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IDENTIFICATION PROBLEMS FOR THE SYSTEM GOVERNED BY ABSTRACT NONLINEAR DAMPED SECOND ORDER EVOLUTION EQUATIONS

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ABSTRACT. Identification problems for the system governed by abstract nonlinear damped second order evolution equations are studied. Since unknown parameters are included in the diffusion operator, we can not simply identify them by using the usual optimal control theories. In this paper we present how to solve our identification problems via the method of transposition.

1. Introduction

In this paper we are concerned with identification problems for the system governed by abstract nonlinear damped second order evolution equations in the Hilbert space H:

$$\begin{cases} \dfrac{d^2y}{dt^2} + A_2(t,q)\dfrac{dy}{dt} + A_1(t,q)y = f(t,q,y) \quad \text{in} \quad (0,T), \\ y(0) = y_0 \in V_1, \quad \dfrac{dy(0)}{dt} = y_1 \in H, \end{cases} \tag{1.1}$$

where $A_1(t,q)$ and $A_2(t,q)$ are time dependent differential operators defined by bilinear forms on Hilbert spaces V_1 and V_2 ($V_1 \subset V_2 \subset H$), respectively, $f(t,q,y)$ is a nonlinear forcing function and these quantities depend on an unknown parameter q, which should be identified by some identification process. At present various theoretical and numerical methods for identifying or estimating unknown parameters have been extensively studied mainly for linear systems (cf. [1], [3]). One of the most powerful tools for identifying unknown parameters is the method of output least-squares, which is well-known as the optimal control theories

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studied by Lions [6]. It has shown its effectiveness in various applications to practical identification problems as in [1] and [3]. Hence we are intent to adopt this method for solving the identification problems of the nonlinear system (1.1) with the output error criterion given by a quadratic cost:

$$J(q) = \frac{1}{2}\|\mathcal{C}y(q) - z_d\|^2_{\mathcal{M}}, \quad q \in Q_{ad} \subset Q, \tag{1.2}$$

where $y(q)$ is a solution of (1.1), $\mathcal{C}$ is an observation operator, $\mathcal{M}$ is a space of observations, Q is a linear space of parameters, Q_{ad} is an admissible set of parameters and z_d is a desired value in $\mathcal{M}$. There are fundamentally two problems of identification for (1.1) with (1.2) such that one is to find an element $\overline{q} \in Q$ such that $J(\overline{q}) = \inf_{q \in Q_{ad}} J(q)$, $Q_{ad} \subset Q$, and the other is to deduce necessary conditions on $\overline{q}$.

For the linear cases where $A_2(t,q) = \gamma A_1(t,q)$ with $\gamma \geq 0$ and $f(t,q,y) = f(t)$, the identification problems have been investigated on the structure of a Gelfand triple $V_1 \hookrightarrow H \hookrightarrow V_1'$ by Ahmed [1]. The studies in [1] are generally enough to cover practical wave and damped wave equations, however, there exist many examples of equations satisfying $A_2(t,q) \neq \gamma A_1(t,q)$, which means damping and diffusion operators have different types of differential orders. Hence it is meaningful and interesting to study the case where $A_1(t,q)$ and $A_2(t,q)$ enjoy different differential orders. To study this case we requires another spatial structure, and so we will introduce a Gelfand five folds $V_1 \hookrightarrow V_2 \hookrightarrow H \hookrightarrow V_2' \hookrightarrow V_1'$. These notations will be explained later.

The purpose of this paper is to solve the identification problems for the system (1.1) with (1.2) on the structure of the Gelfand five folds. We remark that the results are not covered by the corresponding results in Ahmed [1] even if the system (1.1) is linear, i.e., $f(t,q,y) = f(t)$. To solve the identification problems for (1.1) with (1.2), it is fundamental to show that a nonlinear mapping $q \to y(q)$ from a space of parameters to a space of solutions is strongly continuous and weakly Gâteaux differentiable with respect to a topology on the space of solutions. It is not easy due to $f(t,q,y)$ and $A_1(t,q)y$ which does not belong to $L^2(0,T;V_2')$. Anyway to deduce such results we have to prove that the map $q \to y(q)$ is strongly continuous, and here we will do it by utilizing an energy equality for (1.1). We remark that we will prove the strong continuity without the compactness of spaces and monotone properties of operators (see e.g. Banks, Gilliam and Shubov [2]). It is desirable to prove the weak Gâteaux differentiability in the space of solutions, but we can not certain whether it is possible or not. Hence we will prove the differentiability of the map $q \to y(q)$ in a large space than the space of solutions by using the method of transposition to give the exact meaning of Gâteaux derivatives of $y(q)$ with respect to q. For this method, see Lions and Magenes [7]. As a consequence of the continuity and differentiability, the identification problems are established.

This paper is composed of four sections. In Section 2 we review notations, definition, assumptions and an auxiliary theorem. In Section 3 we prove the strong continuity of $y(q)$ in q and the existence of the optimal parameter $\overline{q}$. In Section 4 we deduce the necessary conditions for the optimal parameter $\overline{q}$.

2. Preliminaries

Let X be a Hilbert space. We denote the inner product and induced norm on X by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$, respectively. We denote the dual pairing between X' and X by $\langle\cdot,\cdot\rangle_{X',X}$, where X' is the dual space of X. Let $H = H'$ be a real pivot Hilbert space and denote simply its norm $\|\cdot\|_H$ by $|\cdot|_H$. For $i = 1, 2$, let V_i be real separable Hilbert spaces. From now we do not indicate $i = 1, 2$ explicitly. Assume that each pair (V_i, H) is a Gelfand triple space with a notation, $V_i \hookrightarrow H \equiv H' \hookrightarrow V_i'$, which means that V_i is dense in H and V_i is continuously imbedded in H. Let Q be a normed linear space and Q_{ad} (resp. Q_{bd}) be a convex (resp. bounded) subset of Q. Let $T < \infty$, $\mathbf{R} = (-\infty, \infty)$ and $\mathbf{R}^+ = [0, \infty)$.

For fixed $q \in Q$ let us define the families of bilinear forms $\{a_i(t, q; \phi, \psi)\}_{t\in[0,T]}$ on V_i satisfying conditions:

(2.1) $$a_i(t,q;\phi,\varphi) = a_i(t,q;\varphi,\phi) \quad \text{for all} \quad \phi,\ \psi \in V_i,\ t \in [0,T];$$

(2.2) $$\begin{cases} \text{there exists } c_{i1} > 0 \quad \text{such that} \\ |a_i(t,q;\phi,\varphi)| \le c_{i1}\|\phi\|_{V_i}\|\varphi\|_{V_i} \quad \text{for all} \quad \phi \in V_i,\ t \in [0,T]; \end{cases}$$

(2.3)
$$\begin{cases} \text{there exist } \alpha_i > 0 \text{ and } \lambda_i \in \mathbf{R} \text{ such that} \\ a_i(t,q;\phi,\phi) + \lambda_i|\phi|_H^2 \ge \alpha_i\|\phi\|_{V_i}^2 \quad \text{for all} \quad \phi,\ \psi \in V_i,\ t \in [0,T]; \end{cases}$$

(2.4)
$$\begin{cases} \text{a function} \quad t \to a_1(t,q;\phi,\varphi) \quad \text{is continuously differentiable} \\ \text{and} \quad \text{there exists } c_{12} > 0 \quad \text{such that} \\ |\dfrac{d}{dt}a_1(t,q;\phi,\varphi)| \le c_{12}\|\phi\|_{V_1}\|\varphi\|_{V_1} \quad \text{for all} \quad \phi,\ \psi \in V_1,\ t \in [0,T]. \end{cases}$$

Since all the constants $c_{i1}, \lambda_i, \alpha_i, c_{12}$ in (2.2)-(2.4) depend on q, we assume that there are positive numbers $\tilde{c}_{i1}, \tilde{\lambda}_i, \tilde{\alpha}_i, \tilde{c}_{12}$ such that

$$c_{i1} \le \tilde{c}_{i1}, \quad \lambda_i \le \tilde{\lambda}_i, \quad \alpha_i \ge \tilde{\alpha}_i, \quad c_{12} \le \tilde{c}_{12} \quad \text{on } Q_{bd}. \tag{2.5}$$

From (2.1)-(2.4) we can obtain the symmetric operators $A_i(t,q) \in \mathcal{L}(V_i, V_i')$ and $A_1'(t,q) \in \mathcal{L}(V_1, V_1')$ defined by

$$\langle A_i(t,q)\phi, \varphi\rangle_{V_i', V_i} = a_i(t,q;\phi,\varphi) \quad \text{for all } \phi, \varphi \in V_i \tag{2.6}$$

and

$$\langle A_1'(t,q)\phi, \varphi\rangle_{V_1', V_1} = \frac{d}{dt} a_1(t,q;\phi,\varphi) \quad \text{for all } \phi, \varphi \in V_1, \tag{2.7}$$

where $\mathcal{L}(V_i, V_i')$ is the space of bounded linear operators of V_i to V_i' with the strong operator topology.

We suppose that V_1 is continuously embedded in V_2. Then we see that $V_1 \hookrightarrow V_2 \hookrightarrow H \equiv H' \hookrightarrow V_2' \hookrightarrow V_1'$ and $\langle \phi, \varphi\rangle_{V_1', V_1} = \langle \phi, \varphi\rangle_{V_2', V_2}$ holds for $\phi \in V_2'$ and $\varphi \in V_1$. Also we see that $\langle \phi, \varphi\rangle_{V_1', V_1} = (\phi, \varphi)_H$ for $\phi \in H$ and $\varphi \in V_1$. Let c_1, c_2 (resp. c_2', c_1') be embedding constants for $V_1 \subset V_2, V_2 \subset H$ (resp. $H \subset V_2', V_2' \subset V_1'$), respectively.

