

Theoretical Perspectives for Analyzing Explanation, Justification and Argumentation in Mathematics Classrooms*

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Current interest in mathematics learning that focuses on understanding, mathematical reasoning and meaning making underscores the need to develop ways of analyzing classrooms that foster these types of learning. In this paper, the author show that the constructs of social and socio-mathematical norms, which grew out of taking a symbolic interactionist perspective, and Toulmins scheme for argumentation, as elaborated for mathematics education by Krummheuer [The ethnology of argumentation. In: The emergence of mathematical meaning: Interaction in classroom cultures (1995, pp. 229-269). Hillsdale, NJ: Erlbaum], provide us with means to analyze aspects of explanation, justification and argumentation in mathematics classrooms, including means through which they can be fostered. Examples from a variety of classrooms are used to clarify how these notions can inform instruction at all levels, from the elementary grades through university-level mathematics.

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ZDM Classification: C60

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INTRODUCTION

Current emphasis on mathematical reasoning as central to what it means to engage in mathematical activity (National Council of Teachers of Mathematics 2000; Yackel & Hanna 2003) coupled with similar emphasis on classroom discourse as a primary means

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of mathematical learning (Kieran, Forman, & Sfard 2001; Lampert & Cobb 2003) has resulted in growing interest in investigating explanation, justification, and argumentation in the mathematics classroom.

Issues of what it might mean to explain and justify and develop arguments, and how students might learn to do so, are of primary concern to researchers and classroom teachers alike. In this paper my purpose is to explicate two theoretical perspectives that have proven useful in making sense of explanation, justification and argumentation that occur in mathematics classrooms. They are symbolic interactionism (Blumer 1969) and Toulmins (1969) argumentation scheme as elaborated for mathematics education by Krummheuer (1995). I discuss each perspective in turn and provide examples from classroom-based research to clarify and illustrate how these perspectives are used to develop analyses.

Background

The research activity that my colleagues¹ and I engage in is classroom based and involves conducting classroom teaching experiments that range from six weeks to an entire school year or university term. It involves developing instructional sequences and approaches as well as investigating teaching and learning as it occurs in the classroom. In this type of research, called developmental research (Gravemeijer 1994), the researcher conducts ongoing analyses of classroom activity and uses the results to inform instructional planning and decision-making. It also involves retrospective analyses that attempt to explain the nature of the learning that took place and to explicate significant aspects of the learning situation.

The complexity of the classroom presents a challenge for the researcher not only for data analysis, but also for conceptualizing the analysis process itself. Through our work, we have come to understand the importance of taking into account both the social aspects of learning, including social interaction, and the individual aspects of learning (Cobb & Bauersfeld 1995; Cobb, Yackel & Wood 1989; Yackel, Cobb, Wood, Wheatley & Merkel 1990; Yackel & Rasmussen 2002; Yackel, Rasmussen & King 2001). Accordingly, we have developed an interpretive framework for analyzing classrooms that coordinates a psychological perspective with a sociological perspective. Further, we have explicated theoretical constructs within that framework in terms of our experiential base (Cobb & Yackel 1996; Yackel & Cobb 1996). Of the various sociological perspectives, we have taken symbolic interactionism as a theoretical lens for two reasons. First, it is compatible

¹ The colleagues that I have worked with to conduct and analyze teaching experiments over the past two decades include Paul Cobb, Terry Wood, Koeno Gravemeijer, Diana Underwood Gregg, Chris Rasmussen, and Michelle Stephan.

with psychological constructivism, which forms the theoretical basis for our investigation of individual learning (Cobb & Bauersfeld 1995). Second, as Voigt (1996) points out, the symbolic interactionist approach is particularly useful when studying learning in inquiry mathematics classrooms² because it emphasizes both the individual's sense-making processes and the social processes without giving primacy to either one. Thus, we do not attempt to deduce an individual's learning from social processes or vice versa. Instead, individuals are seen to develop personal meanings as they participate in the ongoing negotiation of classroom norms and classroom mathematical practices³. A key aspect of our framework is that we posit a reflexive relationship between the sociological constructs and their psychological counterparts.

Mathematical explanation, justification and argumentation can be studied from a variety of perspectives. Following the interactionist perspective, my interest in this paper is in treating them as interactional accomplishments and not as logical arguments. Two constructs from the interpretive framework that are particularly relevant to issues of explanation, justification and argumentation are social norms and socio-mathematical norms. Our investigations have pointed to the importance of these constructs in clarifying both the functions that explanation, justification and argumentation serve and the means by which they might be fostered in the classroom. A further implication of treating explanation, justification and argumentation as interactional accomplishments is that detailed analyses focus on what the participants take as acceptable, individually and collectively, and not on whether an argument might be considered valid from a mathematical point of view. Krummheuers elaboration of Toulmins argumentation scheme proves useful as an analytic tool for this purpose. In addition, this argumentation scheme is useful as a methodological tool to clarify how individual students' explanations and the learning of the class as a collective are interactively constituted (Yackel 1997).

