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Derivations of Free Joins of Algebras*

Dept. of Mathematics, Chungbuk National Univ. **Jae-young Han**
hjy@chungbuk.ac.kr

Dept. of Mathematics, Chungbuk National Univ. **Sook-Ja Nam**
nskz@chungbuk.ac.kr

Dept. of Mathematics, Chungbuk National Univ. **Yeon-hee Kim**
kmhe@chungbuk.ac.kr

In this paper, we will prove that a free join algebra and a universal derivation module of its subalgebras have a universal derivation module induced by its subalgebras.

Key words: (universal) derivation module, tensor algebra, free join algebra, fractional extension

0. Introduction

Let R be a commutative ring with identity and A a uitary algebra over R which is not necessarily commutative. For an (A, A)-bimodule M, an R-linear mapping $d: A \to M$ is called an R-derivation if $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. A pair (M, d) of an (A, A)-bimodule M and an R-derivation $d: A \to M$ is called a derivation module of A. An (A, A)-bimodule homomorphism $f: (M, d) \to (N, \delta)$ is a derivation module homomorphism if $f \cdot d = \delta$. A derivation module (U, d) is called a universal derivation module if for any derivation module (N, δ) of A, there exists a unique derivation module homomorphism $f: (M, d) \to (N, \delta)$.

An R-algebra A is called a tensor algebra of an R-module of M over R if for any R-algebra C and an R-linear mapping $f: M \to C$, there exists a unique R-algebra homomorphism $g: A \to C$ extending f.

* 충북대학교 기초과학연구소 연구비에 의하여 연구되었음.

Every tensor algebra of an R-module M over R is generated by M and it is unique up to algebra isomorphism.

1. Free Joins of Algebras

An R-algebra A is called a free join of a family $(A_\alpha)_{\varepsilon\in I}$ of its subalgebras if any algebra C and any family $(f_\alpha)_{\alpha\in I}$, $f_\alpha: A_\alpha \to C$ of algebra homomorphims, there exists a unique algebra homomorphism $f: A \to C$ extending f_α for each $\alpha \in I$.

Proposition 1. Let A be a free join of a family $(A_\alpha)_{\varepsilon\in I}$ of its subalgebras. If R is a direct summand of each A_α and there exists an R-module homomorphism $g_\alpha : A \to R$ for each $\alpha \in I$, then for any finite sequence $\beta = (\alpha_1, \cdots, \alpha_k)$ where the α_i are all different, the mapping $f: A_{\alpha_1} \cdots A_{\alpha_k} \to A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_k}$ given by $a_{\alpha_1} \cdots a_{\alpha_k} \to a_{\alpha_1} \otimes \cdots \otimes a_{\alpha_k}$ is an R-module homomorphism.

Lemma 1. Let A be a free join of a family $(A_\alpha)_{\varepsilon\in I}$ of its subalgebras and $T(A_\alpha)$ and $T(B)$ be the tensor algebras of A_α and $B = \bigoplus_{\alpha\to I} A_\alpha$ respectively. If $h_\alpha: T(A_\alpha) \to A_\alpha$ is an algebra homomorphism extending the identity mapping i_{A_α} for each $\alpha \in I$ and $h: T(B) \to A$ is an algebra homomorphism extending h_α for each $\alpha \in I$, then h is onto and $\ker h$ is the ideal of $T(B)$ generated by $\sum_{\alpha\in I} \ker h_\alpha$.

Theorem 1. Let A be a free join of a family $(A_\alpha)_{\varepsilon\in I}$ of its subalgebras and $a_{\alpha_1}, \cdots, a_{\alpha_k}$ a finite sequence such that $\alpha_i \neq \alpha_{i+1}$, $i = 1, \cdots, k-1$, then the R-linear mapping $f: A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_k} \to A_{\alpha_1} \cdots A_{\alpha_k}$ given by $a_{\alpha_1} \otimes \cdots \otimes a_{\alpha_k} \to a_{\alpha_1} \cdots a_{\alpha_k}$ is an R-module isomorphism.