Let us define a Hilbert space and its inner product by

$$\begin{aligned} \mathcal{W}(0,T) &= \{g | g \in L^2(0,T;V_1),\ g' \in L^2(0,T;V_2),\ g'' \in L^2(0,T;V_1')\}, \\ (g,h)_{\mathcal{W}} &= \int_0^T (g,h)_{V_1} dt + \int_0^T (g',h')_{V_2} dt + \int_0^T (g'',h'')_{V_1'} dt, \end{aligned}$$

where $' = \frac{d}{dt}$ and $'' = \frac{d^2}{dt^2}$ whose derivatives are taken in the sense of distribution. This space $\mathcal{W}(0,T)$ will be a space of solutions later.

Now let us consider a Cauchy problem for abstract nonlinear damped second order evolution equations given by

$$\begin{cases} y'' + A_2(t,q)y' + A_1(t,q)y = f(t,q,y) \quad \text{in } (0,T), \\ y(0) = y_0 \in V_1,\ y'(0) = y_1 \in H, \end{cases} \tag{2.8}$$

where $A_i(t,q)$ are operators defined in (2.6), $f : [0,T] \times Q \times V_2 \to V_2'$ and $q \in Q$ is fixed.

For the nonlinear term $f(t,q,y)$ in (2.8), we impose the following assumptions:

(A1) For each $(q,y) \in Q \times V_2$, $t \to f(t,q,y) : [0,T] \to V_2'$ is (strongly) measurable;

(A2) For each $q \in Q$ there exists $\beta \in L^2(0,T;\mathbf{R}^+)$ such that

$$\|f(t,q,y) - f(t,q,z)\|_{V_2'} \le \beta(t)\|y - z\|_{V_2} \quad \text{a.e. } t \in [0,T] \text{ for all } y, z \in V_2;$$

(A3) For each $q \in Q$ there exists $\gamma \in L^2(0,T;\mathbf{R}^+)$ such that

$$\|f(t,q,0)\|_{V_2'} \leq \gamma(t) \text{ a.e. } t \in [0,T].$$

Since the functions β and γ in (A2) and (A3) depend on q, we also assume that these functions are locally bounded with respect to q, i.e., there exist $\tilde{\beta}$ and $\tilde{\gamma} \in L^2(0,T;\mathbf{R}^+)$ such that

$$\beta(t) \leq \tilde{\beta}(t), \quad \gamma(t) \leq \tilde{\gamma}(t) \text{ on } Q_{bd}. \tag{2.9}$$

Now let us define a *weak solution* of (2.8), which will be a solution of the Cauchy problem (2.8). This definition is similar to the one given in [4].

DEFINITION 2.1. A function $y = y(q)$ is said to be a weak solution of (2.8) if $y \in \mathcal{W}(0,T)$ and y satisfies

$$\begin{cases} \langle y''(\cdot), \phi\rangle_{V_1',V_1} + a_2(\cdot,q;y'(\cdot),\phi) + a_1(\cdot,q;y(\cdot),\phi) \\ = \langle f(\cdot,q,y(\cdot)),\phi\rangle_{V_2',V_2} \text{ for all } \phi \in V_1 \text{ in the sense of } \mathcal{D}'(0,T) \\ y(0) = y_0 \in V_1, \quad y'(0) = y_1 \in H, \end{cases}$$

where $\mathcal{D}'(0,T)$ is the space of distributions on $(0,T)$.

The results of existence, uniqueness and regularity for the weak solutions for (2.8) are follows from Theorem 2.2. For the complete and detailed proof we refer to Ha and Nakagiri [5].

THEOREM 2.2. *Assume that* (2.1)-(2.4) *and* (A1)-(A3) *hold. If* $y_0 \in V_1, y_1 \in H$, *then the problem* (2.8) *has a unique weak solution* $y \in \mathcal{W}(0,T) \cap C([0,T];V_1) \cap C^1([0,T];H)$. *The solution* y *satisfies the energy equality*

$$\begin{aligned} & a_1(t,q;y(t),y(t)) + |y'(t)|_H^2 + 2\int_0^t a_2(\sigma,q;y'(\sigma),y'(\sigma))d\sigma \\ = \; & a_1(0,q;y_0,y_0) + |y_1|_H^2 + \int_0^t a_1'(\sigma,q;y(\sigma),y(\sigma))d\sigma \\ & +2\int_0^t \langle f(\sigma,q,y(\sigma)),y'(\sigma)\rangle_{V_2',V_2} d\sigma \end{aligned} \tag{2.10}$$

and the inequality

$$\begin{aligned} & \|y(t)\|_{V_1}^2 + |y'(t)|_H^2 + \int_0^t \|y'(\sigma)\|_{V_2}^2 d\sigma \\ \leq \; & c(\|y_0\|_{V_1}^2 + |y_1|_H^2 + \|f(\cdot,q,0)\|_{L^2(0,T;V_2')}^2) \end{aligned} \tag{2.11}$$

for all $t \in [0,T]$, *where* $c = c(q) > 0$ *is a constant dependent of* q.

By assumptions (2.5) and (2.9), the constant $c = c(q)$ in (2.11) has a finite upper bound $c(Q_{bd})$ on Q_{bd}, i.e., $c(q) \le c(Q_{bd}) < \infty$ on $q \in Q_{bd}$. Also by (A3) and (2.9) $\|\tilde{\gamma}\|_{L^2(0,T;\mathbf{R}^+)}$ is a finite upper bound $\|f(\cdot,q,0)\|_{L^2(0,T;V_2')}$ on Q_{bd}. Throughout this paper we assume that all conditions (2.1)-(2.4), (2.9) and (A1)-(A3) are satisfied.

3. Continuity on parameters and existence of optimal parameters

For each $q \in Q$ let us consider a nonlinear Cauchy problem involving q in the operators and nonlinear forcing function:

$$(3.1)\qquad \begin{cases} y'' + A_2(t,q)y' + A_1(t,q)y = f(t,q,y) \ \text{ in } \ (0,T), \\ y(0;q) = y_0 \in V_1, \quad y'(0;q) = y_1 \in H. \end{cases}$$

Since there is a unique weak solution $y = y(q)$ to (3.1) for each $q \in Q$ by Theorem 2.1, we have a well-defined mapping $q \to y(q) : Q \to \mathcal{W}(0,T)$. We shall call (3.1) the state equation and $y = y(q)$ the state of (3.1).

In this section we will prove that the map $q \to y(q)$ is strongly continuous. For this, we require the continuity of $A_i(t,q)$ and $f(t,q,y)$ on the parameter q. That is, we assume that

(B1) There are $k_{i1} \in C([0,T] \times \mathbf{R}^+)$ with $k_{i1}(t,0) \equiv 0$ such that

$$|a_i(t,q;\phi,\psi) - a_i(t,p;\phi,\psi)| \le k_{i1}(t,\|q-p\|_Q)\|\phi\|_{V_i}\|\psi\|_{V_i};$$

(B2) There is $k_{12} \in C([0,T] \times \mathbf{R}^+)$ with $k_{12}(t,0) \equiv 0$ such that

$$|\frac{d}{dt}[a_1(t,q;\phi,\psi) - a_1(t,p;\phi,\psi)]| \le k_{12}(t,\|q-p\|_Q)\|\phi\|_{V_1}\|\psi\|_{V_1};$$

(B3) There is a $k \in C([0,T] \times \mathbf{R}^+ \times \mathbf{R}^+)$ with $k(t,0,v) \equiv 0$ such that

$$\|f(t,q,y) - f(t,p,y)\|_{V_2'} \le k(t,\|q-p\|_Q,\|y\|_{V_2}).$$

THEOREM 3.1. *Under the assumptions* (B1)-(B3) *the map* $q \to y(q) : Q \to \mathcal{W}(0,T)$ *is strongly continuous.*

Proof. Let $q_n \to q$, where $q, q_n \in Q$. For $n = 1,2,\ldots$, let $y_n = y(q_n)$ be the weak solutions of

$$(3.2)\qquad \begin{cases} y_n'' + A_2(t,q_n)y_n' + A_1(t,q_n)y_n = f(t,q_n,y_n) \ \text{ in } \ (0,T), \\ y(0;q_n) = y_0 \in V_1, \quad y'(0;q_n) = y_1 \in H. \end{cases}$$

Since $Q_{bd} = \{q_n, q\}$ is bounded in Q, it follows from (2.11) that for each $t \in [0,T]$
(3.3)

$$\|y_n(t)\|_{V_1}^2 + |y_n'(t)|_H^2 + \int_0^t \|y_n'(\sigma)\|_{V_2}^2 d\sigma \le c(\|y_0\|_{V_1}^2 + |y_1|_H^2 + \|\tilde{\gamma}\|_{L^2(0,T;\mathbf{R}^+)}^2),$$

where $c = c(Q_{bd})$. The inequality (3.3) implies that $\{y_n\}$ is bounded in $L^\infty(0,T;V_1)$ and $\{y_n'\}$ is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V_2)$. It also follows from (2.9) that

$$\begin{aligned} \|f(t,q_n,y_n)\|_{V_2'} &\le \|f(t,q_n,0)\|_{V_2'} + \beta(t)\|y_n\|_{V_2} \\ &\le \tilde{\gamma}(t) + \tilde{\beta}(t)\|y_n\|_{V_2} \text{ on } Q_{bd}. \end{aligned} \tag{3.4}$$

Since $\{y_n\}$ is bounded in $L^\infty(0,T;V_1)$, it follows from (3.4) that $\{f(\cdot,q_n, y_n)\}$ is bounded in $L^2(0,T;V_2')$. Since $\{A_1(\cdot,q_n)y_n\}$ and $\{A_2(\cdot,q_n)y_n'\}$ are bounded in $L^2(0,T;V_1')$ by (2.2) and (2.5), it follows from (3.4) that $\{y_n''\}$ is bounded in $L^2(0,T;V_1')$. Hence we can extract a subsequence, denoting it by $\{y_n\}$ again, and choose $z \in \mathcal{W}(0,T)$ and $Y \in L^2(0,T;V_2')$ such that

$$\begin{cases} y_n \to z \text{ weakly in } L^2(0,T;V_1), \\ y_n' \to z' \text{ weakly in } L^2(0,T;V_2), \\ y_n'' \to z'' \text{ weakly in } L^2(0,T;V_1'), \\ f(\cdot,q_n,y_n) \to Y \text{ weakly in } L^2(0,T;V_2'), \\ z(0) = y_0, \ z'(0) = y_1. \end{cases} \tag{3.5}$$

