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CONDITIONAL FOURIER-FEYNMAN TRANSFORM AND CONVOLUTION PRODUCT OVER WIENER PATHS IN ABSTRACT WIENER SPACE : AN L_p THEORY

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Abstract. In this paper, using a simple formula, we evaluate the conditional Fourier-Feynman transforms and the conditional convolution products of cylinder type functions, and show that the conditional Fourier-Feynman transform of the conditional convolution product is expressed as a product of the conditional Fourier-Feynman transforms. Also, we evaluate the conditional Fourier-Feynman transforms of the functions of the forms

$$\exp\left\{\int_0^T \theta(s,x(s))ds\right\}, \qquad \exp\left\{\int_0^T \theta(s,x(s))ds\right\}\phi(x(T)),$$
$$\exp\left\{\int_0^T \theta(s,x(s))d\zeta(s)\right\}, \qquad \exp\left\{\int_0^T \theta(s,x(s))d\zeta(s)\right\}\phi(x(T))$$

which are of interest in Feynman integration theories and quantum mechanics.

1. Introduction and preliminaries

Let $C_0[0,T]$ denote the classical Wiener space, that is, the space of real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. The concept of conditional Wiener integral on Wiener space was introduced by Yeh in [18, 19]. By a conditional Wiener integral we mean the conditional expectation $E[F|X]$ of a real or complex-valued Wiener integrable function F conditioned by a Wiener measurable function X on $C_0[0,T]$,

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which is given as a function on the value space of X. We shall be concerned exclusively with X given by $X(x) = (x(t_1), \ldots, x(t_k))$, where $0 < t_1 < \cdots < t_k = T$ for any fixed positive integer k. In [14], Park and Skoug derived a simple formula for the conditional Wiener integral with the conditioning function X. They then used this formula to express conditional Wiener integrals directly in terms of ordinary Wiener integrals. In [15], Park and Skoug introduced the concept of a conditional Fourier-Feynman transform and the concept of a conditional convolution product and obtained several formulas relating these concepts; in particular see equations (3.13) and (4.11) in [15]. In [5], using ideas and formulas developed in [15], Chang and Skoug examined the effects that drift had on conditional Fourier-Feynman transforms and conditional convolution products. In section 4 of [5] and in section 4 of [15] the authors established various formulas involving conditional transforms and conditional convolution products for functions in the Banach algebra $\mathcal{S}$ which was introduced by Cameron and Storvick in [1].

In [12], Kuelbs and Lepage introduced $C_0(\mathbb{B})$, the space of continuous functions on $[0,T]$ into $\mathbb{B}$ which vanish at 0. In [16], Ryu established various properties involving $C_0(\mathbb{B})$. In [20], Yoo introduced the Banach algebra $\mathcal{S}''_{\mathbb{B}}$ which corresponds to Cameron and Storvick's space $\mathcal{S}''$ in [1]. Chang and his coworkers introduced the class $\mathcal{A}^{(p)}_{n,s}(1 \le p \le \infty)$ of cylinder type functions defined on $C_0(\mathbb{B})$, and then established relationships between the Fourier-Feynman transform and the convolution product of functions in $\mathcal{A}^{(p)}_{n,s}$([4]).

In [2], Chang and his coworkers defined the L_1 conditional Fourier-Feynman transform and the conditional convolution product on the space $C_0(\mathbb{B})$, and they investigate relationships between them.

In this paper, we define the L_p conditional Fourier-Feynman transform on $C_0(\mathbb{B})$ for $1 < p \le \infty$, and we evaluate the L_p conditional Fourier-Feynman transforms and the conditional convolution products of functions in $\mathcal{A}^{(p)}_{n,s}$, and then, we show that the conditional Fourier-Feynman transform of the conditional convolution product can be expressed as a product of the conditional Fourier-Feynman transforms for these functions. Also, for $1 \le p \le \infty$, we evaluate the L_p conditional Fourier-Feynman transforms of the functions of the forms

$$\exp\left\{\int_0^T \theta(s,x(s))ds\right\}, \qquad \exp\left\{\int_0^T \theta(s,x(s))ds\right\}\phi(x(T)),$$
$$\exp\left\{\int_0^T \theta(s,x(s))d\zeta(s)\right\}, \qquad \exp\left\{\int_0^T \theta(s,x(s))d\zeta(s)\right\}\phi(x(T))$$

which are of interest in Feynman integration theories and quantum mechanics.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, let B be a real normed linear space with norm $\|\cdot\|$ and let $\mathcal{B}(B)$ be the Borel σ-field on B. Let $X : (\Omega, \mathcal{A}, P) \to (B, \mathcal{B}(B))$ be a random variable and let $F : \Omega \to \mathbb{C}$ be an integrable function. Let P_X be the probability distribution of X on $(B, \mathcal{B}(B))$ and let $\mathcal{D}$ be the σ-field $\{X^{-1}(A) : A \in \mathcal{B}(B)\}$. Let $P_{\mathcal{D}}$ be the probability measure induced by P, that is, $P_{\mathcal{D}}(E) = P(E)$ for $E \in \mathcal{D}$. By the definition of conditional expectation there exists a $\mathcal{D}$-measurable function $E[F|X]$(the conditional expectation of F given X) defined on Ω such that the relation

$$\int_E E[F|X](\omega)\, dP_{\mathcal{D}}(\omega) = \int_E F(\omega)\, dP(\omega)$$

holds for every $E \in \mathcal{D}$. But there exists a P_X-integrable function ψ defined on B which is unique up to P_X-a.e. such that $E[F|X](\omega) = (\psi \circ X)(\omega)$ for $P_{\mathcal{D}}$-a.e. ω in Ω. ψ is also called the conditional expectation of F given X and without loss of generality, it is denoted by $E[F|X](\xi)$ for $\xi \in B$. Throughout this paper, we will consider the function ψ as the conditional expectation of F given X.