In this paper, I first discuss symbolic interactionism as a theoretical framework. Next, I explain what I mean by explanation and justification and discuss normative aspects of mathematics classrooms relative to explanation and justification. Then I describe how Toulmins argumentation scheme can be used to analyze classroom activity and to document learning in the classroom. I conclude with a discussion of the significance of this work for classroom instruction.

² I follow Richards (1991) in using the label *inquiry* mathematics classrooms to describe those classrooms in which students engage in genuine mathematical discussions with each other and the teacher. See Yackel (2000) for a detailed discussion of the social and sociomathematical norms that characterize inquiry mathematics instruction.

³ The construct of *classroom mathematical practices* refers to the collective development of the classroom community. See Cobb (1998) for an elaboration of this construct and for an illustrative example.

SYMBOLIC INTERACTIONISM AS A THEORETICAL FRAMEWORK

The theory of symbolic interactionism has its roots in the work of George Herbert Mead, John Dewey and others and has been developed extensively by Herbert Blumer (1969). One of its defining principles is the centrality given to the process of interpretation in interaction. To put it another way, the position taken by symbolic interactionism is that, in interacting with one another, individuals have to take account of (interpret) what the other is doing or about to do. Each person's actions are formed, in part, as she changes, abandons, retains, or revises her plans based on the actions of others.

In addition to interpreting the actions of others, individuals engaged in interaction attempt to indicate to others, through their actions, what their own intentions are. Thus, actions have meanings both for the person making them and for the person(s) to whom the action is directed. In this sense there is a joint action that arises by the articulation of the participating actors activity. Blumer (1969) emphasized the collective nature of a joint action as follows:

A joint action, while made up of diverse component acts that enter into its formation, is different from any one of them and from their mere aggregation. The joint action has a distinctive character in its own right, a character that lies in the articulation or linkage as apart from what may be articulated or linked. Thus, joint action may be identified as such and may be spoken of and handled without having to break it down into the separate acts that comprise it (p. 17).

Blumer further pointed out that because the joint action of the collective is an interlinkage of the separate acts of the participants (p. 17) it has to undergo a process of formation. Even though it may be well established as a form of social action, each instance of it has to be formed once again. Consequently, the meanings and interpretations that underlie the joint action are continually subject to challenge. As a result, both individual actions and the joint (collective) action of a group can change over time⁴. Furthermore, this view of joint action supports the position that social rules, norms and values are upheld by a process of social interaction and not the other way around.

In this sense, social interaction is a process that *forms* human conduct rather than simply a setting in which human conduct takes place. As Blumer (1969) stated, One has to *fit* one's own line of activity in some manner to the actions of others. The actions of others have to be taken into account and cannot be merely an arena for the expression of what one is disposed to do or sets out to do (p. 8). Blumer further clarified that the term *symbolic* interactionism refers to the fact that the interaction of interest involves

⁴ See Yackel and Rasmussen (2002) for an elaboration of how this perspective can be used to account for changes in students beliefs in inquiry-based mathematics classrooms.

interpretation of action. Consequently, since attempts to genuinely communicate involve understanding the meanings of another's actions (Rommetveit 1985), for example attempts to explain one's thinking or to understand another's explanation attempts, they involve symbolic interaction.

A second defining principle of symbolic interactionism, in addition to the centrality of interpretation, is that meaning is seen as a social product. Blumer (1969) elaborated this point as follows:

It [symbolic interactionism] does not regard meaning as emanating from the intrinsic makeup of the thing that has meaning, nor does it see meaning as arising through a coalescence of psychological elements in the person. Instead, it sees meaning as arising in the process of interaction between people. The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing. Their actions operate to define the thing for the person. Thus, symbolic interactionism sees meaning as social products, as creations that are formed in and through the defining activities of people as they interact (pp. 4–5).

This view of meaning has important implications for how we interpret the results of classroom discursive activity. Since meanings grow out of social interaction, each individual's personal meanings and understandings are formed in and through the process of interpreting that interaction. Nevertheless, normative understandings are constituted. It is as these normative understandings are constituted that students develop their own interpretations of them. What we mean by saying that these understandings are normative is that there is evidence from classroom dialogue and activity that students' interpretations are compatible. It is in this sense that we say that students' interpretations or meanings are taken-as-shared (*cf.* Cobb, Wood, Yackel & McNeal 1992).

