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ON GEOMETRIC ERGODICITY OF AN AR-ARCH TYPE PROCESS WITH MARKOV SWITCHING

Oesook Lee† and Dong Wan Shin

Abstract. We consider a nonlinear AR-ARCH type process subject to Markov-switching and give sufficient conditions for geometric ergodicity of the process. Existence of moments is also obtained.

1. Introduction

The family of ARCH model, which was introduced by Engle [4] have proven useful in financial applications and have attracted great attention in economics and statistical literature (Tong [22], Bollerslev *et al.* [2], Bougerol and Picard [3], Masry and Tjøstheim [18]), Li and Li [14], Ling [15], Lee and Kim [10], Ling and McAleer [16]). Markov switching model, in which a hidden Markov process governs the behavior of an observable time series was first introduced by Hamilton [8] and gained much attention in recent years (see, e.g, McCulloch and Tsay [19], Hamilton [9], Yang [24], Yao and Attali [26], Francq and Zakoïan [5], Yao [25], Francq *et al.* [6], Zhang and Stine [27], Lee [11]). Given such models, interests are the conditions under which a given model has probabilistic properties such as strict stationarity, geometric ergodicity and existence of higher order moments. Those properties are of great importance in statistical inference for the time series models.

Let $\{U_t : t \geq 0\}$ be an irreducible, aperiodic Markov chain on a finite state space E with stationary n-step transition probability matrix $P^{(n)} = (p_{uv}^{(n)})_{u,v \in E}$. We consider a nonlinear AR-ARCH process with Markov switching $\{y_t\}$ defined for integers $t \geq 1$, by

$$y_t = g_{1,U_t}(y_{t-1}, \ldots, y_{t-p}) + g_{2,U_t}(y_{t-1}, \ldots, y_{t-q})e_t \tag{1.1}$$

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where $\{e_t\}$ is an independent and identically distributed(iid) sequence of random variables with mean zero and variance σ^2, and $\{g_{1,u}\}_{u\in E}$ and $\{g_{2,u}\}_{u\in E}$ are families of measurable functions on R^p and R^q respectively.

If E consists of only one point, equation (1.1) becomes the following ARCH-related system which contains many of the nonlinear classes discussed in the literature;

$$y_t = g_1(y_{t-1}, \dots, y_{t-p}) + g_2(y_{t-1}, \dots, y_{t-q})e_t. \tag{1.2}$$

One of the special type of (1.2) is a double threshold AR-ARCH model obtained by

$$\begin{aligned} y_t &= \phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} y_{t-i} + \epsilon_t, \quad a_{j-1} \le y_{t-b} < a_j \\ \epsilon_t &= \sqrt{h_t}\cdot e_t, \\ h_t &= \alpha_0^{(k)} + \sum_{i=1}^{r} \alpha_i^{(k)} \epsilon_{t-i}^2, \quad b_{k-1} \le \epsilon_{t-d} < b_k \end{aligned} \tag{1.3}$$

where $j = 1, \dots, l_1,\ k = 1, \dots, l_2,\ -\infty = a_0 < \cdots < a_{l_1} = \infty,\ -\infty = b_0 < \cdots < b_{l_2} = \infty, \phi_i^{(j)}, \alpha_i^{(k)}$ are constants with $\alpha_0^{(k)} > 0, \alpha_i^{(k)} \ge 0$ $(1 \le i \le r)$. Stationarity, geometric ergodicity and other probabilistic properties are studied for these models generated by (1.2) (or (1.3)) in Masry and Tjøstheim [18], Li and Li [14], Liu *et al.* [17], Ling [15], Lee [13] etc.

In this paper, we deal with geometric ergodicity of a AR-ARCH type process subject to Markov regime switching given by (1.1) and find sufficient conditions under which the process is geometrically ergodic. AR-ARCH model generated by (1.2) is also considered and geometric ergodicity and existence of moments are studied.

Throughout this paper we assume, without loss of generality, that $p = q$ $(p \ge 1)$.