2. Wiener paths in abstract Wiener space

Let $(\mathcal{H}, \mathbb{B}, m)$ be an abstract Wiener space([13]). Let $\{e_j : j \geq 1\}$ be a complete orthonormal set in the real separable Hilbert space $\mathcal{H}$ such that e_j's are in $\mathbb{B}^*$, the dual space of real separable Banach space $\mathbb{B}$. For each $h \in \mathcal{H}$ and $y \in \mathbb{B}$, define the stochastic inner product $(h, y)^\sim$ by

$$(h, y)^\sim = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^n \langle h, e_j \rangle (y, e_j), & \text{if the limit exists;} \\ 0, & \text{otherwise,} \end{cases}$$

where $(\cdot, \cdot)$ denotes the dual pairing between $\mathbb{B}$ and $\mathbb{B}^*$([11]). Note that for each $h(\neq 0)$ in $\mathcal{H}$, $(h, \cdot)^\sim$ is a Gaussian random variable on $\mathbb{B}$ with mean zero, variance $|h|^2$; also $(h, y)^\sim$ is essentially independent of the choice of the complete orthonormal set used in its definition and further, $(h, \lambda y)^\sim = (\lambda h, y)^\sim = \lambda (h, y)^\sim$ for all $\lambda \in \mathbb{R}$. It is well-known that if $\{h_1, h_2, \ldots, h_n\}$ is an orthogonal set in $\mathcal{H}$, then the random variables $(h_j, \cdot)^\sim$'s are independent. Moreover, if both h and y are in $\mathcal{H}$, then $(h, y)^\sim = \langle h, y \rangle$.

Let $C_0(\mathbb{B})$ denote the set of all continuous functions on $[0, T]$ into $\mathbb{B}$ which vanish at 0. Then $C_0(\mathbb{B})$ is a real separable Banach space with the norm $\|x\|_{C_0(\mathbb{B})} \equiv \sup_{0\leq t\leq T} \|x(t)\|_{\mathbb{B}}$. The minimal σ-field making the mapping $x \to x(t)$ measurable is $\mathcal{B}(C_0(\mathbb{B}))$, the Borel σ-field on $C_0(\mathbb{B})$. Further, Brownian motion in $\mathbb{B}$ induces a probability measure $m_{\mathbb{B}}$ on $(C_0(\mathbb{B}), \mathcal{B}(C_0(\mathbb{B})))$ which is mean-zero Gaussian. We will find a concrete form of $m_{\mathbb{B}}$. Let $\vec{t} = (t_1, t_2, \ldots, t_k)$ be given with $0 = t_0 < t_1 < t_2 < \cdots < t_k \le T$. Let $T_{\vec{t}} : \mathbb{B}^k \to \mathbb{B}^k$ be given by

$$\begin{aligned} &T_{\vec{t}}(x_1, x_2, \ldots, x_k) \\ &= \left((t_1 - t_0)^{\frac{1}{2}} x_1, (t_1 - t_0)^{\frac{1}{2}} x_1 + (t_2 - t_1)^{\frac{1}{2}} x_2, \ldots, \sum_{j=1}^{k} (t_j - t_{j-1})^{\frac{1}{2}} x_j \right) \end{aligned}$$

We define a set function $\nu_{\vec{t}}$ on $\mathcal{B}(\mathbb{B}^k)$ by

$$\nu_{\vec{t}}(B) = \left(\prod_{1}^{k} m \right) \left(T_{\vec{t}}^{-1}(B) \right)$$

for $B \in \mathcal{B}(\mathbb{B}^k)$. Then $\nu_{\vec{t}}$ is a Borel measure. Let $f_{\vec{t}} : C_0(\mathbb{B}) \to \mathbb{B}^k$ be the function defined by

$$f_{\vec{t}}(x) = (x(t_1), x(t_2), \ldots, x(t_k)).$$

For Borel subsets $B_1, B_2, \ldots, B_k$ of $\mathbb{B}$, $f_{\vec{t}}^{-1}(\prod_{j=1}^{k} B_j)$ is called the I-set with respect to $B_1, B_2, \ldots, B_k$. Then the collection $\mathcal{I}$ of all I-sets is a semi-algebra. We define a set function $m_{\mathbb{B}}$ on $\mathcal{I}$ by

$$m_{\mathbb{B}} \left(f_{\vec{t}}^{-1} \left(\prod_{j=1}^{k} B_j \right) \right) = \nu_{\vec{t}} \left(\prod_{j=1}^{k} B_j \right)$$

Then $m_{\mathbb{B}}$ is well-defined and countably additive on $\mathcal{I}$. Using Carathéodory extension process, we have a Borel measure $m_{\mathbb{B}}$ on $\mathcal{B}(C_0(\mathbb{B}))$.

Now, we introduce Wiener integration theorem without proof. For the proof see [16].

THEOREM 1 (Wiener integration theorem). *Let $\vec{t} = (t_1, t_2, \ldots, t_k)$ be given with $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_k \le T$ and let $f : \mathbb{B}^k \to \mathbb{C}$ be a Borel measurable function. Then*

$$\begin{aligned} &\int_{C_0(\mathbb{B})} f(x(t_1), x(t_2), \ldots, x(t_k)) \, dm_{\mathbb{B}}(x) \\ &\stackrel{*}{=} \int_{\mathbb{B}^k} (f \circ T_{\vec{t}})(x_1, x_2, \ldots, x_k) \, d \left(\prod_{j=1}^{k} m \right) (x_1, x_2, \ldots, x_k), \end{aligned}$$

where by $\stackrel{*}{=}$ we mean that if either side exists, then both sides exist and they are equal.

Let $\tau : 0 = t_0 < t_1 < \cdots < t_k = T$ be a partition of $[0,T]$ and let x be in $C_0(\mathbb{B})$. Define the polygonal function $[x]$ of x on $[0,T]$ by

$$(1) \qquad [x](t) = \sum_{j=1}^{k} \chi_{(t_{j-1},t_j]}(t) \left[x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}} (x(t_j) - x(t_{j-1})) \right],$$

where $t \in [0,T]$. For each $\vec{\xi} = (\xi_1, \ldots, \xi_k) \in \mathbb{B}^k$, let $[\vec{\xi}]$ be the polygonal function of $\vec{\xi}$ on $[0,T]$ given by (1) with replacing $x(t_j)$ by ξ_j for $j = 0, 1, \ldots, k$ ($\xi_0 = 0$).

The following lemmas are useful to the next sections. For the detailed proof, see [3].

LEMMA 2. *If $\{x(t) : 0 \le t \le T\}$ is the Wiener process on $C_0(\mathbb{B}) \times [0,T]$, then $\{x(t) - [x](t) : t_{j-1} \le t \le t_j\}$, where $j = 1, \ldots, k$, are stochastically independent.*

LEMMA 3. *Let F be defined and integrable on $C_0(\mathbb{B})$. Let $X_\tau : C_0(\mathbb{B}) \to \mathbb{B}^k$ be a random variable given by $X_\tau(x) = (x(t_1), \ldots, x(t_k))$. Then for every Borel measurable subset B of $\mathbb{B}^k$,*

$$\mu_\tau(B) \equiv \int_{X_\tau^{-1}(B)} F(x)\, dm_{\mathbb{B}}(x) = \int_B E[F(x - [x] + [\vec{\xi}])]\, dP_{X_\tau}(\vec{\xi}),$$

where P_{X_τ} is the probability distribution of X_τ on $(\mathbb{B}^k, \mathcal{B}(\mathbb{B}^k))$.