Furthermore it follows from (3.3) and (3.5) that for fixed $t \in [0,T]$

$$\begin{cases} y_n(t) \to z(t) \text{ weakly in } V_1, \\ y_n'(t) \to z'(t) \text{ weakly in } H. \end{cases} \tag{3.6}$$

The first equation of (3.2) is rewritten by

$$\begin{aligned} & y_n'' + A_2(t,q)y_n' + A_1(t,q)y_n \\ = \ & (A_2(t,q) - A_2(t,q_n))y_n' + (A_1(t,q) - A_1(t,q_n))y_n + f(t,q_n,y_n). \end{aligned}$$

Multiplying both sides of the above equality by $\phi \in L^2(0,T;V_1)$ and integrating it over $[0,T]$, we have

$$\begin{aligned} & \int_0^T \langle y_n'' + A_2(t,q)y_n' + A_1(t,q)y_n, \phi\rangle_{V_1',V_1} dt \\ = & \int_0^T \langle (A_2(t,q) - A_2(t,q_n))\phi, y_n'\rangle_{V_2',V_2} dt \end{aligned} \tag{3.7}$$

$$+\int_0^T \langle (A_1(t,q)-A_1(t,q_n))\phi, y_n\rangle_{V_1',V_1}dt$$
$$+\int_0^T \langle f(t,q_n,y_n),\phi\rangle_{V_2',V_2}dt.$$

For a moment let us set $\psi_n = y_n$ when $i=1$ and $\psi_n = y_n'$ when $i=2$. Then by (B1) we deduce

$$|\langle (A_i(t,q)-A_i(t,q_n))\phi, \psi_n\rangle_{V_i',V_i}| \le k_{i1}(t,\|q_n-q\|_Q)\|\phi\|_{V_i}\|\psi_n\|_{V_i}.$$

Since $\{\psi_n\}$ is bounded in $L^2(0,T;V_i)$ and k_{i1} is continuous on Q, the above inequality implies by the Lebesgue dominated convergence theorem that

$$\int_0^T |\langle (A_i(t,q)-A_i(t,q_n))\phi, \psi_n\rangle_{V_i',V_i}|dt \to 0 \quad \text{as} \quad \|q_n-q\|_Q \to 0.$$

Hence, by taking $n\to\infty$ in (3.7) and by using (3.5) we have

$$\int_0^T \langle z'' + A_2(t,q)z' + A_1(t,q)z, \phi\rangle_{V_1',V_1}dt$$
$$= \int_0^T \langle Y,\phi\rangle_{V_2',V_2}dt, \quad \forall\phi\in L^2(0,T;V_1).$$

By applying the similar arguments in [5] to the above equality we can prove that z is a unique weak solution of linear equations:

$$(3.8)\qquad \begin{cases} z''+A_2(t,q)z'+A_1(t,q)z = Y(t) & \text{in} \quad (0,T), \\ z(0)=y_0\in V_1, \quad z'(0)=y_1\in H. \end{cases}$$

Now if $Y(t)=f(t,q,z)$ is shown, then $z=y(q)$ by the uniqueness of solutions to (3.1). It is not a easy task and it is so long. We will prove it by using the fact that $y_n\to z$ converges strongly. From the energy equalities for y_n and z we have

$$(3.9)\quad a_1(t,q_n;y_n(t),y_n(t)) + |y_n'(t)|_H^2 + 2\int_0^t a_2(\sigma,q_n;y_n'(\sigma),y_n'(\sigma))d\sigma$$
$$= a_1(0,q_n;y_0,y_0) + |y_1|_H^2 + \int_0^t a_1'(\sigma,q_n;y_n(\sigma),y_n(\sigma))d\sigma$$
$$+2\int_0^t \langle f(\sigma,q_n,y_n(\sigma)), y_n'(\sigma)\rangle_{V_2',V_2}d\sigma, \quad \forall t\in[0,T]$$

and

$$\tag{3.10} a_1(t,q;z(t),z(t)) + |z'(t)|_H^2 + 2\int_0^t a_2(\sigma,q;z'(\sigma),z'(\sigma))d\sigma$$
$$= a_1(0,q;y_0,y_0) + |y_1|_H^2 + \int_0^t a_1'(\sigma,q;z(\sigma),z(\sigma))d\sigma$$
$$+2\int_0^t \langle Y(\sigma),z'(\sigma)\rangle_{V_2',V_2}d\sigma, \quad \forall t\in[0,T].$$

It happens to omit to write time variables in equations for simplicity. Adding (3.9) to (3.10) we have

$$\tag{3.11} a_1(t,q_n;y_n-z,y_n-z) + |y_n'-z'|_H^2$$
$$+ 2\int_0^t a_2(\sigma,q_n;y_n'-z',y_n'-z')d\sigma$$
$$= Y_n^0 + \sum_{i=1}^4 Y_n^i(t) + \int_0^t a_1'(\sigma,q_n;y_n-z,y_n-z)d\sigma$$
$$+ 2\int_0^t \langle f(\sigma,q_n,y_n)-f(\sigma,q_n,z),y_n'-z'\rangle_{V_2',V_2}d\sigma,$$

where

$$Y_n^0 = a_1(0,q_n;y_0,y_0) + a_1(0,q;y_0,y_0) + 2|y_1|_H^2,$$
$$Y_n^1(t) = -2(y_n',z')_H - 2[a_1(t,q_n;y_n,z) - a_1(t,q;y_n,z)]$$
$$-2a_1(t,q;y_n,z),$$
$$Y_n^2(t) = -4\int_0^t [a_2(\sigma,q_n;y_n',z') - a_2(\sigma,q;y_n',z')]d\sigma$$
$$+2\int_0^t [a_1'(\sigma,q_n;y_n,z) - a_1'(\sigma,q;y_n,z)]d\sigma$$
$$-4\int_0^t a_2(\sigma,q;y_n',z')d\sigma + 2\int_0^t a_1'(\sigma,q;y_n,z)d\sigma,$$

$$Y_n^3(t) = 2\int_0^t \langle f(\sigma,q_n,z)-f(\sigma,q,z),y_n'-z'\rangle_{V_2',V_2}d\sigma$$
$$+2\int_0^t \langle f(\sigma,q,z),y_n'-z'\rangle_{V_2',V_2}d\sigma$$
$$+2\int_0^t \langle f(\sigma,q_n,y_n)+Y,z'\rangle_{V_2',V_2}d\sigma,$$

$$\begin{aligned} Y_n^4(t) &= a_1(t, q_n; z, z) - a_1(t, q; z, z) \\ &\quad +2\int_0^t \left[a_2(\sigma, q_n; z', z') - a_2(\sigma, q; z', z')\right] d\sigma \\ &\quad + \int_0^t \left[a_1'(\sigma, q; z, z) - a_1'(\sigma, q_n; z, z)\right] d\sigma. \end{aligned}$$

We set

$$Y_n(t) = Y_n^0 + \sum_{i=1}^4 Y_n^i(t).$$

Since

$$\begin{aligned} &\left| 2\int_0^t \langle f(\sigma, q_n, y_n) - f(\sigma, q_n, z), y_n' - z' \rangle_{V_2', V_2} d\sigma \right| \\ \le\ & 2\int_0^t \beta(\sigma) \|y_n - z\|_{V_2} \|y_n' - z'\|_{V_2} d\sigma \\ \le\ & \frac{c_1^2}{\epsilon} \int_0^t \beta^2(\sigma) \|y_n - z\|_{V_2}^2 d\sigma + \epsilon \int_0^t \|y_n' - z'\|_{V_2}^2 d\sigma \end{aligned}$$

for any $\epsilon > 0$, it follows from (2.1)-(2.4) and (3.11) that

$$\begin{aligned} (3.12) \qquad & \alpha_1 \|y_n - z\|_{V_1}^2 + |y_n' - y'|_H^2 + (2\alpha_2 - \epsilon) \int_0^t \|y_n' - z'\|_{V_2}^2 d\sigma \\ \le\ & Y_n(t) + (|\lambda_1| T + 2|\lambda_2|) \int_0^t |y_n' - z'|_H^2 d\sigma \\ & + \int_0^t [c_{12} + \epsilon^{-1} c_1^2 \beta^2(\sigma)] \|y_n - z\|_{V_1}^2 d\sigma, \end{aligned}$$

where $\alpha_i,\ \lambda_i,\ c_{12}, \beta$ are dependent on q_n. We can obtain an inequality independent of q_n on Q_{bd} as applying the assumptions (2.5) and (2.9) to (3.12). That is, we have

$$\begin{aligned} (3.13) \qquad & \tilde{\alpha}_1 \|y_n - z\|_{V_1}^2 + |y_n' - y'|_H^2 + (2\tilde{\alpha}_2 - \epsilon) \int_0^t \|y_n' - z'\|_{V_2}^2 d\sigma \\ \le\ & Y_n(t) + (|\tilde{\lambda}_1| T + 2|\tilde{\lambda}_2|) \int_0^t |y_n' - z'|_H^2 d\sigma \\ & + \int_0^t [\tilde{c}_{12} + \epsilon^{-1} c_1^2 \tilde{\beta}^2(\sigma)] \|y_n - z\|_{V_1}^2 d\sigma \ \text{ on } \ Q_{bd}. \end{aligned}$$