EXPLANATION AND JUSTIFICATION

In this paper my interest is in explanation and justification as social constructs rather than as individual constructs. In this case, they are considered to be aspects of the discourse that serve communicative functions and are interactively constituted by the teacher and the students. Explanations and justifications are distinguished, in part, by the functions they serve. Students and the teacher give mathematical explanations to clarify aspects of their mathematical thinking that they think might not be readily apparent to others. They give mathematical justifications in response to challenges to apparent violations of normative mathematical activity (Cobb et al. 1992). For example, consider the task "How can you figure out $16 + 8 + 14$?" If a child responds, "I took one from the 16 and added it to the 14 to get 15 and 15; then I added the 15 and 15 to get 30, and the other 8 to get 38", we would infer that she was explaining her solution to others.

However, a challenge that you first have to add the 16 and the 8 and then add 14 to that sum is a request for a justification.

Classroom norms that relate to mathematical explanation and justification are both social and socio-mathematical in nature. A Norm is a sociological construct and refers to understandings or interpretations that become normative or taken-as-shared by the group. Thus, a norm is not an individual but a collective notion. One way to describe norms, in our case classroom norms, is to describe the expectations and obligations that are constituted in the classroom.

By analyzing data from our early teaching experiments, we were able to identify a number of social norms that characterized classroom interactions. These include that students are expected to develop personally-meaningful solutions to problems, to explain and justify their thinking and solutions, to listen to and attempt to make sense of each other's interpretations of and solutions to problems, and to ask questions and raise challenges in situations of misunderstandings or disagreement. In saying that these social norms characterize the classroom interactions, I mean that these ways of acting and of interpreting the actions of others became taken-as-shared. In subsequent classroom teaching experiments we had the constitution of these norms as an explicit goal. It is evident that each of these norms relates specifically to explanation and justification when taken as social constructs, as described above.

We were also able to identify normative aspects of interactions that are specific to mathematics. These we called socio-mathematical norms (Yackel & Cobb 1996). Normative understandings of what counts as mathematically different, sophisticated, efficient and elegant are examples of socio-mathematical norms. Similarly, what counts as an acceptable mathematical explanation and justification is a socio-mathematical norm. The distinction between social norms and socio-mathematical norms is subtle. For example, the understanding that students are expected to explain their solutions is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a socio-mathematical norm.

We might ask how notions such as what counts as an acceptable explanation come to have meaning for students. To answer this question, we return to the symbolic interactionist position on meaning. This position is that meaning arises through interaction. Accordingly, the meaning of acceptable mathematical explanation is not something that can be outlined in advance for students to apply. Instead, it is formed in and through the interactions of the participants in the classroom. As with all normative understandings, both explicit and implicit negotiations contribute to developing these understandings.

In the following paragraphs I present examples from an elementary school classroom teaching experiment to illustrate the negotiation of the socio-mathematical norm of what

counts as an acceptable mathematical explanation and justification.

This socio-mathematical norm is crucial for the discussion in this paper because it helps to clarify how individual student's explanation attempts and the mathematical practices of the classroom as a collective are interactively constituted.

Initiating the Negotiation of What Constitutes Acceptable Mathematical Explanation and Justification

In the inquiry classrooms that we have studied, whether at the first-grade level or at the university level, it has become normative that explanations and justifications involve describing actions on what we refer to as experientially-real mathematical objects for students. Consequently, explanations that do not carry such significance are frequently challenged. These challenges, in turn, give rise to situations for the teacher and students to negotiate what is acceptable as a mathematical explanation.

In one primary-grade project classroom this negotiation began the first day of the school year⁵. Several activities used at the beginning of the school year involved dot patterns. In one activity, dot patterns were flashed on an overhead screen several times, each time for only a few seconds. The children's task was to figure out how many dots there were and to explain how they saw it. The instructional intent of the task was to foster development of visual imagery and to generate discussions about number relationships for small numbers. For example, one of the first patterns used on the first day is shown in Figure 1.

In response to the question, "How did you see it?", several children gave replies such as, "I looked at them", "With my eyes", "I saw three going in a slant", "I saw a slanted line", and "I counted". These replies formed the basis for initiating the constitution of the task as a mathematical one, meaning that, in this case, the explanations should be numerical. They should be about quantity. Thus, the response, "I saw three going in a slant", was acceptable but "With my eyes", was not. Similarly, "I counted them", was acceptable but I looked at them, was not. This apparently trivial example demonstrates that what is considered mathematical communication in the classroom must be constituted. As students participate in the classroom activity and discussions, they learn what constitutes mathematical explanation in their classroom. At the same time, their activity contributes to what is constituted as acceptable mathematical explanation.

