal{A}^{d+2} \to [0,\infty)$ *such that*

(vi)

$$\widetilde{\varphi}(x_1,\dots,x_d,z,w) := \sum_{j=0}^{\infty} q^{-j}\varphi(q^j x_1,\dots,q^j x_d,q^j z,q^j w) < \infty,$$

(vii)

$$\|d\cdot {}_{d-2}C_{l-2}h(\frac{\mu x_1+\cdots+\mu x_d}{d}+\frac{zw}{d\ {}_{d-2}C_{l-2}})+{}_{d-2}C_{l-1}\sum_{j=1}^{d}\mu h(x_j)$$
$$-l\sum_{1\le j_1<\cdots<j_l\le d}\mu h(\frac{x_{j_1}+\cdots+x_{j_l}}{l})-h(z)h(w)\|$$
$$\le \varphi(x_1,\dots,x_d,z,w),$$

(viii)

$$\|h(q^n u^*)-h(q^n u)^*\| \le \varphi(\underbrace{q^n u,\dots,q^n u}_{d \text{ times}},0,0)$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in \mathcal{U}(\mathcal{A})$, *all* $n = 0,1,2,\dots$, *and all* $x_1,\dots,x_d,z,w \in \mathcal{A}$. *Then there exists a unique* $*$*-homomorphism* $\Theta : \mathcal{A} \to \mathcal{B}$ *such that*

$$\text{(ix)} \qquad \|h(x)-\Theta(x)\| \le \frac{1}{l\cdot {}_{d-1}C_{l-1}}\widetilde{\varphi}(qx,\underbrace{rx,\dots,rx}_{d-1 \text{ times}},0,0)$$

for all $x \in \mathcal{A}$.

Proof. Put $z = w = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (vii). By the same reasoning as the proof of Theorem 1, there exists a unique $\mathbb{C}$-linear involution $\Theta : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (ix). The $\mathbb{C}$-linear mapping $\Theta : \mathcal{A} \to \mathcal{B}$ is given by

$$\text{(9)} \qquad \Theta(x) = \lim_{n\to\infty}\frac{1}{q^n}h(q^n x)$$

for all $x \in \mathcal{A}$.

It follows from (9) that

$$\text{(10)} \qquad \Theta(x) = \lim_{n\to\infty}\frac{h(q^{2n}x)}{q^{2n}}$$

for all $x \in \mathcal{A}$. Let $x_1 = \cdots = x_d = 0$ in (vii). Then we get

$$\|d\cdot {}_{d-2}C_{l-2}h(\frac{zw}{d\cdot {}_{d-2}C_{l-2}})-h(z)h(w)\| \le \varphi(\underbrace{0,\dots,0}_{d \text{ times}},z,w)$$

for all $z, w \in \mathcal{A}$. Since

$$\frac{1}{q^{2n}}\varphi(\underbrace{0,\dots,0}_{d \text{ times}}, q^n z, q^n w) \le \frac{1}{q^n}\varphi(\underbrace{0,\dots,0}_{d \text{ times}}, q^n z, q^n w),$$

(11)

$$\begin{aligned}&\frac{1}{q^{2n}}\|d\cdot {}_{d-2}C_{l-2}h(\frac{1}{d\cdot {}_{d-2}C_{l-2}}q^n z\cdot q^n w) - h(q^n z)h(q^n w)\|\\ &\le \frac{1}{q^{2n}}\varphi(\underbrace{0,\dots,0}_{d \text{ times}}, q^n z, q^n w) \le \frac{1}{q^n}\varphi(\underbrace{0,\dots,0}_{d \text{ times}}, q^n z, q^n w)\end{aligned}$$

for all $z, w \in \mathcal{A}$. By (vi), (10), and (11),

$$\begin{aligned} d\cdot {}_{d-2}C_{l-2}\Theta(\frac{zw}{d\cdot {}_{d-2}C_{l-2}}) &= \lim_{n\to\infty}\frac{d\cdot {}_{d-2}C_{l-2}h(\frac{1}{d\cdot {}_{d-2}C_{l-2}}q^{2n}zw)}{q^{2n}}\\ &= \lim_{n\to\infty}\frac{d\cdot {}_{d-2}C_{l-2}h(\frac{1}{d\cdot {}_{d-2}C_{l-2}}q^{n}z\cdot q^n w)}{q^{n}\cdot q^n}\\ &= \lim_{n\to\infty}(\frac{h(q^n z)}{q^n}\cdot\frac{h(q^n w)}{q^n})\\ &= \lim_{n\to\infty}\frac{h(q^n z)}{q^n}\cdot\lim_{n\to\infty}\frac{h(q^n w)}{q^n}\\ &= \Theta(z)\Theta(w)\end{aligned}$$

for all $z, w \in \mathcal{A}$. But since Θ is $\mathbb{C}$-linear,

$$\Theta(zw) = d\cdot {}_{d-2}C_{l-2}\Theta(\frac{zw}{d\cdot {}_{d-2}C_{l-2}}) = \Theta(z)\Theta(w)$$

for all $z, w \in \mathcal{A}$. Hence the $\mathbb{C}$-linear mapping $\Theta : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism satisfying the inequality (ix), as desired. □