Let us choose ϵ sufficiently small so that $2\tilde{\alpha}_2 - \epsilon > 0$ and take $\alpha = \min\{\tilde{\alpha}_1, 2\tilde{\alpha}_2 - \epsilon, 1\} > 0$. If we set

$$\Phi_n(t) = \|y_n(t) - z(t)\|_{V_1}^2 + |y_n'(t) - z'(t)|_H^2 + \int_0^t \|y_n'(\sigma) - z'(\sigma)\|_{V_2}^2 d\sigma$$

and

$$h(t) = \alpha^{-1}(|\tilde{\lambda}_1|T + 2|\tilde{\lambda}_2| + \tilde{c}_{12} + \epsilon^{-1}c_1^2\tilde{\beta}^2(t)),$$

then (3.13) implies

$$\Phi_n(t) \leq \alpha^{-1}Y_n(t) + \int_0^t h(s)\Phi_n(s)ds \text{ on } Q_{ad} \text{ (omitted below).}$$

Since $Y_n(t)$ is continuous on $[0,T]$ and $h(t) \geq 0$, we can apply the extended Bellman-Gronwall inequality to have

$$\Phi_n(t) \leq \alpha^{-1}Y_n(t) + \alpha^{-1}\int_0^t \exp\left(\int_s^t h(\tau)d\tau\right) h(s)Y_n(s)ds. \tag{3.14}$$

Let $K(t,s) = \exp\left(\int_s^t h(\tau)d\tau\right) h(s)$ and $M_n(t) = \int_0^t K(t,s)Z_n(s)ds$. Then we see easily that

$$|K(t,s)| \leq \exp\left(\|h\|_{L^1(0,T;\mathbf{R}^+)}\right) h(s)$$

and $M_n(t)$ is uniformly bounded on $[0,T]$. Hence, in order to show that $\lim_{n\to\infty} \Phi_n(t) = 0$ for each $t \in [0,T]$, it is sufficient to prove

$$\lim_{n\to\infty} Y_n(t) = 0 \text{ for all } t \in [0,T]. \tag{3.15}$$

Since $|a_1(0,q_n;y_0,y_0) - a_1(0,q;y_0,y_0)| \leq k_{11}(0,\|q_n - q\|_Q)\|y_0\|_{V_1}^2$ by (B1), we have

$$Y_n^0 \to 2a_1(0,q;y_0,y_0) + 2|y_1|_H^2. \tag{3.16}$$

Also since

$$\begin{aligned}&|a_1(t,q_n;y_n,z) - a_1(t,q;y_n,z)| \\ &\leq k_{11}(t,\|q_n - q\|_Q)\|y_n(t)\|_{V_1}\|z(t)\|_{V_1}\end{aligned}$$

by (B1) and $\{y_n(t), z(t)\}$ is uniformly bounded in V_1, it follows from (3.6) and $a_1(t;q;y_n(t),z(t)) = \langle A_1(t,q)z(t), y_n(t)\rangle_{V_1',V_1}$ that

$$Y_n^1(t) \to -2|z(t)|_H^2 - 2a_1(t,q;z(t),z(t)). \tag{3.17}$$

By (B1), we have by the Lebesgue dominated convergence theorem and boundedness of $\{y_n'\}$ in $L^2(0,T;V_2)$ that

$$\begin{aligned} &\left|\int_0^t [a_2(\sigma,q_n;y_n',z') - a_2(\sigma,q;y_n',z')]d\sigma\right| \\ \le\ & \int_0^t k_{21}(\sigma,\|q_n-q\|_Q)\|y_n'\|_{V_2}\|z'\|_{V_2}d\sigma \\ \le\ & \left(\int_0^t k_{21}^2(\sigma,\|q_n-q\|_Q)\|z'\|_{V_2}^2 d\sigma\right)^{1/2}\left(\int_0^t \|y_n'\|_{V_2}^2 d\sigma\right)^{1/2} \to 0. \end{aligned}$$

By the similar calculations, it follows from the assumptions (B1), (B2) and (3.5) that

$$Y_n^2(t) \to -4\int_0^t a_2(\sigma,q;z',z')d\sigma + 2\int_0^t a_1'(\sigma,q;z,z)d\sigma. \tag{3.18}$$

Similarly, it follows from (B1)-(B3) and (3.5)-(3.6) that

$$Y_n^3(t) \to 4\int_0^t \langle Y,z'\rangle_{V_2',V_2}d\sigma \tag{3.19}$$

and

$$Y_n^4(t) \to 0. \tag{3.20}$$

By using all the results (3.10) and (3.16)-(3.20) we have

$$\begin{aligned} & Y_n(t) = Y_n^0 + \sum_{i=2}^{4} Y_n^i(t) \\ \to\ & 2a_1(0,q;y_0,y_0) + |y_1|_H^2 - 2a_1(t,q;z(t),z(t)) - |z'(t)|_H^2 \\ & +2\left\{-2\int_0^t a_2(\sigma,q;z',z')d\sigma + \int_0^t a_1'(\sigma,q;z,z)d\sigma\right\} \\ & +2\left\{2\int_0^t \langle Y(\sigma),z'\rangle_{V_2',V_2}d\sigma\right\} = 0, \end{aligned}$$

which implies

$$\lim_{n\to\infty}\Phi_n(t) = 0, \quad \forall t\in[0,T]. \tag{3.21}$$

Hence by (3.21) we have

(3.22) $y_n \to z$ strongly in $C([0,T];V_1)$ and in $L^2(0,T;V_1)$,

(3.23) $y_n' \to z'$ strongly in $C([0,T];H)$ and in $L^2(0,T;V_2)$.

Now it is easy to verify $Y(t) = f(t, q, z(t))$ in V_2, *a.e.* $t \in [0, T]$ by applying (3.22), (3.23) and (B3) to inequalities:

$$
\begin{aligned}
(3.24) \qquad & \|f(t, q_n, y_n) - f(t, q, z)\|_{V_2'} \\
\le\; & \|f(t, q_n, y_n) - f(t, q_n, z)\|_{V_2'} + \|f(t, q_n, z) - f(t, q, z)\|_{V_2'} \\
\le\; & \tilde{\beta}(t)\|y_n - z\|_{V_2} + k(t, \|q_n - q\|_Q, \|z\|_{V_2}).
\end{aligned}
$$

Finally, let us subtract (3.8) from (3.2) to have

$$
\begin{aligned}
(z - y_n)'' \;=\; & (A_2(t, q) - A_2(t, q_n))y_n' + A_2(t, q)(z' - y_n') \\
& +(A_1(t, q) - A_1(t, q_n))y_n - A_1(t, q)(z - y_n) \\
& +f(t, q, z) - f(t, q_n, y_n).
\end{aligned}
$$

Note that the last term converges to 0 in $L^2(0, T; V_2)$ by (3.24) and (B3). Hence it follows from (B1), (3.22)-(3.23) and (3.24) that $y_n'' \to z''$ strongly in $L^2(0, T; V_1')$. Since any sequence $\{q_n\}$ converging to q in Q has a subsequence $\{q_{n_k}\}$ such that $y(q_{n_k}) \to y(q)$ in $\mathcal{W}(0, T)$ strongly and uniquely, we can conclude that the mapping $q \to y(q)$ is strongly continuous in $\mathcal{W}(0, T)$. □

Now we go back to the identification problems. The parameter q is estimated through the medium of an output error criterion given by

$$
(3.25) \qquad J(q) = \frac{1}{2}\|\mathcal{C}y(q) - z_d\|_{\mathcal{M}}^2 \quad \text{for} \quad q \in Q,
$$

where $\mathcal{M}$ is a Hilbert space of observations, $\mathcal{C} \in \mathcal{L}(\mathcal{W}(0, T), \mathcal{M})$ is an observer and z_d is a desired value belonging to $\mathcal{M}$.

Our purpose of this paper is to find a minimizing element $\overline{q} \in Q_{ad}$ such that

$$
(3.26) \qquad J(\overline{q}) = \min_{q \in Q_{ad}} J(q)
$$

and derive necessary conditions on such $\overline{q}$. We call $\overline{q}$ the optimal parameter and $y = y(\overline{q})$ the optimal state for (3.25).

For the existence of the optimal parameters, we have the following Theorem 3.2 from Theorem 3.1.

THEOREM 3.2. *Assume that* (B1)-(B3) *hold. If* Q_{ad} *is compact in* Q, *then there exists at least one optimal parameter* $\overline{q} \in Q_{ad}$.

Proof. Let $\{q_n\}$ be a minimizing sequence such that

$$
J(q_n) \to \inf_{q \in Q_{ad}} J(q).
$$

Since Q_{ad} is compact, there exist a subsequence $\{q_{nk}\}$ and $\overline{q} \in Q_{ad}$ such that $\|q_{nk} - \overline{q}\|_Q \to 0$. Hence by Theorem 3.1, $y(q_{nk}) \to y(\overline{q})$ in $\mathcal{W}(0, T)$, so that by (3.25) $J(\overline{q}) = \lim_{k\to\infty} J(q_{nk}) = \inf_{q\in Q_{ad}} J(q)$. This proves that $\overline{q}$ is an optimal parameter. □

4. Necessary conditions

The purpose of this section is to deduce necessary conditions for $\overline{q}$ such that $J(\overline{q}) = \inf_{q\in Q_{ad}} J(q)$ with $J(q) = \frac{1}{2}\|\mathcal{C}y(q) - z_d\|^2_{\mathcal{M}}$. Throughout this section we assume that Q_{ad} is bounded, closed and convex in Q and $\overline{q} \in Q_{ad}$.