⁵ It is important to note that students in this class were experiencing an inquiry approach to mathematics instruction for the first time. They had no prior experience explaining their thinking. In fact, as the example shows, they had to learn what aspects of a task might be considered to be mathematical. The teacher, therefore, was in the position of having to initiate the negotiation of norms relative to explaining and what constitutes mathematical explanation.

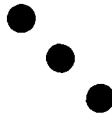


Figure 1. Dot patterns for three shown on the overhead screen

This example illustrates how what is considered to be an acceptable mathematical explanation in the classroom is interactively constituted by the students and the teacher. Children's responses to tasks on the next day provide evidence of the effectiveness of the implicit negotiations begun on the previous day. The next day, the teacher displayed the dot pattern shown in figure 2. This time, children's responses included both counting and non-counting explanations. For example, several students reported that they saw seven because they saw it as three and one and another three, but no one responded with a comment such as, "With my eyes." Nevertheless, as Blumer notes, each instance of collective action has to be formed anew. Consequently, norms relative to acceptable explanation are continually being negotiated and (re)negotiated. In this case, those children who gave responses such as, "Its seven because I saw three and one and another three", did not go on to explain how they arrived at a total of seven based on the three, one, and three. These children seemed to take it for granted that their explanation was sufficient. That is, they took it for granted that the other students would immediately know that $3+1+3=7$. Yet we know from individual interviews conducted with the children at the beginning of the school year and from their mathematical activity in the classroom that some of them would have to count by ones to solve this addition problem. At the same time, all of the children in the class were able to readily see that there were three dots in the first row, one in the second row, and three in the third row (at least one child referred explicitly to the number of dots in each row of the pattern in this way). Thus, it is somewhat questionable whether the explanations referred to above were adequate, at least for some of the children in the class. This points to a crucial aspect of explanation when taken as a collective notion. The adequacy of an explanation depends, not on the speaker, but on the listeners. Furthermore, the conceptual understandings of the individual listeners enable and constrain what is adequate for them personally.

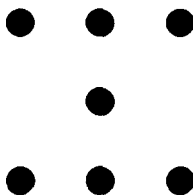


Figure 2. Dot pattern for seven shown on overhead screen

We can now explain the earlier statement that explanations should be about actions on (mathematical) objects that were experientially real to the students. On one hand, we meant that explanations should not simply be repetitions of procedures but should be about actions generated by the students' reasoning. On the other hand, we meant that the actions and the objects acted on should be meaningful conceptually to the students involved in making the explanatory statements and to those attempting to make sense of those statements. To pursue this discussion further we need analytic means to describe the various functions statements serve in argumentation, such as, the function of explanatory relevance. Toulmins argumentation scheme is useful for this purpose.

TOULMINS ARGUMENTATION SCHEME AS ELABORATED BY KRUMMHEUER

Krummheuer's work on argumentation provides the background for the approach taken here. In his study of the ethnology of argumentation, Krummheuer analyzes argumentation using Toulmins scheme of conclusion, data, warrant, and backing. According to this scheme, the conclusion is a statement that is made as though it is certain. It is a claim. In any argument, a claim has to be based on something, some facts, information, or other statements. The support given for the claim is the data. The data can be questioned on several grounds. The validity of the data can be questioned. In that case, different data must be provided or a separate argument, in which the questioned data is now treated as a claim to be established, is needed.

A different type of challenge is made when the explanatory relevance of the data is questioned. In this case a warrant is needed. A warrant explains the legitimacy of the data, that is, why the data are considered to provide support for the conclusion. Backing provides further support for the warrant, that is, the backing indicates why the warrant should be accepted as having authority (Toulmin 1969). Backings refer to general theories, beliefs, and primary strategies (Krummheuer 1995) and are successful to the extent that they are taken-as-shared by those who are involved in the debate (Forman, Larreamendy-Joerns, Stein & Brown 1998). Ball and Bass (2000) have advanced the notion of base of public knowledge as the mathematical knowledge that is available for public use by a particular community in constructing mathematical claims and to justify them to others (p. 201). From the point of view of Toulmins argumentation scheme, the base of public knowledge refers to what might be taken as a backing for a collective argumentation by a mathematics class. An example, cited by Krummheuer, of what might be a backing in a primary-grade class is understanding addition as a counting procedure that can be demonstrated by the use of fingers.