COROLLARY 8. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}&\|d\cdot {}_{d-2}C_{l-2}h(\frac{\mu x_1+\cdots+\mu x_d}{d}+\frac{zw}{d\cdot {}_{d-2}C_{l-2}}) + {}_{d-2}C_{l-1}\sum_{j=1}^{d}\mu h(x_j)\\ &-l\sum_{1\le j_1<\cdots<j_l\le d}\mu h(\frac{x_{j_1}+\cdots+x_{j_l}}{l}) - h(z)h(w)\|\\ &\le \theta(\sum_{j=1}^{d}\|x_j\|^p + \|z\|^p + \|w\|^p),\\ &\|h(q^n u^*) - h(q^n u)^*\| \le dq^{np}\theta\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in \mathcal{U}(\mathcal{A})$, *all* $n = 0, 1, 2, \ldots$, *and all* $x_1, \ldots, x_d, z, w \in \mathcal{A}$. *Then there exists a unique* $*$*-homomorphism* $\Theta : \mathcal{A} \to \mathcal{B}$ *such that*

$$\|h(x) - \Theta(x)\| \leq \frac{q^{1-p}(q^p + (d-1)r^p)\theta}{l \cdot {}_{d-1}C_{l-1}(q^{1-p} - 1)}||x||^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x_1, \ldots, x_d, z, w) = \theta(\sum_{j=1}^d ||x_j||^p + ||z||^p + ||w||^p)$, and apply Theorem 7. □

THEOREM 9. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be a mapping with* $h(0) = 0$ *for which there exists a function* $\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ *satisfying* (vi) *and* (viii) *such that*

$$\begin{aligned}&\|d \cdot {}_{d-2}C_{l-2} h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{zw}{d \cdot {}_{d-2}C_{l-2}}) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j) \\ &- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) - h(z)h(w)\| \\ &\leq \varphi(x_1, \ldots, x_d, z, w)\end{aligned}$$

for $\mu = 1, i$, *and all* $x_1, \ldots, x_d, z, w \in \mathcal{A}$. *If* $h(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{A}$, *then there exists a unique* $*$*-homomorphism* $\Theta : \mathcal{A} \to \mathcal{B}$ *satisfying the inequality* (ix).

Proof. By the same reasoning as the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $\Theta : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (ix).
The rest of the proof is the same as the proofs of Theorems 1 and 7.□

4. Stability of linear $*$-derivations on unital C^*-algebras

We are going to show the generalized Hyers-Ulam-Rassias stability of linear $*$-derivations on unital C^*-algebras.

THEOREM 10. *Let* $h : \mathcal{A} \to \mathcal{A}$ *be a mapping with* $h(0) = 0$ *for which there exists a function* $\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ *satisfying* (vi) *and* (viii) *such* *that*

$$\|d \cdot {}_{d-2}C_{l-2} h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{zw}{d \cdot {}_{d-2}C_{l-2}})$$
$$+ {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j)$$
$$- l \sum_{1 \le j_1 < \cdots < j_l \le d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) - zh(w) - wh(z)\|$$
$$\text{(x)} \qquad \le \varphi(x_1, \ldots, x_d, z, w)$$

for all $\mu \in \mathbb{T}^1$ *and all* $x_1, \ldots, x_d, z, w \in \mathcal{A}$. *Then there exists a unique* $\mathbb{C}$*-linear* $*$*-derivation* $D : \mathcal{A} \to \mathcal{A}$ *such that*

$$\text{(xi)} \qquad \|h(x) - D(x)\| \le \frac{1}{l \cdot {}_{d-1}C_{l-1}} \widetilde{\varphi}(qx, \underbrace{rx, \ldots, rx}_{d-1 \text{ times}}, 0, 0)$$

for all $x \in \mathcal{A}$.

Proof. Put $z = w = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (x). By the same reasoning as the proof of Theorem 1, there exists a unique $\mathbb{C}$-linear involution $D : \mathcal{A} \to \mathcal{A}$ satisfying the inequality (xi). The $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is given by

$$\text{(12)} \qquad D(x) = \lim_{n \to \infty} \frac{1}{q^n} h(q^n x)$$

for all $x \in \mathcal{A}$.