By the definition of conditional expectation and Lemma 3, we have

$$(2) \qquad E[F|X_\tau](\vec{\xi}) = E[F(x - [x] + [\vec{\xi}])] \quad \text{for } P_{X_\tau}\text{-a.e. } \vec{\xi}.$$

The function $E[F|X_\tau]$ is called the conditional Wiener integral of F given X_τ. Note that, for the definition of the conditional Wiener integral, X_τ may be any random variable defined on $C_0(\mathbb{B})$. The equation (2) is called a simple formula for conditional Wiener integral on the space $C_0(\mathbb{B})$.

For $\lambda > 0$ and $\vec{\xi} \in \mathbb{B}^k$, suppose $E[F(\lambda^{-\frac{1}{2}}\cdot)|X_\tau(\lambda^{-\frac{1}{2}}\cdot)](\vec{\xi})$ exists. From (2) we have

$$E[F(\lambda^{-\frac{1}{2}}\cdot)|X_\tau(\lambda^{-\frac{1}{2}}\cdot)](\vec{\xi}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])]$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$. If, for $\vec{\xi} \in \mathbb{B}^k$, $E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])]$ has the analytic extension $J_\lambda(\vec{\xi})$ on $\mathbb{C}_+ \equiv \{\lambda \in \mathbb{C} : \text{Re } \lambda > 0\}$, then we write

$$J_\lambda(\vec{\xi}) = E^{\text{anw}_\lambda}[F|X_\tau](\vec{\xi})$$

for $\lambda \in \mathbb{C}_+$. In this case, we call $J_\lambda(\vec{\xi})$ a version of conditional analytic Wiener integral of F given X_τ.

For non-zero real q and $\vec{\xi} \in \mathbb{B}^k$, if the limit

$$\lim_{\lambda \to -iq} E^{\mathrm{anw}_\lambda}[F|X_\tau](\vec{\xi})$$

exists, where λ approaches to $-iq$ through $\mathbb{C}_+$, then we write

$$\lim_{\lambda \to -iq} E^{\mathrm{anw}_\lambda}[F|X_\tau](\vec{\xi}) = E^{\mathrm{anf}_q}[F|X_\tau](\vec{\xi}).$$

In this case, we call $E^{\mathrm{anf}_q}[F|X_\tau](\vec{\xi})$ a version of conditional analytic Feynman integral of F given X_τ.

3. Conditional Fourier-Feynman transform of cylinder type functions

In this section, we define $L_p (1 < p \leq \infty)$ conditional Fourier-Feynman transform over Wiener paths in abstract Wiener space and evaluate them for cylinder type functions. We also evaluate conditional convolution products for these type of functions.

A subset E of $C_0(\mathbb{B})$ is called a scale-invariant null set if $m_{\mathbb{B}}(\lambda E) = 0$ for any $\lambda > 0$ and a property is said to hold s-a.e. if it holds except for a scale-invariant null set.

For a given real number p with $1 < p \leq \infty$, suppose p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$ (possibly $p' = 1$ if $p = \infty$). Let G_n and G be measurable functions such that, for each $\gamma > 0$,

$$\lim_{n \to \infty} \int_{C_0(\mathbb{B})} |G_n(\gamma x) - G(\gamma x)|^{p'} dm_{\mathbb{B}}(x) = 0.$$

Then we write

$$\underset{n \to \infty}{\mathrm{l.i.m.}} (w_s^{p'})(G_n) \approx G$$

and call G the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by a continuously varying parameter.

Let $\mathcal{H}$ be a real separable infinite dimensional Hilbert space, let n be a positive integer and let $\{h_1, \ldots, h_n\}$ be an orthonormal set in $\mathcal{H}$. For $1 \leq p < \infty$ let $\mathcal{A}_{n,s}^{(p)}$ be the space of all cylinder type functions F defined on $C_0(\mathbb{B})$ of the form

$$F(x) = f((h_1, x(s))^\sim, \ldots, (h_n, x(s))^\sim) \tag{3}$$

for s-a.e. x in $C_0(\mathbb{B})$ where $f : \mathbb{R}^n \to \mathbb{R}$ is in $L_p(\mathbb{R}^n)$ and $s \in (0, T]$. Let $\mathcal{A}_{n,s}^{(0)}$ be the space of all functions of the form (3) with $f \in C_0(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$ which vanish at infinity and let $\mathcal{A}_{n,s}^{(\infty)}$ be the space of all functions of the form (3) with $f \in L_\infty(\mathbb{R}^n)$, the space of essentially bounded functions on $\mathbb{R}^n$. Note that, without loss of generality, we can take f to be Borel measurable.

For convenience, we let

$$\Gamma = \frac{t_{p^*} - t_{p^*-1}}{(t_{p^*} - s)(s - t_{p^*-1})} \tag{4}$$

for $t_{p^*-1} < s < t_{p^*}$ and

$$\phi_\lambda(\vec{u}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{n} u_j^2\right\} \quad (\lambda^{\frac{1}{2}} \in \mathbb{C}_+) \tag{5}$$

where $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\lambda \in \tilde{\mathbb{C}}_+ \equiv \{\lambda \in \mathbb{C} : \text{Re}\,\lambda \geq 0\} - \{0\}$. Also, we let for $\vec{\xi} \in \mathbb{B}^k$

$$\vec{w}_{\vec{\xi}} = (w_{\vec{\xi}1}, \ldots, w_{\vec{\xi}n}) = ((h_1, [\vec{\xi}](s))^\sim, \ldots, (h_n, [\vec{\xi}](s))^\sim) \tag{6}$$

and

$$\vec{w}_y = (w_{y1}, \ldots, w_{yn}) = ((h_1, y(s))^\sim, \ldots, (h_n, y(s))^\sim) \tag{7}$$

for y in $C_0(\mathbb{B})$.