For terminologies and relevant results in Markov chain theory we refer to Meyn and Tweedie [20].

2. Main results

Let $\{U_t : t \ge 0\}$ be an irreducible, aperiodic Markov chain on a finite state space E with stationary n-step transition probability matrix $P^{(n)} = (p_{uv}^{(n)})_{u,v\in E}$. We consider a nonlinear AR-ARCH process with Markov switching $\{y_t\}$ defined for integers $t \ge 1$, by

$$y_t = g_{1,U_t}(y_{t-1}, \dots, y_{t-p}) + g_{2,U_t}(y_{t-1}, \dots, y_{t-p})e_t. \tag{2.4}$$

We assume that $\{e_t\}$ is iid with mean zero and variance σ^2 and $\{e_t\}$ and $\{U_t\}$ are independent. Denote

$$Y_t = (y_t, \dots, y_{t-p+1}), \quad W_t = (U_t, Y_t). \tag{2.5}$$

Then W_t is an aperiodic $E \times R^p$-valued Markov chain.

Recall that verification of geometric ergodicity of a Markov chain $\{W_t\}$ proceeds by proving that the process is ϕ-irreducible aperiodic and by showing the existence of a test function satisfying the following Foster-Lyapounov drift condition (see, e.g., Tjøstheim [21], Meyn and Tweedie [20]):

DRIFT CONDITION: there exists a real valued measurable function $V \geq 1$ such that for some constants $\epsilon > 0, 0 < c < \infty, 0 < \lambda < 1$ and a small set K,

$$E[V(W_t) \mid W_{t-1} = w] \leq \lambda V(w) - \epsilon, \quad x \in K^c$$

and

$$E[V(W_t) \mid W_{t-1} = w] < c, \quad x \in K.$$

We make the following assumptions.

ASSUMPTION I. (a) The iid random variables $\{e_t\}$ have a probability density function f that is continuous and positive over R. (b) The functions $g_{1,u}$ and $g_{2,u}$ are nonperiodic and bounded on compact sets, and $g_{2,u}(z) > 0$ for all $z \in R^p$, $u \in E$.

Let μ_p denote the Lebesgue measure on the Borel σ-field $\mathcal{B}(R^p)$ of R^p and $\|\cdot\|$ be any vector norm on R^p $(p = 1, 2, \dots)$. For any function f, let $E_u[f(U_t)] = E[f(U_t)|U_{t-1} = u]$.

ASSUMPTION II. For each $u \in E$, there exist $a_i(u) \geq 0$, $d_i(u) \geq 0$ $(i = 1, \dots, p)$ such that for $z = (z_1, \dots, z_p)$,

(a) $|g_{1,u}(z)| \leq \sum_{i=1}^p a_i(u)|z_i| + o(\|z\|)$ and

(b) $g_{2,u}^2(z) \leq \sum_{i=1}^p d_i(u) z_i^2 + o(\|z\|^2)$.

THEOREM 2.1. *Let the assumptions I and II hold. If*

$$\sum_{i=1}^p \sup_u E[\sum_{j=1}^p a_i(U_t)a_j(U_t) + \sigma^2 d_i(U_t) \mid U_{t-1} = u] < 1, \tag{2.6}$$

then W_t is geometrically ergodic and $E_{\pi_y}[y_t^2] < \infty$ where π is the unique invariant distribution of W_t and $\pi_y(B) = \pi(E \times B \times R^{p-1}), B \in \mathcal{B}(R)$.

Proof. We first show that $\{W_t\}$ is $\nu \times \mu_p$-irreducible where ν is a counting measure on E and that every compact set is a small set.