One classical method of being able to deduce necessary conditions on $\overline{q}$ is to evaluate the first variation of $J(q)$ around $\overline{q}$. Since $J(q)$ depends on $y(q)$ directly, it is enough to calculate that of $y(q)$ around $\overline{q}$. Hence if $y(q)$ is Gâteaux differentiable at $\overline{q} \in Q_{ad}$ in the direction $q - \overline{q}$ and its Gâteaux derivative $z = d_G y(\overline{q})[q - \overline{q}]$ belongs to $\mathcal{W}(0,T)$, then necessary conditions on $\overline{q}$ can be characterized by the inequality as follows:

(4.1)
$$d_G J(\overline{q})[q-\overline{q}] = \langle \mathcal{C}^*\Lambda_{\mathcal{M}}\mathcal{C}y(\overline{q}) - z_d, z\rangle_{\mathcal{W}(0,T)',\mathcal{W}(0,T)} \geq 0 \quad \text{for all} \quad q \in Q_{ad}.$$

Here in (4.1) $\mathcal{C}^* \in \mathcal{L}(\mathcal{M}', \mathcal{W}(0,T)')$ is the adjoint operator of $\mathcal{C}$ and $\Lambda_{\mathcal{M}}$ is the canonical isomorphism of $\mathcal{M}$ onto $\mathcal{M}'$ in the sense that $\langle \Lambda_{\mathcal{M}}\phi, \phi\rangle_{\mathcal{M}',\mathcal{M}} = \|\phi\|^2_{\mathcal{M}}$ and $\|\Lambda_{\mathcal{M}}\phi\|_{\mathcal{M}'} = \|\phi\|_{\mathcal{M}}$ for all $\phi \in \mathcal{M}$. It may not expected in general that the Gâteaux derivative $d_G y(\overline{q})[q - \overline{q}]$ exists in $\mathcal{W}(0,T)$. This is convinced by the fact that we can not utilize the energy equality for the difference $z_\lambda = [y(\overline{q} + \lambda(q - \overline{q})) - y(\overline{q})]/\lambda$, because $A_1(t,\overline{q})y(\overline{q})$ does not belong to $L^2(0,T;V_2')$. In other words, it is not sure whether z_λ can be estimated in $\mathcal{W}(0,T)$. If it can't, we have to adopt a new space where z_λ can be estimated. Of course, we need a new method other than the classical one. Here we are going to use the method of transposition to estimate z_λ in $L^2(0,T;V_2)$ which is a larger space than $\mathcal{W}(0,T)$. Furthermore we are going to estimate $z_\lambda(T)$ in the space H.

For Gâteaux differentiability of $y(q)$ at $\overline{q}$, we impose the following assumptions:

(A4) For each $y \in V_2$, $f(t,\cdot,y)$ is Gâteaux differentiable on Q_{ad} for a.e. $t \in [0,T]$, and for each $q \in Q_{ad}$, $f(t,q,\cdot)$ is Frêchet differentiable on V_2 for a.e. $t \in [0,T]$ and the Gâteaux derivative $d_{G_q} f(t,q,y)$ and the Frêchet derivative $d_y f(t,q,y)$ are continuous on $Q \times V_2$ for a.e. $t \in [0,T]$. Furthermore for a bounded subset

V_{bd} of V_2, there are $\tilde{\beta}_i \in L^2(0,T;\mathbf{R}^+)$ depending on V_{bd} such that

$$\|d_{G_q} f(t,q,y)\|_{\mathcal{L}(Q,V_2')} \leq \tilde{\beta}_1(t) \text{ for all } (q,y) \in Q_{ad} \times V_{bd} \text{ a.e. } t \in [0,T],$$
$$\|d_y f(t,q,y)\|_{\mathcal{L}(V_2,V_2')} \leq \tilde{\beta}_2(t) \text{ for all } (q,y) \in Q_{ad} \times V_{bd} \text{ a.e. } t \in [0,T];$$

(A5) For t, ϕ, ψ, $a_i(t,\cdot;\phi,\psi)$ is Gâteaux differentiable on Q_{ad} and there are $\tilde{c}_{i3} > 0$ such that the Gâteaux derivatives $d_{G_q} a_i(t,q;\phi,\psi)$ satisfy

$$\|d_{G_q} a_i(t,q;\phi,\psi)\|_{\mathcal{L}(Q,\mathbf{R})} \leq \tilde{c}_{i3}\|\phi\|_{V_i}\|\psi\|_{V_i} \quad \text{for all } (t,q) \in [0,T] \times Q_{ad}.$$

For simplifying notations we will write $d_y f = f_y, d_{G_q} = d_G$. Let us denote the adjoint operator of $f_y(t,q,y) \in \mathcal{L}(V_2,V_2')$ by $f_y^*(t,q,y) \in \mathcal{L}(V_2,V_2')$.

In order to prove the weak Gâteaux differentiability of $q \to (y(T,q), y(q))$ in $H \times L^2(0,T;V_2)$ through the method of transposition, we have to characterize the adjoint equations which stem from estimating z_λ in $L^2(0,T;V_2)$ and $z_\lambda(T)$ in H.

Let $q, \overline{q} \in Q$ and $\lambda \in [0,1]$ be arbitrarily fixed. For each $(q,\lambda) \in Q_{ad} \times [0,1]$ we suppose that there is an operator $L(q,\lambda;\cdot) \in L^2(0,T;\mathcal{L}(V_2,V_2'))$. We consider the following terminal value problem of linear damped second evolution equations depending on $q, \overline{q}, \lambda$ given by

(4.2)
$$\begin{cases} \phi'' - A_2(t,\overline{q})\phi' - A_2'(t,\overline{q})\phi + A_1(t,\overline{q})\phi + L(q,\lambda;t)\phi = g \quad \text{in} \quad (0,T), \\ \phi(T) = 0, \ \phi'(T) = \phi_1, \end{cases}$$

where $\phi_1 \in H$ and $g \in L^2(0,T;V_2')$. To give the problem (4.2) an exact meaning, we suppose additional conditions on the bilinear form a_2:

(4.3)
$$\begin{cases} \text{For all } \ \phi,\psi \in V_2, \quad t \to a_2(t,\overline{q};\phi,\varphi) \ \text{ is continuously differentiable} \\ \text{and there exists } c_{22} > 0 \ \text{ such that} \\ |\dfrac{d}{dt} a_2(t,\overline{q};\phi,\varphi)| \leq c_{22}\|\phi\|_{V_2}\|\varphi\|_{V_2} \ \text{ for all } \ t \in [0,T]. \end{cases}$$

Then it is proved via Theorem 2.1 that the problem (4.2) has a unique weak solution $\phi = \phi(q,\lambda;\phi_1,g) \in \mathcal{W}(0,T)$ due to (4.3) if we consider reversed time flow $t \to T - t$. Furthermore the solution ϕ is estimated by

$$\max_{t \in [0,T]} \|\phi(t)\|_{V_1} \leq \sqrt{c}(|\phi_1|_H + \|g\|_{L^2(0,T;V_2')}), \tag{4.4}$$

where c depends on $q, \overline{q}$ and λ but not on ϕ_1 and g. If there exists $\tilde{\beta}_3 \in L^2(0,T;\mathbf{R}^+)$ such that
(4.5)
$$\|L(q,\lambda;t)\|_{\mathcal{L}(V_2,V_2')} \leq \tilde{\beta}_3(t) \ \text{ a.e. } \ t \in [0,T] \ \text{ for all } \ (q,\lambda) \in Q_{ad} \times [0,1],$$

then the constant $c > 0$ in (4.4) can be taken independent on $q, \overline{q}$ and λ. We denote the constant by $\tilde{c} > 0$.

Next let us explain a modified transposition method suitable for nonlinear equations. For each $(q,\lambda) \in Q_{ad} \times [0,1]$, there exists a unique weak solution $\phi = \phi(q,\lambda;\phi_1,g) \in \mathcal{W}(0,T)$ of (4.2) provided that $\phi_1 \in H$, $g \in L^2(0,T;V_2')$ and $L(q,\lambda;\cdot) \in L^2(0,T;\mathcal{L}(V_2,V_2'))$. We denote by $X[q,\lambda]$ be a set of such all solutions $\phi = \phi(q,\lambda;\phi_1,g) \in \mathcal{W}(0,T)$. Since the problem (4.2) is linear, we can define an inner product on $X[q,\lambda]$ by $(\phi,\psi)_{X[q,\lambda]} = (\phi_1,\psi_1)_H + (g,h)_{L^2(0,T;V_2')}$ for $\phi = \phi(q,\lambda;\phi_1,g)$, $\psi = \phi(q,\lambda;\psi_1,h)$. It is easily verified that $(X[q,\lambda],(\cdot,\cdot)_{X[q,\lambda]})$ is a Hilbert space and the map $\phi = \phi(q,\lambda;\phi_1,g) \to (\phi_1,g)$ of $X[q,\lambda]$ onto $H \times L^2(0,T;V_2')$ is an isomorphism. Let us define the operator $\Phi[q,\lambda;t] : \mathcal{W}(0,T) \to L^2(0,T;V_2')$ by

$$\Phi[q,\lambda;t](\phi) = \phi'' - A_2'(t,\overline{q})\phi - A_2(t,\overline{q})\phi' + A_1(t,\overline{q})\phi + L(q,\lambda;t)\phi. \tag{4.6}$$

By the method of transposition due to Lions and Magenes [7], for a bounded linear functional l defined on $X[q,\lambda]$ there is a unique solution $(\zeta_1,\zeta) \in H \times L^2(0,T;V_2)$ such that
(4.7)
$$(\zeta_1,\phi_1)_H + \int_0^T \langle \zeta(t), \Phi[q,\lambda;t](\phi)(t)\rangle_{V_2,V_2'} dt = l(\phi) \ \text{ for all } \ \phi \in X[q,\lambda].$$

Particularly for $(q,\lambda) = (\overline{q},0) \in Q_{ad} \times [0,1]$ we define the operator $L(\overline{q},0;t)$ by $-f_y^*(t,\overline{q},y(t,\overline{q})) \in \mathcal{L}(V_2,V_2')$, *a.e.* $t \in [0,T]$. Then by (A4), $L(\overline{q},0;\cdot) \in L^2(0,T;\mathcal{L}(V_2,V_2'))$. We also denote by $\Phi[\overline{q},0;t]$ the corresponding operator in (4.6) and by $X[\overline{q},0]$ the solution space for this $(\overline{q},0)$.