Definition 4. Let F be defined on $C_0(\mathbb{B})$ and let X_τ be given as in Lemma 3. For $\lambda \in \mathbb{C}_+$ and for s-a.e. $\vec{\xi} \in \mathbb{B}^k$ let

$$T_\lambda[F|X_\tau](y, \vec{\xi}) = E^{\text{anw}_\lambda}[F(y + \cdot)|X_\tau](\vec{\xi})$$

for s-a.e. $y \in C_0(\mathbb{B})$ if it exists. For non-zero real q and for s-a.e. $\vec{\xi} \in \mathbb{B}^k$, we define the L_1 conditional Fourier-Feynman transform $T_q^{(1)}[F|X_\tau]$ of F given X_τ by the formula

$$T_q^{(1)}[F|X_\tau](y, \vec{\xi}) = \lim_{\lambda \to -iq} T_\lambda[F|X_\tau](y, \vec{\xi})$$

if it exists for s-a.e. $y \in C_0(\mathbb{B})$ and for $1 < p \leq \infty$ we define the L_p conditional Fourier-Feynman transform $T_q^{(p)}[F|X_\tau]$ of F given X_τ by the formula

$$T_q^{(p)}[F|X_\tau](\cdot, \vec{\xi}) \approx \underset{\lambda \to -iq}{\text{l.i.m.}} (w_s^{p'})(T_\lambda[F|X_\tau](\cdot, \vec{\xi})),$$

where λ approaches to $-iq$ through $\mathbb{C}_+$.

THEOREM 5. *Let* $F \in \mathcal{A}_{n,s}^{(p)}(1 \le p \le \infty)$ *be given by* (3). *Let* X_τ *be given as in Lemma 3 and let* $t_{p^*-1} < s \le t_{p^*}$ *for some* $p^* \in \{1, \dots, k\}$. *Then, for any* $\lambda \in \mathbb{C}_+$ *and for s-a.e.* $\vec{\xi} \in \mathbb{B}^k$, $T_\lambda[F|X_\tau](y, \vec{\xi})$ *exists for s-a.e.* $y \in C_0(\mathbb{B})$ *and* $T_\lambda[F|X_\tau](\cdot, \vec{\xi}) \in \mathcal{A}_{n,s}^{(p)}$. *Moreover, when* $t_{p^*-1} < s < t_{p^*}$, *we have*

$$T_\lambda[F|X_\tau](y, \vec{\xi}) = (\phi_{\lambda\Gamma} * f)(\vec{w}_y + \vec{w}_{\vec{\xi}}) \tag{8}$$

and, when $s = t_{p^*}$, *we have*

$$T_\lambda[F|X_\tau](y, \vec{\xi}) = F(y + [\vec{\xi}]) = f(\vec{w}_y + \vec{w}_{\vec{\xi}}), \tag{9}$$

where Γ, $\phi_{\lambda\Gamma}$, $\vec{w}_{\vec{\xi}}$ *and* $\vec{w}_y$ *are given by* (4), (5), (6) *and* (7), *respectively.*

Proof. Let $t_{p^*-1} < s < t_{p^*}$ and let $\lambda > 0$. Using the same method in the proof of Theorem 4.2 in [2], we have

$$E[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])] = (\phi_{\lambda\Gamma} * f)(\vec{w}_y + \vec{w}_{\vec{\xi}})$$

for s-a.e. $\vec{\xi} \in \mathbb{B}^k$ and for s-a.e. $y \in C_0(\mathbb{B})$. By Morera's theorem, we have (8). The fact $T_\lambda[F|X_\tau](\cdot, \vec{\xi}) \in \mathcal{A}_{n,s}^{(p)}$ follows from Young's inequality in [7, p.232].

When $s = t_{p^*}$, the equation (9) follows easily since $E[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])] = F(y + [\vec{\xi}])$ for $\lambda > 0$. □

THEOREM 6. *For* $1 \le p \le 2$ *let* $F \in \mathcal{A}_{n,s}^{(p)}$ *be given by* (3) *and let* $\frac{1}{p} + \frac{1}{p'} = 1 (p' = \infty$ *if* $p = 1)$. *Let* X_τ *be given as in Lemma 3 and let* $t_{p^*-1} < s \le t_{p^*}$ *for some* $p^* \in \{1, \dots, k\}$. *Then, for any non-zero real* q, *and for s-a.e.* $\vec{\xi} \in \mathbb{B}^k$, $T_q^{(p)}[F|X_\tau](y, \vec{\xi})$ *exists for s-a.e.* $y \in C_0(\mathbb{B})$. *Moreover, when* $t_{p^*-1} < s < t_{p^*}$, *we have*

$$T_q^{(p)}[F|X_\tau](y, \vec{\xi}) = (\phi_{-iq\Gamma} * f)(\vec{w}_y + \vec{w}_{\vec{\xi}})$$

with $T_q^{(p)}[F|X_\tau](\cdot, \vec{\xi}) \in \mathcal{A}_{n,s}^{(p')}$ *(in fact,* $T_q^{(1)}[F|X_\tau](\cdot, \vec{\xi}) \in \mathcal{A}_{n,s}^{(0)}$*) and, when* $s = t_{p^*}$, *we have*

$$T_q^{(p)}[F|X_\tau](y, \vec{\xi}) = F(y + [\vec{\xi}]) = f(\vec{w}_y + \vec{w}_{\vec{\xi}}) \tag{10}$$

with $T_q^{(p)}[F|X_\tau](\cdot, \vec{\xi}) \in \mathcal{A}_{n,s}^{(p)}$, *where* Γ, $\phi_{-iq\Gamma}$, $\vec{w}_{\vec{\xi}}$ *and* $\vec{w}_y$ *are given by* (4), (5), (6) *and* (7), *respectively.*

Proof. When $p = 1$, the result follows from Theorem 4.2 in [2].

Let $1 < p \le 2$, $t_{p^*-1} < s < t_{p^*}$. Then, for $\gamma > 0$, for s-a.e. $\vec{\xi}$ in $\mathbb{B}^k$ and by Theorems 1, 5, we have

$$\begin{aligned} &\lim_{\lambda \to -iq} \int_{C_0(\mathbb{B})} |(\phi_{\lambda\Gamma} * f)(\gamma \vec{w}_y + \vec{w}_{\vec{\xi}}) - (\phi_{-iq\Gamma} * f)(\gamma \vec{w}_y + \vec{w}_{\vec{\xi}})|^{p'} \\ &dm_{\mathbb{B}}(y) \\ = &\lim_{\lambda \to -iq} \int_{\mathbb{B}} |(\phi_{\lambda\Gamma} * f)(\gamma\sqrt{s}((h_1, x_1)^\sim, \ldots, (h_n, x_1)^\sim) + \vec{w}_{\vec{\xi}}) \\ &-(\phi_{-iq\Gamma} * f)(\gamma\sqrt{s}((h_1, x_1)^\sim, \ldots, (h_n, x_1)^\sim) + \vec{w}_{\vec{\xi}})|^{p'} dm(x_1) \\ \le &\lim_{\lambda \to -iq} \left(\frac{1}{2\pi s\gamma^2}\right)^{\frac{n}{2}} \|(\phi_{\lambda\Gamma} * f)(\cdot + \vec{w}_{\vec{\xi}}) - (\phi_{-iq\Gamma} * f)(\cdot + \vec{w}_{\vec{\xi}})\|_{p'}^{p'} \end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$, by Lemmas 1.1, 1.2 in [10] and since $(h_j, \cdot)^\sim$ is normally distributed with mean 0, variance 1.