Using Toulmins Scheme to Analyze Argumentation

We return to the example from the primary-grade classroom given above to illustrate how Toulmins scheme can be used to analyze argumentation. In that example, several students explained that they saw seven dots in Figure 2 because they saw three dots, one dot and another three dots. Above, I remarked that this explanation may have been inadequate for those children who did not immediately know that three and one and three add to seven. Using the language of Toulmins argumentation scheme, we would say that the conclusion is, [There are] seven [dots]. The data is, there are three and one and another three.

This data is adequate for those children who know immediately that $3+1+3=7$. However, those children who do not just know may require a warrant. A warrant would provide the explanatory relevance of the data, that is why three and one and three have anything to do with the conclusion, seven. For these children, counting the totality would be further (perhaps necessary) backing.

An important aspect of this approach to analyzing argumentation is that the need for warrants and backing and the various functions that statements serve may differ from one student to another. Consequently, analysis of the argumentation cannot be made without considering the participants and the interpretations they make. In this sense argumentation is an interactional accomplishment and is collective in nature. Statements do not have a function apart from the interaction in which they are situated. Thus, what constitutes data, warrants, and backing is not predetermined but is negotiated by the participants as they interact.

This approach to argumentation is useful for documenting the collective learning of a class because it provides a way to demonstrate changes that take place over time. Further, it helps to clarify the relationship between the individual and the collective, that is, between the explanations and justifications that individual students give in specific instances and the classroom mathematical practices that become taken-as-shared. Using the language of argumentation, we would say that the evolution of classroom mathematical practices parallels (is enabled and constrained by) an evolution of what students take as data, warrants and backing. This, in turn, is closely related to the individual students' conceptual understandings and possibilities. As mathematical practices become taken-as-shared in the classroom, they are beyond justification, and hence, what is required as data, warrants and backing evolves. Similarly, the types of rationales that are given as data, warrants, and backing for explanations and justifications contribute to the development of what is taken-as-shared by the classroom community.

For example, in the primary-grade classroom discussed earlier, explanations of solution methods that involved thinking strategies became more cryptic over time.

Several weeks into the school year the teacher posed tasks using a double tens frame. The teacher placed chips in each frame of the double tens frame and flashed the image briefly, several times, on the overhead screen. The task was to figure out how many chips there were and to explain how you figured it out. The teacher used chips of two colors, a different color for each of the two frames. In doing so, she created a situation where students might interpret the task as finding the sum of two quantities, the quantities in the two frames. An important feature of the instructional activity was that the tasks were sequenced so that the configurations in subsequent tasks might be thought of as related to those in prior tasks. We would say that the teacher created opportunities for students to relate tasks to each other, without requiring them to do so. In the ninth lesson of the school year the teacher posed several such tasks. The first two tasks were as follows.

Task 1: Five chips in the first frame and five in the second frame.

Task 2: Six chips in the first frame and five in the second frame.

Several children gave responses in which they related tasks. For example, when called upon to explain his solution to Task 2, Hakeem replied, “Five and five is ten and..., it was one more dot on the six. One more dot on the five and that made six and I put them together and that made 11”. Later, when questioned by another student he elaborated, “Because watch. I got 10 and then she [the teacher] added one more and look, then it went to the six and this was five and then this goes to this (demonstrating with his hands, first holding up five fingers and then changing to six) and that made, that made 11”. This elaboration was particularly informative in the class discussion because Hakeem had previously explained his solution to Task 1 by holding up both hands and remarking, “Then I looked at all of them. Thats how I know it was ten”.

Thus, his two explanations taken together could be interpreted by others as clarifying both that he saw the tasks as related and also how he used the relationship to solve the second task. In this sense, he was providing both data for his claim and a warrant. Such direct references to previous problems became important in the class discussion as other students took note and subsequently attempted to use similar strategies. In this lesson, one boy specifically referred to Hakeem’s solution to Task 2 as insightful, describing it as a genius solution.

Over the next few weeks, increasingly many children began to give solutions based on relating the task at hand to prior tasks. In addition, the way they explained their thinking became more cryptic. For example, in contrast to Hakeem’s explanation in which he explicitly said, “[there was]one more dot on the five and that made 6”, explanations (for other problems) such as, “Its just one more, You had 11 and all you did was add one more, One more. So 16”, became commonplace. These slight differences might seem insignificant. Yet, following Krummheuers analysis of argumentation, we infer that,

increasingly, these cryptic statements served as data for the conclusion without the need of a warrant or of additional backing. Hakeem's explicit reference to one more dot on the five provides a rationale for why it is that five and five is ten is relevant. For some children, this additional information would have served the function of a needed warrant. This evolution in what constituted sufficiency for the students is precisely what makes it possible for us to refer to the practice of solving a task by relating it to prior tasks as a taken-as-shared classroom mathematical practice.