LEMMA 4.1. *Assume that* (B1)-(B3), (A4), (A5) *and* (4.3) *hold. Then the map* $q \to (y(T,q),y(q)) : Q \to H \times L^2(0,T;V_2)$ *is weakly Gâteaux differentiable at* $\overline{q} \in Q_{ad}$ *in the direction* $q - \overline{q}$, $q \in Q_{ad}$. *Furthermore* $(z_1,z) = (d_G y(T,\overline{q})[q-\overline{q}], d_G y(\overline{q})[q-\overline{q}]) \in H \times L^2(0,T;V_2)$ *satisfies the*

transposed equation:
(4.8)

$$\begin{cases} -(z_1,\phi_1)_H + \displaystyle\int_0^T \langle z(t), \Phi[\overline{q},0;t](\phi)(t)\rangle_{V_2,V_2'}dt \\ = -\displaystyle\int_0^T d_G a_2(t,\overline{q};y'(t,\overline{q}),\phi(t))[q-\overline{q}]dt \\ \quad -\displaystyle\int_0^T d_G a_1(t,\overline{q};y(t,\overline{q}),\phi(t))[q-\overline{q}]dt \\ \quad +\displaystyle\int_0^T \langle d_G f(t,\overline{q},y(t,\overline{q}))[q-\overline{q}],\phi(t)\rangle_{V_2',V_2}dt \quad \text{for all} \quad \phi \in X[\overline{q},0], \end{cases}$$

where

$$\Phi[\overline{q},0;t](\phi)(t) = \phi''(t) - A_2'(t,\overline{q})\phi(t) - A_2(t,\overline{q})\phi'(t) + A_1(t,\overline{q})\phi(t) \\ - f_y^*(t,\overline{q},y(t,\overline{q}))\phi(t).$$

Proof. Let us show the weak Gâteaux differentiability of $q \to (y(T,q), y(q))$ at $\overline{q}$ in the direction $q-\overline{q}$, $q \in Q_{ad}$. For this, let us set $q_\lambda = \overline{q}+\lambda(q-\overline{q})$, $\lambda \in [0,1]$. Then $q_\lambda \in Q_{ad}$ due to convexity of Q_{ad} and $\|q_\lambda - \overline{q}\|_Q = \lambda\|q-\overline{q}\|_Q \to 0$ as $\lambda \to 0$. Since $\{q_\lambda\}_{\lambda\in[0,1]} \subset Q_{ad}$ and Q_{ad} is bounded, by Theorem 3.1 we have

$$(4.9) \qquad y(q_\lambda) \to y(\overline{q}) \text{ strongly in } \mathcal{W}(0,T) \text{ as } \lambda \to 0.$$

We set $y_\lambda = y(q_\lambda)$ for $\lambda \in [0,1]$ and $\overline{y} = y(\overline{q})$, which are the weak solutions to (3.26) for given parameters q_λ and $\overline{q}$, respectively. We remark that $y_\lambda, \overline{y} \in \mathcal{W}(0,T)\cap C([0,T];V_1)\cap C^1([0,T];H)$. Referring to (3.3) we have a uniform boudedness as follows:

$$(4.10) \qquad \sup\{\|y_\lambda(t)\|_{V_1}^2 + |y_\lambda'(t)|_H^2 + \int_0^t \|y_\lambda'(\sigma)\|_{V_2}^2 d\sigma \\ : (t,q,\lambda) \in [0,T]\times Q_{ad}\times[0,1]\} < \infty.$$

For $\lambda \in (0,1]$ the quotient $z_\lambda = (y_\lambda - \overline{y})/\lambda$ satisfies the following equations

$$\begin{cases} z_\lambda'' + A_2(t,\overline{q})z_\lambda' + A_1(t,\overline{q})z_\lambda \\ \quad = -\dfrac{A_2(t,q_\lambda)-A_2(t,\overline{q})}{\lambda}y_\lambda' - \dfrac{A_1(t,q_\lambda)-A_1(t,\overline{q})}{\lambda}y_\lambda \\ \qquad +\dfrac{f(t,q_\lambda,\overline{y})-f(t,\overline{q},\overline{y})}{\lambda} + \dfrac{f(t,q_\lambda,y_\lambda)-f(t,q_\lambda,\overline{y})}{\lambda}, \\ z_\lambda(0) = z_\lambda'(0) = 0 \end{cases}$$

in the weak sense, or equivalently

(4.11)
$$\begin{cases} z_\lambda'' + A_2(t,\overline{q})z_\lambda' + A_1(t,\overline{q})z_\lambda - \int_0^1 f_y(t,q_\lambda,\theta y_\lambda + (1-\theta)\overline{y})d\theta\, z_\lambda \\ \quad = -\dfrac{A_2(t,q_\lambda) - A_2(t,\overline{q})}{\lambda} y_\lambda' - \dfrac{A_1(t,q_\lambda) - A_1(t,\overline{q})}{\lambda} y_\lambda \\ \quad + \dfrac{f(t,q_\lambda,\overline{y}) - f(t,\overline{q},\overline{y})}{\lambda}, \\ z_\lambda(0) = z_\lambda'(0) = 0. \end{cases}$$

Now we define a linear operator $L(q,\lambda;t)$ by

$$L(q,\lambda;t) = -\int_0^1 f_y^*(t,q_\lambda,\theta y_\lambda + (1-\theta)\overline{y})d\theta \quad \text{for all} \quad (\lambda,q) \in [0,1] \times Q_{ad}.$$

Note that $L(q,0;t) = -f_y^*(t,\overline{q},\overline{y})$ for all $q \in Q_{ad}$. Since $q_\lambda \in Q_{ad}$ and

$$\sup\{\|\theta y_\lambda(t) + (1-\theta)\overline{y}(t)\|_{V_2} : (t,\theta,q,\lambda) \in [0,T]\times[0,1]\times Q_{ad}\times[0,1]\} < \infty$$

due to (4.10), it follows from (A4) that

$$\begin{aligned} (4.12) \qquad \|L(q,\lambda;t)\|_{\mathcal{L}(V_2,V_2')} &\le \int_0^1 \tilde{\beta}_2(t)d\theta \\ &= \tilde{\beta}_2(t) \quad \text{a.e. } t \ \text{ for all } (q,\lambda) \in Q_{ad} \times [0,1], \end{aligned}$$

so that $L(q,\lambda;\cdot) \in L^2(0,T;\mathcal{L}(V_2,V_2'))$. Then for each $(q,\lambda) \in Q_{ad} \times [0,1]$ there is a unique weak solution $\phi = \phi(q,\lambda;\phi_1,q) \in \mathcal{W}(0,T)$ such that

(4.13)
$$\begin{cases} \Phi[q,\lambda;t](\phi) = g, \ L(q,\lambda;t) = -\int_0^1 f_y^*(t,q_\lambda,\theta y_\lambda + (1-\theta)\overline{y})d\theta, \\ \phi(T) = 0, \ \phi'(T) = \phi_1 \end{cases}$$

provided that $\phi_1 \in H,\ g \in L^2(0,T;V_2')$. Furthermore, since (4.5) is satisfied by (4.12), we have the following uniform estimate of ϕ independent of (q,λ):

$$(4.14) \qquad \max_{t\in[0,T]} \|\phi(t)\|_{V_1} \le \sqrt{\tilde{c}}(|\phi_1|_H + \|g\|_{L^2(0,T;V_2')}).$$

By multiplying $\phi \in \mathcal{W}(0,T)$ with $\phi(T) = 0, \phi'(T) = \phi_1$ on the both sides of (4.11) and using integration by parts on $[0,T]$, we obtain

$$
\begin{aligned}
(4.15)\qquad & -(z_\lambda(T),\phi_1)_H + \int_0^T \langle z_\lambda, \Phi[q,\lambda;t](\phi)\rangle_{V_2,V_2'} dt \\
= & -\int_0^T \frac{a_1(t,q_\lambda;y_\lambda,\phi) - a_1(t,\overline{q};y_\lambda,\phi)}{\lambda} dt \\
& -\int_0^T \frac{a_2(t,q_\lambda;y'_\lambda,\phi) - a_2(t,\overline{q};y'_\lambda,\phi)}{\lambda} dt \\
& +\int_0^T \langle \frac{f(t,q,\overline{y}) - f(t,\overline{q},\overline{y})}{\lambda}, \phi\rangle_{V_2',V_2} dt, \\
= & \ I_1(\phi) + I_2(\phi) + I_3(\phi).
\end{aligned}
$$