Proof. Let h be the algebra homomorphism in Lemma 1. Let $f = h \mid A_{\alpha_1} \otimes \cdots$

$\otimes A_{\alpha_k}$. Since $\ker f \subset \ker h_\alpha$ and $\ker h \cap (A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_k}) = \emptyset$, $\ker f \cap (A_{\alpha_1} \otimes \cdots \otimes A_{\alpha_k}) = \emptyset$. Hence this mapping is one to one. This implies that f is an isomorphism.

Theorem 2. Let A be a free join of a family $(A_\alpha)_{\varepsilon \in I}$ of its subalgebras and (U_α, d_α) a universal derivation module of A_α for each $\alpha \in I$. If $U = \bigoplus_{\alpha \in I} (A \underset{A_\alpha}{\otimes} U_\alpha \underset{\alpha \in I}{\otimes} A)$ and $D: A \to U$ is the R-derivation defined by $\sum_{\alpha \in I} a_1 \cdots a_k \to \sum_{\alpha \in I} (\sum_{i=1}^{k} a_1 \cdots a_{i-1} \otimes d_{\alpha_i}(a_i) \otimes a_{i+1} \cdots a_k)$ where $a_i \in A_{\alpha_i}$, $\alpha_i \in I$, then (U, D) is a universal derivation module of A.

Proof. Let $\phi'_\alpha : A_{\alpha_1} \times \cdots \times A_{\alpha_k} \to U$ be an R-multilinear mapping given by

$$(a_1, \cdots, a_k) \to a_1 \cdots a_{i-1} \otimes d_{\alpha_i}(a_i) \otimes a_{i+1} \cdots a_k$$

where $a_i \in A_{\alpha_i}$ and D_{α_i} the R-linearlization of ϕ'_α. Define a mapping $D: A \to U$ by

$$D(\sum_{\alpha \in I} a_1 \cdots a_k) = \sum_{\alpha \in I} (\sum_{i=1}^{k} D_{\alpha_i}(a_1 \cdots a_k)), \quad a_i \in A_{\alpha_i}$$

Then D is an R-derivation same as above. To show that (U, D) is a universal derivation module of A, let (M, δ)be any derivation module of A and $\delta_\alpha = \delta | A_\alpha$. Since each δ_α is an R-derivation and (U_α, D_α) is a universal derivation module of A_α, there exists a unique (A_α, A_α)-bimodule homomorphism $f_\alpha: A_\alpha \to M$ such that $f_\alpha \cdot d_\alpha = \delta_\alpha$. Let $g'_\alpha : A \times U_\alpha \times A \to M$ be an A-multilinear mapping given by $(a, u_\alpha, b) \to a f_\alpha(u_\alpha) b$ where $u_\alpha \in U_\alpha$, $a, b \in A$. Let g_α be the A-liberalization of g'_α. Then each g_α is an (A, A)-bimodule homomorphism. Define a mapping $g : U \to M$ by $\sum_\alpha (a_\alpha \otimes u_\alpha \otimes b_\alpha) \to \sum_\alpha g_\alpha (a_\alpha \otimes u_\alpha \otimes b_\alpha)$ where a_α, $b_\alpha \in A$ and $\sum_\alpha (a_\alpha \otimes u_\alpha \otimes b_\alpha) \in A \underset{A_\alpha}{\otimes} U_\alpha \underset{A_\alpha}{\otimes} A$.

Then h is an R-derivation module homomorphism such that $g \cdot D = \delta$. The uniqueness of such homomorphism from the fact that each $A \underset{A_\alpha}{\otimes} U_\alpha \underset{A_\alpha}{\otimes} A$ is generated by $D_\alpha(A_\alpha)$ and hence U is generated by $D(A)$. We proved that (U, d) constructed in this way is a universal derivation module of A. ■

2. Extension of Algebras

Let E be a unitary extension algebra of an R-algebra. An ideal I of A is said to be E-dense if $EI = IE = E$. An ideal I of A which contains an E-dense ideal J is also E-dense, since $EI \supset EJ = E$, $IE \supset JE = E$. If I and J are E-dense ideals of A, then IJ and JI are E-dense.