When $s = t_{p^*}$, the result follows trivially by Theorem 5. □

REMARK 1. (10) holds for $1 \le p \le \infty$.

Next we establish an inverse conditional Fourier-Feynman transform theorem for functions in $\mathcal{A}_{n,s}^{(p)}$.

THEOREM 7. *For $1 \le p < \infty$ let $F \in \mathcal{A}_{n,s}^{(p)}$ be given by (3) and let X_τ be given as in Lemma 3. Let q be a non-zero real number and let $\vec{w}_y$ be given by (7). For $(\vec{\xi}_1, \vec{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k$ and for $y \in C_0(\mathbb{B})$, let $F_{\vec{\xi}_1,\vec{\xi}_2}(y) = f(\vec{w}_y + \vec{w}_1 + \vec{w}_2)$, where $\vec{w}_1 = (w_{11}, \ldots, w_{1n}) = ((h_1, [\vec{\xi}_1](s))^\sim, \ldots, (h_n, [\vec{\xi}_1](s))^\sim)$, $\vec{w}_2 = (w_{21}, \ldots, w_{2n}) = ((h_1, [\vec{\xi}_2](s))^\sim, \ldots, (h_n, [\vec{\xi}_2](s))^\sim)$. Then we have*

(i) *for s-a.e. $(\vec{\xi}_1, \vec{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k$ and for $\gamma > 0$, we have*

$$\lim_{\lambda \to -iq} \int_{C_0(\mathbb{B})} |T_{\bar{\lambda}}[T_\lambda[F|X_\tau](\cdot, \vec{\xi}_1)|X_\tau](\gamma y, \vec{\xi}_2) - F_{\vec{\xi}_1,\vec{\xi}_2}(\gamma y)|^p dm_{\mathbb{B}}(y) = 0$$

and

(ii) *for s-a.e. $(\vec{\xi}_1, \vec{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k$ and for s-a.e. $y \in C_0(\mathbb{B})$, we have*

$$T_{\bar{\lambda}}[T_\lambda[F|X_\tau](\cdot, \vec{\xi}_1)|X_\tau](y, \vec{\xi}_2) \longrightarrow F_{\vec{\xi}_1,\vec{\xi}_2}(y),$$

where λ approaches to $-iq$ through $\mathbb{C}_+$.

Proof. Let $t_{p^*-1} < s < t_{p^*}$ for some $p^* \in \{1, \ldots, k\}$ and let $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$. Then for $\lambda \in \mathbb{C}_+$, for s-a.e. $(\vec{\xi}_1, \vec{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k$ and for s-a.e. $y \in C_0(\mathbb{B})$, using the same method in the proof of Theorem 4.4 in [2], we have

$$\begin{aligned} &T_{\bar{\lambda}}[T_\lambda[F|X_\tau](\cdot,\vec{\xi}_1)|X_\tau](y,\vec{\xi}_2) \\ &= \left(\frac{|\lambda|^2\Gamma}{4\pi\mathrm{Re}\lambda}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u})\exp\left\{-\frac{|\lambda|^2\Gamma}{4\mathrm{Re}\lambda}\sum_{j=1}^n (u_j - w_{yj} - w_{1j} - w_{2j})^2\right\} d\vec{u} \end{aligned}$$

where Γ is given by (4). Let $\gamma > 0$ and for $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ let

$$\begin{aligned} &k_{\vec{\xi}_1,\vec{\xi}_2}(\lambda,\vec{v}) \\ &= \left(\frac{|\lambda|^2\Gamma}{4\pi\mathrm{Re}\lambda}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u})\exp\left\{-\frac{|\lambda|^2\Gamma}{4\mathrm{Re}\lambda}\sum_{j=1}^n (u_j - v_j - w_{1j} - w_{2j})^2\right\} d\vec{u}. \end{aligned}$$

Then, by Wiener integration theorem(Theorem 1) and the change of variable theorem, we have

$$\begin{aligned} &\int_{C_0(\mathbb{B})} |T_{\bar{\lambda}}[T_\lambda[F|X_\tau](\cdot,\vec{\xi}_1)|X_\tau](\gamma y,\vec{\xi}_2) - F_{\vec{\xi}_1,\vec{\xi}_2}(\gamma y)|^p dm_{\mathbb{B}}(y) \\ = &\int_{\mathbb{B}} |k_{\vec{\xi}_1,\vec{\xi}_2}(\lambda,\gamma\sqrt{s}((h_1,x_1)^\sim,\ldots,(h_n,x_1)^\sim)) \\ &-f(\gamma\sqrt{s}((h_1,x_1)^\sim,\ldots,(h_n,x_1)^\sim) + \vec{w}_1 + \vec{w}_2)|^p dm(x_1) \\ = &\left(\frac{1}{2\pi s\gamma^2}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} |k_{\vec{\xi}_1,\vec{\xi}_2}(\lambda,\vec{z}) - f(\vec{z} + \vec{w}_1 + \vec{w}_2)|^p \\ &\times \exp\left\{-\frac{1}{2s\gamma^2}\sum_{j=1}^n z_j^2\right\} d\vec{z} \end{aligned}$$

where $\vec{z} = (z_1, \ldots, z_n)$, since $(h_j, \cdot)^\sim$ is normally distributed with mean 0 and variance 1. Let $\epsilon = (\frac{2\mathrm{Re}\lambda}{|\lambda|^2\,\Gamma})^{\frac{1}{2}}$ and let $\varphi_\epsilon(\vec{u}) = \frac{1}{\epsilon^n}\phi_1(\frac{\vec{u}}{\epsilon})$, where ϕ_1 is given by (5). Then, $\int_{\mathbb{R}^n} \phi_1(\vec{u})d\vec{u} = 1$ and hence in view of Theorem 1.18 in [17], we have

$$\begin{aligned} &\lim_{\lambda\to -iq} \int_{C_0(\mathbb{B})} |T_{\bar{\lambda}}[T_\lambda[F|X_\tau](\cdot,\vec{\xi}_1)|X_\tau](\gamma y,\vec{\xi}_2) - F_{\vec{\xi}_1,\vec{\xi}_2}(\gamma y)|^p dm_{\mathbb{B}}(y) \\ \le &\lim_{\lambda\to -iq} \left(\frac{1}{2\pi s\gamma^2}\right)^{\frac{n}{2}} \|(f * \varphi_\epsilon)(\cdot + \vec{w}_1 + \vec{w}_2) - f(\cdot + \vec{w}_1 + \vec{w}_2)\|_p^p \\ = &\ 0 \end{aligned}$$

where λ approaches to $-iq$ through $\mathbb{C}_+$. This proves (i). Also, (ii) follows from Theorem 1.25 in [17].