Let us estimate $I_i(\phi), i = 1,2,3$ by using (A4)-(A5) and (4.10). Noting that $\overline{q} + \theta\lambda(q-\overline{q}) \in Q_{ad}$ for all $\theta \in [0,1]$, we have

$$
\begin{aligned}
|I_1(\phi)| &= \left|\int_0^T d_G a_1(t,\overline{q}+\theta_1\lambda(q-\overline{q});y_\lambda,\phi)[q-\overline{q}]dt\right| \quad (\exists\theta_1 \in [0,1]) \\
&\le \tilde{c}_{13}\|q-\overline{q}\|_Q \max_{t\in[0,T]}\|y_\lambda(t)\|_{V_1} \max_{t\in[0,T]}\|\phi(t)\|_{V_1} \\
&\le \tilde{c}_1 \max_{t\in[0,T]}\|\phi(t)\|_{V_1},
\end{aligned}
$$

$$
\begin{aligned}
|I_2(\phi)| &= \left|\int_0^T d_G a_2(t,\overline{q}+\theta_2\lambda(q-\overline{q});y'_\lambda,\phi)[q-\overline{q}]dt\right| \quad (\exists\theta_2 \in [0,1]) \\
&\le \tilde{c}_{23}\sqrt{T}\|q-\overline{q}\|_Q\|y'_\lambda\|_{L^2(0,T;V_2)} \max_{t\in[0,T]}\|\phi(t)\|_{V_2} \\
&\le c_1\tilde{c}_{23}\sqrt{T}\|q-\overline{q}\|_Q\|y'_\lambda\|_{L^2(0,T;V_2)} \max_{t\in[0,T]}\|\phi(t)\|_{V_1} \\
&\le \tilde{c}_2 \max_{t\in[0,T]}\|\phi(t)\|_{V_1}
\end{aligned}
$$

and

$$
\begin{aligned}
|I_3(\phi)| &= \left|\int_0^T \langle d_G f(t,\overline{q}+\theta_3\lambda(q-\overline{q}),\overline{y})[q-\overline{q}],\phi\rangle_{V_2',V_2} dt\right| (\exists\theta_3 \in [0,1]) \\
&\le \|q-\overline{q}\|_Q \int_0^T \tilde{\beta}_1(t)\|\phi(t)\|_{V_2} dt \\
&\le c_1\sqrt{T}\|q-\overline{q}\|_Q\|\tilde{\beta}_1\|_{L^2(0,T;\mathbf{R}^+)} \max_{t\in[0,T]}\|\phi(t)\|_{V_1} \\
&\le \tilde{c}_3 \max_{t\in[0,T]}\|\phi(t)\|_{V_1}.
\end{aligned}
$$

Hence

$$|I_1(\phi)| + |I_2(\phi)| + |I_3(\phi)| \leq (\tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3) \max_{t\in[0,T]} \|\phi(t)\|_{V_1}.$$

Note that $\tilde{c}_i, i = 1,2,3$ are positive constants independent of $q \in Q_{ad}$. Now we can properly choose g, ϕ_1 in (4.13) to obtain that $\{z_\lambda\} \subset L^2(0,T;V_2)$ and $\{z_\lambda(T)\} \subset H$ are bounded. Let us choose the terminal conditions ϕ_0, ϕ_1 and forcing function g in (4.13) as follows:

(i) $\phi_1 = 0, \quad g(t) = z_\lambda(t)$;

(ii) $\phi_1 = 0, \quad g(t) = A_2(t, q_\lambda)z_\lambda(t)$;

(iii) $\phi_1 = -z_\lambda(T), \quad g(t) = 0$.

Since $z_\lambda,\ A_2(t,q_\lambda)z_\lambda \in L^2(0,T;V_2')$ and $z_\lambda \in C([0,T];V_1)$, three terminal value problems corresponding to (i), (ii) and (iii) have unique weak solutions $\phi = \phi(q,\lambda;\phi_1,g) \in \mathcal{W}(0,T)\cap C([0,T];V_1)\cap C^1([0,T];H)$ satisfying (4.14).

The case (i): applying this condition to (4.15) and referring to (4.4) and (4.15), we obtain

$$\begin{aligned} \|z_\lambda\|^2_{L^2(0,T;H)} &= \int_0^T \langle z_\lambda, z_\lambda\rangle_{V_2,V_2'} dt = \sum_{i=1}^3 I_i(\phi) \\ &\leq (\tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3) \max_{t\in[0,T]} \|\phi(t)\|_{V_1} \\ &\leq \tilde{c}_0 \|z_\lambda\|_{L^2(0,T;V_2')} \leq \tilde{c}_0 c_2' \|z_\lambda\|_{L^2(0,T;H)}, \end{aligned}$$

where $\tilde{c}_0 = [\tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3]\sqrt{\tilde{c}}$. Therefore $\|z_\lambda\|_{L^2(0,T;H)} \leq \tilde{c}_0 c_2'$.

The case (ii): it follows from (4.14) and (2.3) that

$$\begin{aligned} &\alpha_2 \|z_\lambda\|^2_{L^2(0,T;V_2)} - \lambda_2 \|z_\lambda\|^2_{L^2(0,T;H)} \\ \leq\ & \int_0^T a_2(t, q_\lambda; z_\lambda, z_\lambda) dt \\ =\ & \int_0^T \langle z_\lambda, A_2(t,q_\lambda)z_\lambda\rangle_{V_2,V_2'} dt = \sum_{i=1}^3 I_i(\phi) \\ \leq\ & \tilde{c}_0 \|A_2(\cdot, q_\lambda)z_\lambda\|_{L^2(0,T;V_2')} \leq \tilde{c}_0 \tilde{c}_{21} \|z_\lambda\|_{L^2(0,T;V_2)}, \end{aligned}$$

where α_2 and λ_2 are dependent on q_λ in (2.3). Thus we have

$$\tilde{\alpha}_2 \|z_\lambda\|^2_{L^2(0,T;V_2)} \leq \tilde{\lambda}_2 \|z_\lambda\|^2_{L^2(0,T;H)} + \tilde{c}_0 \tilde{c}_{21} \|z_\lambda\|_{L^2(0,T;V_2)}$$

where $\tilde{\alpha}_2$ and $\tilde{\lambda}_2$ are the constants corresponding to $Q_{bd} = Q_{ad}$ in (2.5). This quadratic inequality on $\|z_\lambda\|_{L^2(0,T;V_2)}$ implies that $\{z_\lambda\}$ is bounded in $L^2(0,T;V_2)$ because of $\tilde{\alpha}_2 > 0$.

The case (iii) : it follows from (4.14) and (4.15) that

$$|z_\lambda(T)|_H^2 = \sum_{i=1}^{3} I_i(\phi) \le (\tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3) \max_{t\in[0,T]} \|\phi(t)\|_{V_1} \le \tilde{c}_0 |z_\lambda(T)|_H,$$

whence $\{z_\lambda(T)\}$ is bounded in H. Therefore we can extract a subsequence $\{z_\lambda\}$, denoting it by $\{z_\lambda\}$ again, and $z \in L^2(0,T;V_2), z_1 \in H$ such that

(4.16) $$z_\lambda \to z \quad \text{weakly in} \quad L^2(0,T;V_2),$$

(4.17) $$z_\lambda(T) \to z_1 \quad \text{weakly in} \quad H.$$

In the following calculations we fix $\phi \in X[\overline{q},0] \subset \mathcal{W}(0,T)$ with $L(\overline{q},0;t) = -f_y^*(t,\overline{q},\overline{y})$. For a moment we put $\psi = \overline{y}, \psi_\lambda = y_\lambda$ if $i = 1$ and $\psi = \overline{y}', \psi_\lambda = y'_\lambda$ if $i = 2$. Then $-I_i(\phi)$ is written by

$$\begin{aligned} -I_i(\phi) &= \int_0^T \frac{a_i(t,q_\lambda;\psi_\lambda,\phi) - a_i(t,\overline{q};\psi_\lambda,\phi)}{\lambda} dt \\ &= \int_0^T \langle \frac{[A_i(t,q_\lambda) - A_i(t,\overline{q})]\phi}{\lambda}, \psi \rangle_{V_i',V_i} dt \\ &\quad + \int_0^T \langle \frac{[A_i(t,q_\lambda) - A_i(t,\overline{q})]\phi}{\lambda}, \psi_\lambda - \psi \rangle_{V_i',V_i} dt = J_{i1} + J_{i2}. \end{aligned}$$

By (A5), J_{i1} converges to

(4.18) $$\int_0^T d_G a_i(t,\overline{q};\psi,\phi)[q-\overline{q}]dt \quad \text{as} \quad \lambda \to 0.$$

By (A5), there are $\theta_i \in (0,1)$ such that

$$J_{i2} = \int_0^T d_G a_i(t,\overline{q} + \theta_i\lambda(q-\overline{q});\phi,\psi_\lambda - \psi)[q-\overline{q}]dt.$$

It follows from the strong convergence of $\psi_\lambda \to \psi$ due to (4.9) that

(4.19) $$|J_{i2}| \le \tilde{c}_{i3} \|\phi\|_{L^2(0,T;V_i)} \|\psi_\lambda - \psi\|_{L^2(0,T;V_i)} \to 0 \quad \text{as} \quad \lambda \to 0.$$

Hence by (4.18) and (4.19) we have

(4.20) $$I_i(\phi) \to \int_0^T -d_G a_i(t,\overline{q};\psi,\phi)[q-\overline{q}]dt \quad \text{as} \quad \lambda \to 0.$$

By (A4), for fixed $t \in [0,T]$ we have

$$f_y^*(t,q_\lambda,\theta y_\lambda + (1-\theta)\overline{y}) \to f_y^*(t,\overline{q},\overline{y}) \quad \text{for all} \quad \theta \in [0,1] \quad \text{as} \quad \lambda \to 0$$

in $\mathcal{L}(V_2, V_2')$. Since this convergence is uniform for θ, we also have

$$L(q,\lambda;t) \to -f_y^*(t,\overline{q},\overline{y}) \ \text{ a.e. } \ t\in[0,T] \ \text{ as } \ \lambda \to 0,$$

so that by (4.12) and the Lebesgue dominated convergence theorem

$$L(q,\lambda;\cdot)\phi \to -f_y^*(\cdot,\overline{q},\overline{y})\phi \ \text{ strongly in } \ L^2(0,T;V_2') \ \text{ as } \ \lambda\to 0.$$

Hence from (4.16) we have

$$\begin{aligned}(4.21)\qquad & \int_0^T \langle z_\lambda, L(q,\lambda;t)\phi\rangle_{V_2',V_2}dt \\ & \to \ -\int_0^T \langle z, f_y^*(t,\overline{q},\overline{y})\phi\rangle_{V_2',V_2}dt \ \text{ as } \ \lambda\to 0.\end{aligned}$$

Finally, by similar calculations as above it follows from (A4) that

$$\begin{aligned}(4.22)\qquad & \int_0^T \langle \frac{f(t,q_\lambda,\overline{y})-f(t,\overline{q},\overline{y})}{\lambda}, \phi\rangle_{V_2',V_2}dt \\ & \to \int_0^T \langle d_G f(t,\overline{q},\overline{y})[q-\overline{q}],\phi\rangle_{V_2',V_2}dt \ \text{ as } \ \lambda\to 0.\end{aligned}$$

Therefore, taking $\lambda\to 0$ in (4.15) and using (4.20)-(4.22) and (4.16)-(4.17), (z_1, z) satisfies an integral equation:

(4.23)

$$\begin{cases} -(z_1,\phi_1)_H + \displaystyle\int_0^T \langle z, \Phi[\overline{q},0;t](\phi)\rangle_{V_2,V_2'}dt \\ = -\displaystyle\int_0^T d_G a_2(t,\overline{q};y'(\overline{q}),\phi)[q-\overline{q}]dt \\ \quad -\displaystyle\int_0^T d_G a_1(t,\overline{q};y(\overline{q}),\phi)[q-\overline{q}]dt \\ \quad +\displaystyle\int_0^T \langle d_G f(t,\overline{q},y(\overline{q})[q-\overline{q}],\phi\rangle_{V_2',V_2}dt \ \text{ for all } \ \phi\in X[\overline{q},0], \end{cases}$$

where $\Phi[\overline{q},0;t](\phi) = \phi'' - A_2'(t,\overline{q})\phi - A_2(t,\overline{q})\phi' + A_1(t,\overline{q})\phi - f_y^*(t,\overline{q},y(\overline{q}))\phi$.