For fixed $u_1, u_2, \ldots, u_p$, $x = (x_p, \ldots, x_1)$, $z = (z_p, \ldots, z_1)$, define

$$\begin{aligned}
& h(x, z|u_1, \ldots, u_p) \\
(2.7) \qquad = \; & f(g_{2,u_1}^{-1}(x_p, \ldots, x_1)(z_1 - g_{1,u_1}(x_p, \ldots, x_1))) \\
& \Pi_{i=2}^{p} f(g_{2,u_i}^{-1}(z_{i-1}, \ldots, z_1, x_p, \ldots, x_i)) \\
& (z_i - g_{1,u_i}(z_{i-1}, \ldots, z_1, x_p, \ldots, x_i)).
\end{aligned}$$

Since the density function f is continuous and positive everywhere, we have that if $\mu_p(A) > 0$ and C is a compact subset of R^p, then

$$(2.8) \qquad \int_A h(x, z|u_1, \ldots, u_p) d\mu_p(z) > 0$$

and

$$(2.9) \qquad \inf_{x \in C} \int_A h(x, z|u_1, \ldots, u_p) d\mu_p(z) > 0.$$

(see, for example, Bhattacharya and Lee [1], Lee [12])

For any $E' \subset E$, choose $u_1, \ldots, u_t$ in E such that $t \geq p$, $u_t \in E'$ and

$$p_{u_0 u_1} \cdots p_{u_{t-1} u_t} > 0.$$

Let $Y_t(u_1, \ldots, u_t)$ be Y_t given $U_1 = u_1, \ldots, U_t = u_t$.

Note that the following two equalities hold: for $W_0 = (u_0, x)$,

$$\begin{aligned}
(2.10) \quad & P(W_t \in E' \times A|W_0) \\
= & \sum_{u_t \in E'} \sum_{u_{t-1} \in E} \cdots \sum_{u_1 \in E} p_{u_0 u_1} \cdots p_{u_{t-1} u_t} P(Y_t(u_1, \ldots, u_t) \in A|W_0)
\end{aligned}$$

and

$$\begin{aligned}
& P(Y_t(u_1, \ldots, u_t) \in A|Y_{t-p} = w) \\
(2.11) \qquad = \; & \int_A h(w, z|u_{t-p+1}, \ldots, u_t) d\mu_p(z) > 0, \quad \forall w.
\end{aligned}$$

Combining (2.7)-(2.11), desired results follow.

Let $\eta_i = \sup_u E[\beta_i(U_t) + \sigma^2 d_i(U_t) \mid U_{t-1} = u]$, where

$$\beta_i(u) = \sum_{j=1}^{p} a_i(u) a_j(u).$$

Choose $\delta > 0$ so that $\sum \eta_i + \delta = 1$. Now define a test function $V : E \times R^p \to R$ by

$$V(u, z_1, \ldots, z_p) = \sum_{i=1}^{p} \gamma_i z_i^2 + 1, \tag{2.12}$$

where γ_1 is any positive real number and $\gamma_{i+1} = \gamma_1(1 - \eta_1 - \cdots - \eta_i - i\delta/p)$, $i = 1, 2, \ldots, p-1$. Then

$$\gamma_{i+1} + \gamma_1\eta_i \le \gamma_i(1 - \frac{\delta}{p}), \quad 1 \le i \le p-1$$
$$\gamma_1\eta_p \le \gamma_p(1 - \frac{\delta}{p}),$$

and hence we obtain that for $z = (z_1, \ldots, z_p)$,

$$\begin{aligned}
& E[V(W_t) \mid W_{t-1} = (u, z_1, \ldots, z_p)] \\
= \;& E[\gamma_1(g_{1,U_t}(z) + g_{2,U_t}(z)e_t)^2 + \sum_{i=2}^{p} \gamma_i z_{i-1}^2 \mid U_{t-1} = u] + 1 \\
\le \;& \gamma_1 E_u[g_{1,U_t}^2(z) + g_{2,U_t}^2(z)\sigma^2] + \sum_{i=2}^{p} \gamma_i z_{i-1}^2 + 1 \\
\le \;& \gamma_1 \sum_{i=1}^{p} E_u[\beta_i(U_t) + \sigma^2 d_i(U_t)] z_i^2 + \sum_{i=2}^{p} \gamma_i z_{i-1}^2 \\
& + \gamma_1 E_u[R_{U_t}(z)] + 1 \\
\le \;& \gamma_1 \sum_{i=1}^{p} \eta_i z_i^2 + \sum_{i=2}^{p} \gamma_i z_{i-1}^2 + \gamma_1 E_u[R_{U_t}(z)] + 1 \\
\le \;& (1 - \frac{\delta}{p}) \sum_{i=1}^{p} \gamma_i z_i^2 + \gamma_1 E_u[R_{U_t}(z)] + 1 \\
\le \;& V(u, z)(1 - \frac{\delta}{p} + \frac{\gamma_1 E_u[R_{U_t}(z)] + \delta/p}{V(u, z)}),
\end{aligned} \tag{2.13}$$