I have used this example to illustrate that by analyzing arguments in terms of what is taken as conclusions, data, warrants, and backing we are able to show evolution over time. This evolution in both what is required and what is given in terms of data, warrants and backing demonstrates progress for the classroom as a collective. Further, it demonstrates how the understandings of individual children and of the classroom as a whole evolve interactively.

SIGNIFICANCE FOR CLASSROOM INSTRUCTION

The significance of the preceding discussion for classroom instruction is that it suggests ways classroom teachers and university instructors might conceptualize aspects of classroom discourse. Such conceptualizations are not only useful to analyze ongoing instruction but are potentially useful to plan for future instruction. In this section of the paper, I describe examples from university-level instruction in differential equations that was deliberately planned to give attention to social and sociomathematical norms for explanation, justification and argumentation and I draw attention to the instructional benefits that ensued.

These examples are taken from several different differential equations classes taught by my colleague, Chris Rasmussen, to university students majoring in engineering or mathematics. Over the past several years, Rasmussen has conducted a series of classroom teaching experiments to investigate the teaching and learning of differential equations using his own classes as sites for investigation. A specific goal in each teaching experiment was to constitute social and sociomathematical norms characteristic of inquiry instruction. Regarding explanation and justification, these include that students explain and justify their thinking, that they listen to and attempt to make sense of the explanations of others, and that explanations describe actions on objects that are experientially real for them. Analysis of data from the first teaching experiments shows both that these norms were constituted and how the instructor initiated their negotiation (Yackel & Rasmussen 2002; Yackel, Rasmussen & King 2001).

The following example, taken from the second class session of the first teaching

experiment, illustrates the constitution of social norms that foster explanation. The instructor began the class with a brief statement of the expectations he had for the students' mathematical activity. Then he orchestrated a whole class discussion of approximately twenty minutes in which he and the students discussed the rationale underlying a differential equation they had used in the previous class session to indicate the rate of change of the recovered population in the context of an infectious disease. The crucial aspect of this segment for purposes of this paper is the explicit attention the instructor gave throughout to the negotiation of social norms. Most of his comments were explicitly or implicitly directed towards expectations. For example, he made statements such as:

“Okay, can you explain to us then why it was $1/14$ times I?”

“What do the rest of the people think about that?”

“Is that similar to what you were thinking?”

“Anyone want to add to that explanation?”, “Expand on it a little bit?”

“So lets put that question out”. “So your question is....”

“Is that what I heard you say?”

In making these remarks, the instructor was attempting to influence the interpretations students made of how to engage in discussion. From this perspective, it might seem that the instructor is the only one who contributes to the renegotiation of social norms. However, norms are constituted as individuals interact. In this case, as the episode evolved, students contributed to the negotiation of the norms by increasingly acting in accordance with the expectations. As the discussion progressed, students not only responded to the instructors questions, they initiated comments that showed that they were beginning to change their understanding of the classroom participation structure. For example, a few minutes into the discussion one student said, “I didn't quite understand what he [another student] said” and a few seconds later explained what he did understand and then said, “What I don't understand, what I was asking about....”

As the episode continued, several students asked questions, offered explanations, and asked for elaborations and clarification. In doing so, they too were contributing to the ongoing constitution of the social norms that students are expected to explain their thinking to others and ask questions and raise challenges when they do not understand. Further, analyses of classroom interactions throughout the semester provide evidence that these norms became well established. Furthermore, although I have not provided evidence here, it became normative that explanations were about actions on (mathematical) objects that were experientially real for the students.

The next example, which is taken from a more recent teaching experiment, is included to clarify and illustrate the benefits that ensued from deliberate emphasis on

argumentation. In this teaching experiment, Rasmussen's instructional approach had an even stronger emphasis on justification and argumentation than previously observed. The approach involved short segments of small group work followed by longer segments of whole class discussion. Typically groups worked for little more than two minutes before sharing their thinking. Whole class discussions then might continue for as much as 15 minutes before another short segment of two to four minutes of small group work took place. Typically, this cycle was repeated four to five times in an 80-minute class session. Furthermore the students task during small group work typically was to think about some question or issue rather than to solve a specific problem.