It follows from (12) that

$$\text{(13)} \qquad D(x) = \lim_{n \to \infty} \frac{h(q^{2n} x)}{q^{2n}}$$

for all $x \in \mathcal{A}$. Let $x_1 = \cdots = x_d = 0$ in (x). Then we get

$$\|d \cdot {}_{d-2}C_{l-2} h(\frac{zw}{d \cdot {}_{d-2}C_{l-2}}) - zh(w) - wh(z)\| \le \varphi(\underbrace{0, \ldots, 0}_{d \text{ times}}, z, w)$$

for all $z, w \in \mathcal{A}$. Since

$$\frac{1}{q^{2n}} \varphi(\underbrace{0, \ldots, 0}_{d \text{ times}}, q^n z, q^n w) \le \frac{1}{q^n} \varphi(\underbrace{0, \ldots, 0}_{d \text{ times}}, q^n z, q^n w),$$

$$\begin{aligned}&\frac{1}{q^{2n}}\|d\cdot {}_{d-2}C_{l-2}h(\frac{1}{d\cdot {}_{d-2}C_{l-2}}q^n z\cdot q^n w)\\&-q^n zh(q^n w)-q^n wh(q^n z)\|\\&\le \frac{1}{q^{2n}}\varphi(\underbrace{0,\dots,0}_{d\text{ times}},q^n z,q^n w)\le \frac{1}{q^n}\varphi(\underbrace{0,\dots,0}_{d\text{ times}},q^n z,q^n w)\end{aligned}\tag{14}$$

for all $z, w \in \mathcal{A}$.

By (x), (13), and (14),

$$\begin{aligned}d\cdot {}_{d-2}C_{l-2}D(\frac{zw}{d\cdot {}_{d-2}C_{l-2}})&=\lim_{n\to\infty}\frac{d\cdot {}_{d-2}C_{l-2}h(\frac{1}{d\cdot {}_{d-2}C_{l-2}}q^{2n}zw)}{q^{2n}}\\&=\lim_{n\to\infty}\frac{d\cdot {}_{d-2}C_{l-2}h(\frac{1}{d\cdot {}_{d-2}C_{l-2}}q^n z\cdot q^n w)}{q^n\cdot q^n}\\&=\lim_{n\to\infty}(\frac{q^n z}{q^n}\cdot\frac{h(q^n w)}{q^n}+\frac{q^n w}{q^n}\cdot\frac{h(q^n z)}{q^n})\\&=zD(w)+wD(z)\end{aligned}$$

for all $z, w \in \mathcal{A}$. But since D is $\mathbb{C}$-linear,

$$D(zw)=d\cdot {}_{d-2}C_{l-2}D(\frac{zw}{d\cdot {}_{d-2}C_{l-2}})=zD(w)+wD(z)$$

for all $z, w \in \mathcal{A}$. Hence the $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is a $*$-derivation satisfying the inequality (xi), as desired. □

COROLLARY 11. *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}&\|d\cdot {}_{d-2}C_{l-2}h(\frac{\mu x_1+\cdots+\mu x_d}{d}+\frac{zw}{d\cdot {}_{d-2}C_{l-2}})+{}_{d-2}C_{l-1}\sum_{j=1}^{d}\mu h(x_j)\\&-l\sum_{1\le j_1<\cdots<j_l\le d}\mu h(\frac{x_{j_1}+\cdots+x_{j_l}}{l})-zh(w)-wh(z)\|\\&\le\theta(\sum_{j=1}^{d}\|x_j\|^p+\|z\|^p+\|w\|^p),\\&\|h(q^n u^*)-h(q^n u)^*\|\le dq^{np}\theta\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in \mathcal{U}(\mathcal{A})$, *all* $n = 0, 1, 2, \ldots$, *and all* $x_1, \ldots, x_d, z, w \in \mathcal{A}$. *Then there exists a unique* $\mathbb{C}$*-linear* $*$*-derivation* $D : \mathcal{A} \to \mathcal{A}$ *such that*

$$\|h(x) - D(x)\| \leq \frac{q^{1-p}(q^p + (d-1)r^p)\theta}{l \cdot {}_{d-1}C_{l-1}(q^{1-p} - 1)} \|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x_1, \ldots, x_d, z, w) = \theta(\sum_{j=1}^{d} \|x_j\|^p + \|z\|^p + \|w\|^p)$, and apply Theorem 10. □

THEOREM 12. *Let* $h : \mathcal{A} \to \mathcal{A}$ *be a mapping with* $h(0) = 0$ *for which there exists a function* $\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ *satisfying* (vi) *and* (viii) *such that*

$$\begin{aligned} &\|d \cdot {}_{d-2}C_{l-2} h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{zw}{d \cdot {}_{d-2}C_{l-2}}\right) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j) \\ &- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) - z h(w) - w h(z)\| \\ &\leq \varphi(x_1, \ldots, x_d, z, w) \end{aligned}$$

for $\mu = 1, i$, *and all* $x_1, \ldots, x_d, z, w \in \mathcal{A}$. *If* $h(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{A}$, *then there exists a unique* $\mathbb{C}$*-linear* $*$*-derivation* $D : \mathcal{A} \to \mathcal{A}$ *satisfying the inequality* (xi).

Proof. By the same reasoning as the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ satisfying the inequality (xi).

The rest of the proof is the same as the proofs of Theorems 1 and 10. □

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