where $R_u(z) = 2o(\|z\|)(\sum_{i=1}^{p} a_i(u)|z_i|) + (o(\|z\|))^2 + \sigma^2 o(\|z\|^2)$. Since every norm on R^p is equivalent, $E_u[R_{U_t}(z)]/V(u, z) \to 0$ as $\|z\| \to \infty$.

Therefore for any $\epsilon > 0$ we can choose ρ, $1 - \delta/p < \rho < 1$ and $M < \infty$ so that the next two inequalities hold;

$$E[V(W_t) | W_{t-1} = (u, z)] \le \rho V(u, z) - \epsilon, \quad \|z\| > M \tag{2.14}$$

$$\sup_{\|z\| \le M} E[V(W_t) | W_{t-1} = (u, z)] < \infty. \tag{2.15}$$

Geometric ergodicity of W_t is obtained from (2.14) and (2.15) and finiteness of second-moment of y_t can be derived from (2.12), (2.14) and Theorem 2 of Tweedie [23]. □

THEOREM 2.2. *In addition to the assumption I, we assume that for each $u \in E$, there exists $0 < \lambda_u < 1$ such that for $z = (z_1, \ldots, z_p)$,*

$$|g_{1,u}(z)| \le \lambda_u \max_{1\le i\le p}\{|z_i|\} + o(\|z\|) \quad \text{and} \tag{2.16}$$

$$g_{2,u}(z) = o(\|z\|). \tag{2.17}$$

Then the process given in (2.5) *is geometrically ergodic.*

Proof. Let $\lambda = \sup_u \lambda_u$. Choose $\gamma_1 > 0$ arbitrary but fixed and take $\gamma_{i+1} = \lambda^{\frac{1}{p}}\gamma_i, \quad i = 1, 2, \ldots, p-1$. Define a test function $V_1(\cdot)$ on $E \times R^p$ to R by $V_1(u, z) = \max_{1\le i\le p}\{\gamma_i|z_i|\} + 1$. Then for any $\epsilon > 0$, there is $\rho < 1$ such that

$$\begin{aligned}
& E[V_1(W_t)|W_{t-1} = (u, z_1, \ldots, z_p)] \\
\le\ & E_u[\max\{\gamma_1|g_{1,U_t}(z)|, \gamma_2|z_1|, \ldots, \gamma_p|z_{p-1}|\}] \\
& + \gamma_1 E_u[g_{2,U_t}(z)]E|e_t| + 1 \\
\le\ & \max\{\gamma_1\lambda\max\{|z_i|\}, \gamma_2|z_1|, \ldots, \gamma_p|z_{p-1}|\} + K(z) \\
\le\ & \lambda^{\frac{1}{p}}V_1(u, z) + K(z) - \lambda^{\frac{1}{p}} \\
\le\ & \rho V_1(u, z) - \epsilon, \quad \|z\| > M_1
\end{aligned} \tag{2.18}$$

for some sufficiently large $M_1 < \infty$, where

$$K(z) = \gamma_1 o(\|z\|) + \gamma_1 o(\|z\|)E|e_t| + 1.$$

Last inequality in (2.18) follows from the fact that $K(z)/V_1(u, z) \to 0$ as $\|z\| \to \infty$. Since $\sup_{\|z\|\le M_1} E[V_1(W_t) \mid W_{t-1} = (u, z)] < \infty$, inequalities in drift condition with V_1 as its test function are satisfied and geometric ergodicity of Y_t is obtained. □