The second class session of the semester began with a discussion of a predator-prey problem described by a pair of differential equations. Students did not yet have any analytic techniques for recovering the solutions to the differential equations but used informal and qualitative reasoning to make sense of the situation. Throughout the discussion the instructor repeatedly asked the students the reasons for their claims. When students gave reasons, he did not evaluate their validity but instead asked other students what they thought and solicited other arguments. For example he made remarks such as, "What do you think of his idea?" and "Okay, that's nice. Let me hear another argument for...." From the point of view of norms, we would say that the instructor was initiating the expectations that students are to provide arguments to justify their claims, that they are to attempt to make sense of others arguments, and that there might be more than one argument to support a given claim. From the point of view of argumentation, we would say that the discussion focused on data, warrants, and backing to support claims but not on the claims themselves.

The instructor then posed a simplified problem that involved only one species and asked students to take a few minutes, sketch for yourself and talk about it with the folks in your group what your graphs [for population versus time] look like and the reasons for why you think the graphs look like that. After less than two minutes the instructor called the class together, drew a linear graph with positive slope on the chalkboard and asked students to give reasons for why they would agree or disagree with that graph. What is interesting about this episode is that the instructor did not ask students to talk about the graphs they had sketched. Using the language of argumentation, he did not ask them for their conclusions. Instead, he proposed a conclusion (claim) and asked them to provide reasons, that is, data, warrants, and backing to support the proposed claim. As the discussion progressed, a number of students offered reasons for why they agreed or disagreed with a linear graph. As a result, several important issues related to logistic growth problems were raised by the students, including that population growth is dependent on the existing population and that it is reasonable to think about a growth rate parameter for a particular species.

I have included this episode to illustrate that it became normative in this class that students' mathematical activity was centered around attempting to develop reasons for claims, that those reasons are themselves open to further challenge and debate, and that additional mathematical ideas and tools may need to be developed to provide adequate support for or against various claims. Because of the continual emphasis on reasoning, whole class discussions resulted in the emergence of key concepts, including slope fields, bifurcation diagrams and phase planes. See Stephan and Rasmussen (2002) for a detailed discussion of the mathematical concepts that emerged over a six-week period of this differential equations class based on analyses of the argumentation that occurred in the class sessions.

In this sense, the instructional approach seems to have considerable potential for in-depth conceptual development that grows out of student's discursive activity. The emphasis on reasons did not have the effect of bringing closure to the discussions, but of creating opportunities to advance the mathematical agenda. This result is important because it shows how what is typically called mathematical content develops in classrooms where instruction follows reformed approaches, such as those advocated by *Principles and Standards* (National Council of Teachers of Mathematics 2000). In this sense it debunks those critics who argue that mathematical content is compromised when instruction takes students' sense-making seriously.

CONCLUSIONS

My purpose in this paper has been to elaborate theoretical perspectives for analyzing explanation, justification and argumentation in the mathematics classroom and to use classroom examples to illustrate how these perspectives can be used for analysis. First, I have shown how the theoretical perspective of symbolic interactionism is useful to analyze norms related to explanation, justification, and argumentation. In particular, I have presented an example from a primary-grade classroom to demonstrate how the sociomathematical norm of what counts as an acceptable mathematical explanation is interactively constituted.

Then, I showed how Toulmin's argumentation scheme can be used to document evolution over time. For this purpose, I provided an example from the same classroom to show that what the students required and gave as data, warrants and backing evolved as the school year progressed. Finally, I have presented examples from a university-level differential equations class to demonstrate that instruction deliberately planned to give attention to social and sociomathematical norms for explanation and justification that are characteristic of inquiry instruction, and that focuses on data, warrants, and backing for

claims, rather than on the claims themselves, results in the emergence of powerful mathematical concepts. In this way, I have demonstrated how theoretical perspectives can inform both teachers and researchers as they plan and conduct mathematics instruction from the primary grades through university-level mathematics. In particular, I have shown how instruction that emphasizes mathematical reasoning as central to what it means to engage in mathematical activity might be realized in the classroom.