Suppose that $s = t_{p^*}$ for some $p^* \in \{1, \dots, k\}$ and let $\gamma > 0$. Then, for $\lambda \in \mathbb{C}_+$, for s-a.e. $(\vec{\xi_1}, \vec{\xi_2}) \in \mathbb{B}^k \times \mathbb{B}^k$ and for s-a.e. $y \in C_0(\mathbb{B})$, we have

$$T_{\bar{\lambda}}[T_\lambda[F|X_\tau](\cdot, \vec{\xi_1})|X_\tau](\gamma y, \vec{\xi_2}) = F_{\vec{\xi_1}, \vec{\xi_2}}(\gamma y)$$

by Theorem 5. Then (i) follows trivially and (ii) follows when $\gamma = 1$. □

REMARK 2. (ii) of Theorem 7 holds for $p = \infty$.

4. Conditional convolution product and conditional transformation of conditional convolution product

Now we define conditional convolution product and investigate relationships between conditional convolution product and conditional Fourier Feynman transform.

DEFINITION 8. Let X_τ be given as in Lemma 3 and let F, G be defined on $C_0(\mathbb{B})$. We define the conditional convolution product $[(F * G)_\lambda|X_\tau]$ of F, G given X_τ by the formula, for s-a.e. $\vec{\xi} \in \mathbb{B}^k$

$$[(F * G)_\lambda|X_\tau](y, \vec{\xi}) = \begin{cases} E^{\mathrm{anw}_\lambda}\left[F\left(\dfrac{y+\cdot}{2^{\frac{1}{2}}}\right)G\left(\dfrac{y-\cdot}{2^{\frac{1}{2}}}\right)\middle|X_\tau\right](\vec{\xi}), & \lambda \in \mathbb{C}_+; \\ E^{\mathrm{anf}_q}\left[F\left(\dfrac{y+\cdot}{2^{\frac{1}{2}}}\right)G\left(\dfrac{y-\cdot}{2^{\frac{1}{2}}}\right)\middle|X_\tau\right](\vec{\xi}), & \lambda = -iq, \quad q \in \mathbb{R} - \{0\} \end{cases}$$

if they exist for s-a.e. $y \in C_0(\mathbb{B})$.

Using similar method in the proof Theorem 4.5 in [2], we have the following theorem.

THEOREM 9. *Let $F_1 \in \mathcal{A}_{n,s}^{(p_1)}$ and $F_2 \in \mathcal{A}_{n,s}^{(p_2)}$ be given by (3) with replacing f by f_1 and f_2, respectively, and let $\frac{1}{p_1} + \frac{1}{p_1'} = 1 (1 \le p_1, p_2 \le \infty)$. Let X_τ be given as in Lemma 3 and let $t_{p^*-1} < s \le t_{p^*}$ for some $p^* \in \{1, \dots, k\}$. Then, for λ in $\mathbb{C}_+$ and for s-a.e. $\vec{\xi} \in \mathbb{B}^k$, $[(F_1 * F_2)_\lambda|X_\tau](y, \vec{\xi})$ exists for s-a.e. $y \in C_0(\mathbb{B})$. Moreover, when $t_{p^*-1} < s < t_{p^*}$,* we have

$$[(F_1 * F_2)_\lambda|X_\tau](y, \vec{\xi}) = \int_{\mathbb{R}^n} f_1\left(\frac{\vec{w}_y + \vec{w}_{\vec{\xi}} + \vec{l}}{2^{\frac{1}{2}}}\right) f_2\left(\frac{\vec{w}_y - \vec{w}_{\vec{\xi}} - \vec{l}}{2^{\frac{1}{2}}}\right) \phi_{\lambda\Gamma}(\vec{l}) d\vec{l}$$

with $[(F_1*F_2)_\lambda|X_\tau](\cdot,\vec{\xi}) \in \mathcal{A}_{n,s}^{(1)}$ *if* $p_2 \le p_1'$ *and* $[(F_1*F_2)_\lambda|X_\tau](\cdot,\vec{\xi}) \in \mathcal{A}_{n,s}^{(p_2)}$ *if* $p_2 \ge p_1'$, *where* Γ, $\phi_{\lambda\Gamma}$, $\vec{w}_{\vec{\xi}}$ *and* $\vec{w}_y$ *are given by* (4), (5), (6) *and* (7), *respectively, and when* $s = t_{p^*}$, *we have*

$$[(F_1 * F_2)_\lambda|X_\tau](y,\vec{\xi}) = \left[F_1\left(\frac{1}{2^{\frac{1}{2}}}(y+[\vec{\xi}])\right)\right]\left[F_2\left(\frac{1}{2^{\frac{1}{2}}}(y-[\vec{\xi}])\right)\right].$$

The following corollary follows from Theorem 9 immediately.

COROLLARY 10. *Let* $F_1, F_2 \in \cup_{1\le p\le\infty}\mathcal{A}_{n,s}^{(p)}$. *Let* X_τ *be given as in Lemma 3 and let* $s = t_{p^*}$ *for some* $p^* \in \{1,\dots,k\}$. *Then, for* λ *in* $\mathbb{C}_+^\sim$ *and for s-a.e.* $\vec{\xi} \in \mathbb{B}^k$, $[(F_1 * F_2)_\lambda|X_\tau](y,\vec{\xi})$ *exists for s-a.e.* $y \in C_0(\mathbb{B})$ *and it is given by*

$$[(F_1 * F_2)_\lambda|X_\tau](y,\vec{\xi}) = \left[F_1\left(\frac{1}{2^{\frac{1}{2}}}(y+[\vec{\xi}])\right)\right]\left[F_2\left(\frac{1}{2^{\frac{1}{2}}}(y-[\vec{\xi}])\right)\right].$$

THEOREM 11. *Let* X_τ *be given as in Lemma 3 and let* $t_{p^*-1} < s < t_{p^*}$ *for some* $p^* \in \{1,\dots,k\}$. *For s-a.e.* $\vec{\xi}$ *in* $\mathbb{B}^k$ *and for* λ *in* $\mathbb{C}_+^\sim$, *we have the followings*:

1. *if* $F_1, F_2 \in \mathcal{A}_{n,s}^{(1)}$, *then* $[(F_1 * F_2)_\lambda|X_\tau](\cdot,\vec{\xi}) \in \mathcal{A}_{n,s}^{(1)}$,
2. *if* $F_1, F_2 \in \mathcal{A}_{n,s}^{(2)}$, *then* $[(F_1 * F_2)_\lambda|X_\tau](\cdot,\vec{\xi}) \in \mathcal{A}_{n,s}^{(0)}$,
3. *if* $F_1 \in \mathcal{A}_{n,s}^{(1)}$ *and* $F_2 \in \mathcal{A}_{n,s}^{(2)}$, *then* $[(F_1 * F_2)_\lambda|X_\tau](\cdot,\vec{\xi}) \in \mathcal{A}_{n,s}^{(2)}$,
4. *if* $F_1 \in \mathcal{A}_{n,s}^{(1)}$ *and* $F_2 \in \mathcal{A}_{n,s}^{(1)} \cap \mathcal{A}_{n,s}^{(2)}$, *then* $[(F_1 * F_2)_\lambda|X_\tau](\cdot,\vec{\xi}) \in \mathcal{A}_{n,s}^{(1)} \cap \mathcal{A}_{n,s}^{(2)}$,
5. *if* $F_1 \in \mathcal{A}_{n,s}^{(1)}$ *and* $F_2 \in \mathcal{A}_{n,s}^{(0)}$, *then* $[(F_1 * F_2)_\lambda|X_\tau](\cdot,\vec{\xi}) \in \mathcal{A}_{n,s}^{(0)}$.

Proof. Let F_1, F_2 be given by (3) with replacing f by f_1, f_2, respectively. For λ in $\mathbb{C}_+$ let

$$h(\lambda,\vec{u},\vec{w}_{\vec{\xi}}) = \int_{\mathbb{R}^n} f_1\left(\frac{\vec{u}+\vec{w}_{\vec{\xi}}+\vec{l}}{2^{\frac{1}{2}}}\right) f_2\left(\frac{\vec{u}-\vec{w}_{\vec{\xi}}-\vec{l}}{2^{\frac{1}{2}}}\right)\phi_{\lambda\Gamma}(\vec{l})d\vec{l}$$

where $\vec{u}$ in $\mathbb{R}^n$ and Γ, $\phi_{\lambda\Gamma}$, $\vec{w}_{\vec{\xi}}$ are given by (4), (5), (6), respectively.

1. By Theorem 9, it suffices to show that $h(\lambda,\cdot,\vec{w}_{\vec{\xi}})$ is in $L_1(\mathbb{R}^n)$ for $\lambda \in \mathbb{C}_+^\sim$. But this fact follows from the following;

$$\begin{aligned}&\int_{\mathbb{R}^n} |h(\lambda,\vec{u},\vec{w}_{\vec{\xi}})|d\vec{u}\\ \le\ & \left(\frac{|\lambda|\Gamma}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|f_1\left(\frac{\vec{u}+\vec{w}_{\vec{\xi}}+\vec{l}}{2^{\frac{1}{2}}}\right) f_2\left(\frac{\vec{u}-\vec{w}_{\vec{\xi}}-\vec{l}}{2^{\frac{1}{2}}}\right)\right| d\vec{l}d\vec{u}\end{aligned}$$

$$= \left(\frac{|\lambda|\Gamma}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left| f_1\left(\vec{v}_1 + \frac{\vec{w}_{\vec{\xi}}}{2^{\frac{1}{2}}}\right) f_2\left(\vec{v}_2 - \frac{\vec{w}_{\vec{\xi}}}{2^{\frac{1}{2}}}\right)\right| d\vec{v}_1 d\vec{v}_2$$
$$= \left(\frac{|\lambda|\Gamma}{2\pi}\right)^{\frac{n}{2}} \|f_1\|_1 \|f_2\|_1 ,$$

where $\frac{\vec{u}+\vec{l}}{2^{\frac{1}{2}}} = \vec{v}_1$ and $\frac{\vec{u}-\vec{l}}{2^{\frac{1}{2}}} = \vec{v}_2$, by the change of variable theorem.

2. Note that, for $\lambda \in \mathbb{C}_+^{\sim}$, $h(\lambda, \cdot, \vec{w}_{\vec{\xi}})$ is $L_\infty(\mathbb{R}^n)$ since

$$|h(\lambda, \vec{u}, \vec{w}_{\vec{\xi}})| \le \left(\frac{|\lambda|\Gamma}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \left| f_1\left(\frac{\vec{u}+\vec{w}_{\vec{\xi}}+\vec{l}}{2^{\frac{1}{2}}}\right) f_2\left(\frac{\vec{u}-\vec{w}_{\vec{\xi}}-\vec{l}}{2^{\frac{1}{2}}}\right)\right| d\vec{l}$$
$$\le \left(\frac{|\lambda|\Gamma}{\pi}\right)^{\frac{n}{2}} \|f_1\|_2 \|f_2\|_2$$

for $\vec{u} \in \mathbb{R}^n$ by Hölder inequality. Thus we have $h(\lambda, \cdot, \vec{w}_{\vec{\xi}}) \in C_0(\mathbb{R}^n)$ from a standard argument.

3. By Theorem 9, it suffices to show that $h(\lambda, \cdot, \vec{w}_{\vec{\xi}})$ is in $L_2(\mathbb{R}^n)$ for $\lambda \in \mathbb{C}_+^{\sim}$. But this fact follows from the following;

$$\int_{\mathbb{R}^n} |h(\lambda, \vec{u}, \vec{w}_{\vec{\xi}})|^2 d\vec{u}$$
$$\le \left(\frac{|\lambda|\Gamma}{2\pi}\right)^{n} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \left| f_1\left(\frac{\vec{u}+\vec{w}_{\vec{\xi}}+\vec{l}}{2^{\frac{1}{2}}}\right) f_2\left(\frac{\vec{u}-\vec{w}_{\vec{\xi}}-\vec{l}}{2^{\frac{1}{2}}}\right)\right| d\vec{l}\right]^2 d\vec{u}$$
$$= \left(\frac{|\lambda|\Gamma}{2\pi}\right)^{n} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \left| f_1\left(\frac{\vec{u}+\vec{w}_{\vec{\xi}}+\vec{l}}{2^{\frac{1}{2}}}\right) f_2\left(\frac{\vec{u}-\vec{w}_{\vec{\xi}}-\vec{l}}{2^{\frac{1}{2}}}\right)\right| d\vec{l}\right]$$
$$\times \left[\int_{\mathbb{R}^n} \left| f_1\left(\frac{\vec{u}+\vec{w}_{\vec{\xi}}+\vec{z}}{2^{\frac{1}{2}}}\right) f_2\left(\frac{\vec{u}-\vec{w}_{\vec{\xi}}-\vec{z}}{2^{\frac{1}{2}}}\right)\right| d\vec{z}\right] d\vec{u}.$$