An extension E of an R-algebra A is called a fractional extension of A if for any p, $q \in A$, there exists E-dense ideals I and J such that pI, $Jq \in A$.

Proposition 2. Let Q be a two-sided quotient algebra of an R-algebra A with relative to a multiplicative subset of S without zero divisor. Then Q is a fractional extension of A.

An (A, A)-bimodule M is said to be E-torsion free if for any E-dense ideal I of A and $x \in M$, $Ix = 0$ implies $x = 0$ and $xI = 0$ implies $x = 0$. For example every (E, E)-bimodule is an E-torsion free (A, A)-bimodule, when E is a fractional extension of A.

Lemma 2. Let E be a fractional extension of an R-algebra A.

(1) For any E-torsion free (A, A)-bimodule M, an (A, A)-bimodule homomorphism $f : M \to E \bigotimes_A M \bigotimes_A E$ given by $f(x) = 1 \otimes x \otimes 1$, $x \in M$ is one to one.

(2) Every (A, A)-bimodue homomorphism $f : M \to N$ of (E, E)-bimodules is an (E, E)-bimodules homomorphism.

Theorem 3. Let E be a fractional extension of an R-algebra A, and M an (E, E)-bimodule. If R-derivation d, $\delta : E \to M$ are equal on A, then $d = \delta$.

Proof. Let I be an E-dense ideal of A such that $Iq \subset A$, $q \in E$. Since $d - \delta$ is an R-derivation of E, $(d-\delta)(bq) = b(d-\delta)(q) + (d-\delta)(b)q$, $b \in I$. Since $bq \in A$, $b \in A$, we have $b(d-\delta)(q) = 0$ for all $q \in E$. Hence $I(d-\delta)q = 0$. By the fact M is an E-torsion free (A, A)-bimodule, $(d-\delta)(b)q = 0$, $q \in E$. ■

An (A, A)-bimodule H is called an injective hull of an (A, A)-bimodule M if H

is a left(right) injective hull of the left $A\bigotimes_R A^{OP}$-module (right $A^{OP}\bigotimes_R A$-module) M.

Lemma 3. Let E be a fractional extension of an R-algebra A, and M an E-torsion free (A, A)-bimodule. Then every injective hull of M is E-torsion free.

Lemma 4. Let E be a fractional extension of an R-algebra A, and M an E-torsion free (A, A)-bimodule, and I an E-dense ideal A. If $\phi: I \to M$ is an (A, A)-bimodule homomorphism, then there exists a unique (A, A)-bimodule homomorphism $f: A \to M$ extending ϕ.

Theorem 4. Let E be a fractional extension of an R-algebra A, and $f: M \to E\bigotimes_A M\bigotimes_A E$ an (A, A)-bimodule homomorphism given by $f(x) = 1\otimes x\otimes 1$ for all $x \in M$. Then for any R-derivation $d: A \to M$, there exists a unique R-derivation $\delta: E \to E\bigotimes_A M\bigotimes_A E$ such that $\delta | A = f\cdot d$.

Proof. Let $q \in E$, and I an E-dense ideal of A such that $qI \subset A$. Define a mapping $f_{I,q}: I \to E\bigotimes_A M\bigotimes_A E$ by $b \to 1\otimes d(ab)\otimes 1 - q\otimes d(b)\otimes 1$ for all $b \in I$. Then $f_{I,q}$ is a right A-module homomorphism. Indeed for any $a, b \in I, r, s \in R$, $f_{I,q}(ra+sb) = rf_{I,q}(a) + sf_{I,q}(b)$.