REFERENCES

- Ball, D. L. & Bass, H. (2000): Making believe. The collective construction of public knowledge in the elementary classroom. In: D. Phillips (Ed.), *Yearbook of the National Society for the Study of Education. Constructivism in education* (pp.193–224). Chicago: University of Chicago Press.
- Blumer, H. (1969): *Symbolic interactionism*. Engelwood Cliffs, NJ: Prentice-Hall.
- Cobb, P. (1998): Analyzing the mathematical learning of the classroom community: the case of statistical data analysis. In: A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education, Vol. 1* (pp. 33–48). Stellenbosch, South Africa: University of Stellenbosch. MATHDI 1999c.02170
- Cobb, P. & Bauersfeld, H. (1995): Introduction: The coordination of psychological and sociological perspectives in mathematics education. In: P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 1–16). Hillsdale, NJ: Erlbaum.
- Cobb, P.; Wood, T.; Yackel, E. & McNeal, B. (1992): Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal* 29(3), 573–604. MATHDI 1993c.37203
- Cobb, P. & Yackel, E. (1996): Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist* 31, 175–190.
- ____ (1997): Constructivist, emergent, and sociocultural perspectives in the context of developmental research. In: D. T. Owens, M. K. Reed & G. M. Millsaps (Eds.), *17th. annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA XVII). Proceedings. Vol. 1. Plenary lectures, discussion groups, research papers, oral reports, and poster presentations* (pp. 3–29). Columbus, OH: North American Chapter of the International Group for the Psychology of Mathematics Education. MATHDI 1997b.01485
- Cobb, P.; Yackel, E. & Wood, T. (1989): Young childrens emotional acts while doing mathematical problem solving. In: D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 117–148). New York: Springer-Verlag.

- Forman, E. A.; Larreamendy-Joerns, J.; Stein, M. K. & Brown, C. A. (1998): You're going to want to find which and prove it: Collective argumentation in a mathematics classroom. *Learning and Instruction* **8(6)**, 527–548.
- Gravemeijer, K. (1994): Educational development and educational research in mathematics education. *Journal for Research in Mathematics Education* **25(5)**, 443–471. MATHDI 1995c.01615
- Kieran, C.; Forman, E. & Sfard, A. (2001): Bridging the individual and the social: discursive approaches to research in mathematics education [Special issue]. *Educational Studies in Mathematics* **46**, 1–3.
- Krummheuer, G. (1995): The ethnology of argumentation. In: P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229–269). Hillsdale, NJ: Erlbaum.
- Lampert, M. & Cobb, P. (2003): Communication and language. In: J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to NCTMs principles and standards* (pp. 237–249). Reston, VA: National Council of Teachers of Mathematics.
- National Council of Teachers of Mathematics (2000): *Principles and standards for school mathematics*. Reston, VA: Author. MATHDI 1999f.04754
- Richards, J. (1991): Mathematical discussions. In: E. von Glasersfeld (Ed.), *Constructivism in mathematics education* (pp. 13–52). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Rommetveit, R. (1985): Language acquisition as increasing linguistic structuring of experience and symbolic behavior control. In: J. V. Wertsch (Ed.), *Culture, communication, and cognition* (pp. 183–205). Cambridge: Cambridge University Press.
- Stephan, M. & Rasmussen, C. (2002): Classroom mathematical practices in differential equations. *Journal of Mathematical Behavior* **21**, 459–490.
- Toulmin, S. (1969): *The uses of argument*. Cambridge, UK: Cambridge University Press.
- Voigt, J. (1996): Negotiation of mathematical meaning in classroom processes: Social interaction and learning mathematics. In: L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin & B. Greer (Eds.), *Theories of mathematical learning* (pp. 21–50). Mahwah, NJ: Erlbaum.
- Yackel, E. (1996): Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education* **22(4)**, 458–477. MATHDI 1997f.04566
- Yackel, E. (1997): *Explanation as an interactive accomplishment: A case study of one second-grade mathematics classroom*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL.
- Yackel, E. (2000): *Creating a mathematics classroom environment that fosters the development of mathematical argumentation*. Paper presented at the Ninth International Congress of Mathematical Education, Tokyo.
- Yackel, E. (2001): Explanation, justification and argumentation in mathematics classrooms. In: M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International*

Group for the Psychology of Mathematics Education, Vol. 1 (pp. 9–24). Utrecht, The Netherlands: Freudenthal Institute.

- Yackel, E.; Cobb, P.; Wood, T.; Wheatley, G. & Merkel, G. (1990): The importance of social interactions in childrens construction of mathematical knowledge. In: T. Cooney (Ed.), *Teaching and learning mathematics in the 1990s*. Reston, VA: National Council of Teachers of Mathematics.
- Yackel, E. & Hanna, G. (2003): Reasoning and proof. In: J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to NCTMs principles and standards* (pp. 227–236). Reston, VA: National Council of Teachers of Mathematics.
- Yackel, E. & Rasmussen, C. (2002): Beliefs and norms in the mathematics classroom. In: G. Leder, E. Pehkonen & G. Toerner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 313–330). The Netherlands: Kluwer Academic Publishers. MATHDI 2003c.02416
- Yackel, E.; Rasmussen, C. & King, K. (2001): Social and sociomathematical norms in an advanced undergraduate mathematics course. *Journal of Mathematical Behavior* 19, 1–13.