COROLLARY 2.1. *Suppose that the assumptions I and II*(a) *and equation* (2.17) *hold.*

(1) *If* $\sup_u \sum_{i=1}^{p} a_i(u) < 1$, *the geometric ergodicity of* W_t *follows.*

(2) *If* $\sum_{i=1}^{p} \sup_u E[a_i(U_t)|U_{t-1} = u] < 1$, *the geometric ergodicity of* W_t *follows.*

Proof. (1) Since

$$|g_{1,u}(z)| \quad \le \quad \sum_{i=1}^{p} a_i(u)|z_i| + o(\|z\|)$$

$$\leq \quad (\sum_{i=1}^{p} a_i(u)) \max_{1\leq i\leq p}\{|z_i|\} + o(\|z\|), \tag{2.19}$$

the conclusion follows from Theorem 2.2.

(2) Take a test function by $V(z) = \sum \gamma_i |z_i|$ where γ_i is the same as defined in the proof of Theorem 2.1 with $\eta_i = \sup_u E[a_i(U_t)|U_{t-1} = u]$. Then the remaining part of the proof is similar to that of Theorem 2.1 and is omitted. □

REMARK 1. Note that none of three conditions : inequality (2.6) in Theorem 2.1 and two inequalities in Corollary 2.1 (1) and (2) is superior than the other.

Now consider a nonlinear AR-ARCH type model given by

$$y_t = g_1(y_{t-1}, \ldots, y_{t-p}) + g_2(y_{t-1}, \ldots, y_{t-p})e_t, \tag{2.20}$$

which is a special case of (2.4). Let

$$Y_t = (y_t, y_{t-1}, \ldots, y_{t-p+1}). \tag{2.21}$$

For the case in (2.20), we give the following assumption:

ASSUMPTION III. There exist vectors $(a_1, \ldots, a_p)$, $(d_1, \ldots, d_p)$ with $d_i \geq 0$, $(i = 1, \ldots, p)$ such that for $z = (z_1, \ldots, z_p)$,

(a) $g_1(z) = \sum_{i=1}^{p} a_i z_i + o(\|z\|)$ and

(b) $g_2^2(z) = \sum_{i=1}^{p} d_i z_i^2 + o(\|z\|^2)$.

Masry and Tjøstheim [18] showed that under the assumptions I and III, $\rho(AA^t) + \max_{1\leq i\leq p} d_i \cdot \sigma^2 < 1$ is sufficient for geometric ergodicity of $\{Y_t\}$, where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & \cdots & 1 & a_p \end{pmatrix}$$

But $\rho(AA^t) \geq 1$ for $p > 1$.

THEOREM 2.3. *Suppose that the assumptions I and III hold.*

(1) *If* $(\sum_{i=1}^{p} |a_i|)^2 + \sigma^2 \sum_{i=1}^{p} d_i < 1$, Y_t *is geometric ergodic and* $E_{\pi_y}[y_t^2] < \infty$.

(2) *If* $E[e_t^3] = 0$ *and* $(1+3\sigma^2)(\sum_{i=1}^{p} |a_i|)^4 + (E[e_t^4]+3\sigma^2)(\sum_{i=1}^{p} d_i)^2 < 1$, *then* Y_t *is geometrically ergodic and* $E_{\pi_y}[y_t^4] < \infty$.