Finally let us consider a linear functional l on $X[\overline{q},0]$ defined by

$$\begin{aligned}(4.24)\qquad l(\phi) \ = \ & -\int_0^T d_G a_2(t,\overline{q};y'(\overline{q}),\phi)[q-\overline{q}]dt \\ & -\int_0^T d_G a_1(t,\overline{q};y(\overline{q}),\phi)[q-\overline{q}]dt \\ & +\int_0^T \langle d_G f(t,\overline{q},y(\overline{q})[q-\overline{q}],\phi\rangle_{V_2',V_2}dt.\end{aligned}$$

Similarly to the above processes we can show that l is a bounded linear functional on $X[\overline{q},0]$. Therefore (z_1, z) becomes a unique solution of (4.23) in the sense of (4.7). □

Lemma 4.1 leads us to treat two kinds of observation operators as follows:

1. we take $\mathcal{C}_1 \in \mathcal{L}(L^2(0,T;V_2),\mathcal{M})$ and observe $z(q) = \mathcal{C}_1 y(q)$;
2. we take $\mathcal{C}_2 \in \mathcal{L}(H,\mathcal{M})$ and observe $z(q) = \mathcal{C}_2 y(T,q)$.

In the following two cases, we suppose all assumptions used in Theorems 2.1, 3.1 and Lemma 4.1.

(1) The case of $\mathcal{C}_1 \in \mathcal{L}(L^2(0,T;V_2),\mathcal{M})$

In this case the necessary condition on $\overline{q}$ is given by

$$\begin{aligned}&(\mathcal{C}_1 y(\overline{q}) - z_d, \mathcal{C}_1 z)_{\mathcal{M}}\\ &= \int_0^T \langle \mathcal{C}_1^* \Lambda_{\mathcal{M}}(\mathcal{C}_1 y(\overline{q}) - z_d), z\rangle_{V_2',V_2} dt \geq 0 \quad \text{for all} \quad q \in Q_{ad},\end{aligned}$$

where $z = d_G y(\overline{q})[q-\overline{q}]$ is the solution of (4.8).

Let us consider the equations, which are called the adjoint state equations, described by

$$\begin{cases} \Phi[\overline{q},0;t]\xi = \mathcal{C}_1^* \Lambda_{\mathcal{M}}(\mathcal{C}_1 y(\overline{q}) - z_d) \quad \text{in} \quad (0,T),\\ \xi(T) = \xi'(T) = 0,\end{cases}$$

where $L(\overline{q},0;t) = -f_y^*(t,\overline{q},y(\overline{t,q}))$. Since

$$\mathcal{C}_1^* \Lambda_{\mathcal{M}}(\mathcal{C}_1 y(\overline{q}) - z_d) \in L^2(0,T;V_2')$$

and $L(\overline{q},0;t) \in L^2(0,T;\mathcal{L}(V_2,V_2'))$, there is a unique weak solution $\xi = \xi(\overline{q}) \in \mathcal{W}(0,T)$. Hence we take $\xi = \phi$ in (4.8) to yield

$$\begin{aligned}&\int_0^T [d_G a_2(t,\overline{q};y'(\overline{q}),\xi(\overline{q}))[q-\overline{q}] + d_G a_1(t,\overline{q};y(\overline{q}),\xi(\overline{q}))[q-\overline{q}]\\ &-\langle d_G f(t,\overline{q},y(\overline{q}))[q-\overline{q}],\xi(\overline{q})\rangle_{V_2',V_2}]dt \leq 0 \quad \text{for all } q \in Q_{ad}.\end{aligned}$$

Summarizing these we have the following theorem.

THEOREM 4.2. *Assume that the observer $\mathcal{C}_1 \in \mathcal{L}(L^2(0,T;V_2),\mathcal{M})$. Then the optimal parameter $\overline{q}$ is characterized by the state and adjoint state equations and inequality as follows:*

$$\begin{cases} y'' + A_2(t,\overline{q})y' + A_1(t,\overline{q})y = f(t,\overline{q},y) \quad in \quad (0,T),\\ y(0) = y_0 \in V_1, \quad y'(0) = y_1 \in H,\end{cases}$$

$$\begin{cases} \xi'' - A_2(t,\overline{q})\xi' + (A_1(t,\overline{q}) - A_2'(t,\overline{q}))\xi \\ \qquad = f_y^*(t,\overline{q},y)\xi + \mathcal{C}_1^*\Lambda_{\mathcal{M}}(\mathcal{C}_1 y - z_d) \ \ in \ \ (0,T), \\ \xi(T) = \xi'(T) = 0, \end{cases}$$

$$\int_0^T [d_G a_2(t,\overline{q};y',\xi)[q-\overline{q}] + d_G a_1(t,\overline{q};y,\xi)[q-\overline{q}] \\ \qquad - \langle d_G f(t,\overline{q},y)[q-\overline{q}],\xi\rangle_{V_2',V_2}]dt \leq 0 \ \ for\ all \ \ q \in Q_{ad}.$$

(2) The case of $\mathcal{C}_2 \in \mathcal{L}(H,\mathcal{M})$

In this case the necessary condition on $\overline{q}$ is written by

$$(\mathcal{C}_2 y(T,\overline{q}) - z_d, \mathcal{C}_2 z(T))_{\mathcal{M}} \\ = (\mathcal{C}_2^*\Lambda_{\mathcal{M}}(\mathcal{C}_2 y(T,\overline{q}) - z_d), z(T))_H \geq 0 \ \ \text{for all} \ \ q \in Q_{ad},$$

where $z(T) = d_G y(T,\overline{q})[q-\overline{q}]$ is the solution of (4.8).

We consider the adjoint state equations described by

$$\begin{cases} \Phi[\overline{q},0;t]\xi = 0 \ \ \text{in} \ \ (0,T), \\ \xi(T) = 0, \ \ \xi'(T) = \mathcal{C}_2^*\Lambda_{\mathcal{M}}(\mathcal{C}_2 y(T,\overline{q}) - z_d), \end{cases}$$

where $L(\overline{q},0;t) = -f_y^*(t,\overline{q},y(t,\overline{q}))$. Since $\mathcal{C}_2^*\Lambda_{\mathcal{M}}(\mathcal{C}_2 y(T,\overline{q}) - z_d) \in H$, there exists a unique weak solution $\xi = \xi(\overline{q}) \in \mathcal{W}(0,T)$. Hence we take $\xi = \phi$ in (4.8) to yield

$$\int_0^T [d_G a_2(t,\overline{q};y'(\overline{q}),\xi(\overline{q}))[q-\overline{q}] + d_G a_1(t,\overline{q};y(\overline{q}),\xi(\overline{q}))[q-\overline{q}] \\ \qquad - \langle d_G f(t,\overline{q},y(\overline{q}))[q-\overline{q}],\xi(\overline{q})\rangle_{V_2',V_2}]dt \geq 0, \ \ \forall q \in Q_{ad}.$$

Therefore we have the following theorem.

THEOREM 4.3. *Assume that the observer $\mathcal{C}_2 \in \mathcal{L}(H,\mathcal{M})$. Then the optimal parameter $\overline{q}$ is characterized by the state and adjoint state equations and inequality as follows:*

$$\begin{cases} y'' + A_2(t,\overline{q})y' + A_1(t,\overline{q})y = f(t,\overline{q},y) \ \ in \ \ (0,T), \\ y(0) = y_0 \in V_1, \ \ y'(0) = y_1 \in H, \end{cases}$$

$$\begin{cases} \xi'' - A_2(t,\overline{q})\xi' + (A_1(t,\overline{q}) - A_2'(t,\overline{q}))\xi \\ \qquad = f_y^*(t,\overline{q},y)\xi \ \ in \ \ (0,T), \\ \xi(T) = 0, \ \ \xi'(T) = \mathcal{C}_2^*\Lambda_{\mathcal{M}}(\mathcal{C}_2 y(T,\overline{q}) - z_d), \end{cases}$$

$$\int_0^T [d_G a_2(t,\overline{q};y',\xi)[q-\overline{q}] + d_G a_1(t,\overline{q};y,\xi)[q-\overline{q}] \\ \qquad - \langle d_G f(t,\overline{q},y)[q-\overline{q}],\xi\rangle_{V_2',V_2}]dt \geq 0 \ \ for\ all \ \ q \in Q_{ad}.$$

REMARK 4.4. If we make more strong assumption such as $A_1(t,q) \in \mathcal{L}(V_1, V_2')$ in Lemma 4.1, then we can obtain that $q \to y(q)$ is weakly Gâteaux differentiable on Q into $\mathcal{W}(0,T)$.

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