For all $c \in A$, $f_{I,q}(ra+sb) = rf_{I,q}(a) + sf_{I,q}(b)$. By Lemma 4 there exists a unique A-module homomophism $g_{I,q}: A \to E\bigotimes_A M\bigotimes_A E$ extending $f_{I,q}$ for all $q \in E$. To Show that $g_{I,q}$ is independent of the choice of an E-dense ideal of A, let J be any E-dense ideal of A such that $qJ \subset A$. Since $I \cap J$ is an E-dense ideal of A, there exists a unique right A-module homomorphism $g_{I\cap J,q}: A \to E\bigotimes_A M\bigotimes_A E$ extending the left A-module homomorphism $f_{I\cap J,q}: I\cap J \to E\bigotimes_A M\bigotimes_A E$ given by $c \to 1\otimes d(qc)\otimes 1 - q\otimes d(c)\otimes 1$ for all $c \in I\cap J$. Then $f_{I,q}| I\cap j = f_{I\cap J,q} = f_{J,q}| I\cap J$. Hence $g_{I,q}$ and $g_{J,q}$ are right A-module homomorphism extending a right A-module homomorphism $f_{I\cap J,q}$. By the uniqueness of such A-module homomorphism, $g_{I,q} = g_{J,q}$. Let

$g_q = g_{I,q}$ for all $q \in E$. Define a mapping $\delta: E \to E \underset{A}{\otimes} M \underset{A}{\otimes} E$ by $\delta(q) = g_q(1)$, $q \in E$. Then δ is an R-derivation.

Let $g_a = f_{A,a}: A \to E \underset{A}{\otimes} M \underset{A}{\otimes} E$ be a right A-module homomorphism given by $f(b) = 1 \otimes d(ab) \otimes 1 - a \otimes d(b) \otimes 1$, $b \in A$. Then $\delta(a) = g_a(1) = 1 \otimes d(a) \otimes 1 - a \otimes d(1) \otimes 1$, $b \in A$. Hence $\delta: E \to E \underset{A}{\otimes} M \underset{A}{\otimes} E$ is R-derivation such that $\delta | A = f \cdot d$. By Theorem 3, δ is a unique R-derivation of E satisfying the given condition.

Theorem 5. Let E be a fractional extension of an R-algebra A, (U, d) a universal derivation module of A, and $\delta: E \to E \underset{A}{\otimes} M \underset{A}{\otimes} E$ an R-derivation such that $\delta | A = f \cdot d$. Here $f: U \to E \underset{A}{\otimes} U \underset{A}{\otimes} E$ is an (A, A)-bimodule homomorphism given by $f(x) = 1 \otimes x \otimes 1$ for all $x \in U$. Then $(E \underset{A}{\otimes} U \underset{A}{\otimes} E, \delta)$ is a universal derivation module of E.

Proof. Let (V, τ) be any derivation module of A, and let $\tau' = \tau \mid A$. Since τ' is an R-derivation of A, there exists an (A, A)-bimodule homomorphism $g: U \to V$ such that $g \cdot d = \tau'$. Let ϕ be the A-liberalization of A-multilinear mapping $\psi: E \times U \times E \to M$ given by $(p, x, q) \to pg(x)q$, $p, q \in E$. Then ϕ is an (E, E)-bimodules homomorphism. Furthermore $\phi \cdot \delta$ and τ are R-derivation of E such that $\phi \cdot \delta = \tau$ on A. By Theorem 3, $\phi \cdot \delta = \tau$ on E. The uniqueness of such module homomorphism follows from the fact $(E \underset{A}{\otimes} U \underset{A}{\otimes} E, \delta)$ is a universal derivation module of E.

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다원환의 자유결합의 미분

충북대학교 수학과 **한재영**
충북대학교 수학과 **남숙자**
충북대학교 수학과 **김연희**

이 논문에서는 다원환의 자유결합의 대수적 구조를 규명하여 분수확대체로서의 미분가군의 일반적인 특성을 연구하고 있다. 보편적 범주 내에서의 미분과 미분가군의 대수적 형태는 대수적 결합의 기본 원칙을 충실히 보존한다는 원칙을 밝혔을 뿐만 아니라 대수적 동형 개념으로 수학의 우주적 균형이론을 실질적으로 보여주고 있다.

주제어 : 대수적 미분, 범다원환, 텐서다원환, 자유결합다원환, 가환미분다원환, 다항식다원환, 분수확대체