Proof. (1) If E has only one point, equation (2.4) becomes (2.20). Results follow from the relation

$$\sum_{i=1}^{p}(\sum_{j=1}^{p}|a_i a_j| + \sigma^2 d_i) = (\sum_{i=1}^{p}|a_i|)^2 + \sigma^2 \sum_{i=1}^{p} d_i.$$

(2) Since $E[e_t] = E[e_t^3] = 0$, we have that

$$\begin{aligned} & E[(g_1(z) + g_2(z)e_t)^4] \\ \leq\ & (1+3\sigma^2)g_1^4 + (E[e_t^4] + 3\sigma^2)g_2^4 \\ \leq\ & (1+3\sigma^2)\sum_{i=1}^{p}\alpha_i z_i^4 + (E[e_t^4] + 3\sigma^2)\sum_{i=1}^{p}\delta_i z_i^4 + L(z), \end{aligned} \tag{2.22}$$

where $\alpha_i = \sum_{j=1}^{p}\beta_i\beta_j$, $\delta_i = \sum_{j=1}^{p} d_i d_j$, $\beta_i = \sum_{j=1}^{p}|a_i a_j|$, and $L(z) = (1+3\sigma^2)(4(\sum a_i z_i)^3 o(\|z\|) + 6(\sum a_i z_i)^2(o(\|z\|))^2 + 4(\sum a_i z_i)(o(\|z\|))^3 + (o(\|z\|))^4) + (E[e_t^4] + 3\sigma^2)(2(\sum d_i z_i^2)o(\|z\|^2) + (o(\|z\|^2))^2)$. Note that $\sum \alpha_i = (\sum |a_i|)^4$ and $\sum \delta_i = (\sum d_i)^2$.

Let $\eta_i = (1+3\sigma^2)\alpha_i + (E[e_t^4] + 3\sigma^2)\delta_i$ and $\delta = 1 - \sum \eta_i > 0$. Choose $\gamma_1 > 0$ arbitrary and $\gamma_{i+1} = \gamma_1(1 - \eta_1 - \cdots - \eta_i - i\delta/p)$ $(i = 1, \ldots, p-1)$. Now define

$$V_3(z) = \sum_{i=1}^{p}\gamma_i z_i^4 + 1.$$

Then we have

$$\begin{aligned} & E[V_3(Y_t)|Y_{t-1} = z] \\ \leq\ & \gamma_1 \sum_{i=1}^{p}[(1+3\sigma^2)\alpha_i + (E[e_t^4] + 3\sigma^2)\delta_i]z_i^4 + \sum_{i=2}^{p}\gamma_i z_{i-1}^4 + 1 + \gamma_1 L(z). \end{aligned}$$

From assumption,

$$\begin{aligned} & (1+3\sigma^2)\sum \alpha_i + (E[e_t^4] + 3\sigma^2)\sum \delta_i \\ & = (1+3\sigma^2)(\sum_{i=1}^{p}|a_i|)^4 + (E[e_t^4] + 3\sigma^2)(\sum_{i=1}^{p} d_i)^2 < 1. \end{aligned}$$

Since $L(z)/\|z\|^4 \to 0$ as $\|z\| \to \infty$, the remaining part of the proof is the same as that of Theorem 2.1 and details are omitted. □

EXAMPLE. Suppose that the assumptions I and III hold and e_t has the standard normal distribution. Then $(\sum |a_i|)^2 + \sum d_i < 1$ implies that Y_t is geometrically ergodic and $E_{\pi_y}[y_t^2] < \infty$. $4(\sum |a_i|)^4 + 6(\sum d_i)^2 < 1$ ensures the finiteness of the fourth-moment of y_t; $E_{\pi_y}[y_t^4] < \infty$.

REMARK 2. It is known that the geometric ergodicity of a Markov chain implies the absolute regularity with a geometric convergence rate, and absolute regularity is stronger than strong mixing. Therefore the geometric ergodic process holds limiting theorems for absolutely regular process and/or strong mixing process. On the other hand, due to Theorem 4.2 of Glynn and Meyn [7], we can conclude that under the assumption of Theorem 2.1, if $h^2 \leq V + c$ hold for some constant c, the functional central limit theorem holds for h.

References

[1] R. N. Bhattacharya and C. Lee, *On geometric ergodicity of nonlinear autoregressive models*, Statist. Probab. Lett. **22** (1995), 311–315.

[2] T. Bollerslev, R. T. Chou and K. F. Kroner, *Arch modeling in finance*, J. Econometrics **52** (1992), 5–59.

[3] P. Bougerol and N. Picard, *Strict stationarity of generalized autoregressive processes*, Ann. Prob. **20** (1992), 1714–1730.

[4] R. F. Engle, *Autoregressive conditional heteroscedasticity with estimates of the variance of the United Kingdom inflation*, Econometrica **50** (1982), 987–1007.

[5] C. Francq and J-M. Zakoïan, *Stationarity of multivariate Markov switching ARMA models*, J. Econometrics **102** (2001), 339–364.

[6] C. Francq, M. Roussignol, and J-M. Zakoïan, *Conditional heteroskedasticity driven by hidden Markov chains*, J. Time Ser. Anal. **22** (2001), no. 2, 197–220.

[7] P. W. Glynn and S. P. Meyn, *A Lyapounov bound for solutions of the Poisson equation*, Ann. Prob. **2** (1996), no. 2, 916–931.

[8] J. D. Hamilton, *A new approach to the economic analysis of nonstationary time series and business cycle*, Econometrica **57** (1989), no. 2, 357–384.

[9] ———, *Specification testing in Markov switching time series models.*, J. Econometrics **70** (1996), 127–157.

[10] O. Lee and J. Kim, *Strict stationarity and functional central limit theorem for ARCH/GARCH models*, Bull. Korean Math. Soc. **38** (2001), no. 3, 495–504.

[11] O. Lee, *Probabilistic properties of a nonlinear ARMA processes with Markov switching*, submitted for publication, 2002.

[12] ———, *Irreducibility of ARMA(p,q) process with Markov switching*, submitted for publication, 2002.

[13] ———, *On strict stationarity of nonlinear ARMA processes with nonlinear GARCH innovations*, submitted for publication, 2002.

[14] C. W. Li and W. K. Li, *On a double threshold autoregressive conditional heteroscedastic time series model*, J. Appl. Econometrics **11** (1996), 253–274.

[15] S. Ling, *On the probabilistic properties of a double threshold ARMA conditional heteroskedastic model*, J. Appl. Probab. **36** (1999), 688–705.

[16] S. Ling and M. McAleer, *Stationarity and existence of moments of a family of GARCH processes*, J. Econometrics **106** (2002), 109–117.

[17] J. Liu, W. Li and C. W. Li, *On a threshold autoregression with conditional heteroscedastic variances*, J. Statist. Plann. Inference **62** (1997), 279–300.

[18] E. Masry and D. Tjøstheim, *Nonparametric estimation and identification of nonlinear ARCH time series*, Econometric theory **11** (1995), 258–289.
[19] R. McCulloch and R. Tsay, *Statistical analysis of economic time series via Markov switching models*, J. Time Ser. Anal. **15** (1994), 523–539.
[20] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springer, London, 1993.
[21] D. Tjøstheim, *Nonlinear time series and Markov chains*, Adv. Appl. Probab. **22** (1990), 587–611.
[22] H. Tong, *Nonlinear Time Series: A dynamical system approach*, Oxford University Press, Oxford, 1990.
[23] R. L. Tweedie, *Invariant Measures for Markov chains with no irreducibility assumptions*, J. Appl. Probab. **25A** (1988), 275–285.
[24] M. Yang, *Some properties of vector auroregressive processes with Markov-switching coefficients*, Econometric Theory **16** (2000), 23–43.
[25] J. F. Yao, *On square integrability of an AR process with Markov switching*, Statist. Probab. Lett. **52** (2001), no. 3, 265–270.
[26] J. F. Yao and J. G. Attali, *On stationarity of nonlinear AR processes with Markov switching*, Adv. Appl. Probab. **32** (2000), 394–407.
[27] J. Zhang and R. Stine, *Autocovariance structure of Markov regime switching models and model selection*, J. Time Ser. Anal. **22** (2001), no. 1, 107–124.

Department of Statistics
Ewha Womans University
Seoul 120-750, Korea
E-mail: oslee@ewha.ac.kr
shindw@ewha.ac.kr