Let $\vec{r} = \frac{\vec{u}+\vec{l}}{2^{\frac{1}{2}}}$ and $\vec{s} = \frac{\vec{u}+\vec{z}}{2^{\frac{1}{2}}}$. Then we have

$$\int_{\mathbb{R}^n} |h(\lambda, \vec{u}, \vec{w}_{\vec{\xi}})|^2 d\vec{u}$$
$$\le \left(\frac{|\lambda|\Gamma}{2\pi}\right)^{n} 2^n \int_{\mathbb{R}^n} |f_1(\vec{r} + 2^{-\frac{1}{2}}\vec{w}_{\vec{\xi}})| \int_{\mathbb{R}^n} |f_1(\vec{s} + 2^{-\frac{1}{2}}\vec{w}_{\vec{\xi}})|$$
$$\times \int_{\mathbb{R}^n} |f_2(\sqrt{2}\vec{u} - \vec{r} - 2^{-\frac{1}{2}}\vec{w}_{\vec{\xi}})||f_2(\sqrt{2}\vec{u} - \vec{s} - 2^{-\frac{1}{2}}\vec{w}_{\vec{\xi}})| d\vec{u} d\vec{s} d\vec{r}$$
$$\le \left(\frac{|\lambda|\Gamma}{2\pi}\right)^{n} 2^{\frac{n}{2}} \|f_1\|_1^2 \|f_2\|_2^2$$

by Hölder inequality.

4. It follows from 1 and 3.
5. It follows immediately and it is trivial.

□

By Theorems 5 and 9, using similar method in the proof of Theorem 4.8 in [2], we have the following theorem.

THEOREM 12. *Let $F_1, F_2 \in \cup_{1\le p\le\infty}\mathcal{A}_{n,s}^{(p)}$ be given by (3) with replacing f by f_1, f_2, respectively, and let X_τ be given as in Lemma 3. Then, for λ in $\mathbb{C}_+$ and for s-a.e. $(\vec{\xi}_1, \vec{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k$, we have*

$$\begin{aligned} &T_\lambda[[(F_1 * F_2)_\lambda | X_\tau](\cdot, \vec{\xi}_1)|X_\tau](y, \vec{\xi}_2) \\ =& \left[T_\lambda[F_1|X_\tau]\left(\frac{y}{2^{\frac{1}{2}}}, \frac{\vec{\xi}_2 + \vec{\xi}_1}{2^{\frac{1}{2}}}\right)\right]\left[T_\lambda[F_2|X_\tau]\left(\frac{y}{2^{\frac{1}{2}}}, \frac{\vec{\xi}_2 - \vec{\xi}_1}{2^{\frac{1}{2}}}\right)\right], \end{aligned}$$

for s-a.e. $y \in C_0(\mathbb{B})$.

The following theorem shows that the conditional Fourier-Feynman transform of conditional convolution product of some cylinder type functions is a product of conditional transforms. For convenience, if $\lambda = -iq(q \in \mathbb{R} - \{0\})$, then we denote $[(F_1 * F_2)_\lambda | X_\tau](\cdot, \vec{\xi}_1)$ by $[(F_1 * F_2)_q | X_\tau](\cdot, \vec{\xi}_1)$ in the following theorem.

THEOREM 13. *Let X_τ be given as in Lemma 3 and let q be a non-zero real number.*

1. *Let $F_1, F_2 \in \mathcal{A}_{n,s}^{(1)}$. Then, for s-a.e. $(\vec{\xi}_1, \vec{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k$, we have*

$$\begin{aligned} &T_q^{(1)}[[(F_1 * F_2)_q | X_\tau](\cdot, \vec{\xi}_1)|X_\tau](y, \vec{\xi}_2) \\ =& \left[T_q^{(1)}[F_1|X_\tau]\left(\frac{y}{2^{\frac{1}{2}}}, \frac{\vec{\xi}_2 + \vec{\xi}_1}{2^{\frac{1}{2}}}\right)\right]\left[T_q^{(1)}[F_2|X_\tau]\left(\frac{y}{2^{\frac{1}{2}}}, \frac{\vec{\xi}_2 - \vec{\xi}_1}{2^{\frac{1}{2}}}\right)\right] \end{aligned}$$

   for s-a.e. $y \in C_0(\mathbb{B})$.

2. *Let $F_1 \in \mathcal{A}_{n,s}^{(1)}$ and $F_2 \in \mathcal{A}_{n,s}^{(2)}$. Then, for s-a.e. $(\vec{\xi}_1, \vec{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k$, we have*

$$\begin{aligned} &T_q^{(2)}[[(F_1 * F_2)_q | X_\tau](\cdot, \vec{\xi}_1)|X_\tau](y, \vec{\xi}_2) \\ =& \left[T_q^{(1)}[F_1|X_\tau]\left(\frac{y}{2^{\frac{1}{2}}}, \frac{\vec{\xi}_2 + \vec{\xi}_1}{2^{\frac{1}{2}}}\right)\right]\left[T_q^{(2)}[F_2|X_\tau]\left(\frac{y}{2^{\frac{1}{2}}}, \frac{\vec{\xi}_2 - \vec{\xi}_1}{2^{\frac{1}{2}}}\right)\right] \end{aligned}$$

   for s-a.e. $y \in C_0(\mathbb{B})$.

3. *Let* $F_1 \in \mathcal{A}_{n,s}^{(1)}$ *and* $F_2 \in \mathcal{A}_{n,s}^{(1)} \cap \mathcal{A}_{n,s}^{(2)}$. *Then, for s-a.e.* $(\vec{\xi}_1, \vec{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k$, *we have*

$$\begin{aligned}&T_q^{(1)}[[(F_1 * F_2)_q|X_\tau](\cdot, \vec{\xi}_1)|X_\tau](y, \vec{\xi}_2)\\ = &\left[T_q^{(1)}[F_1|X_\tau]\left(\frac{y}{2^{\frac{1}{2}}}, \frac{\vec{\xi}_2 + \vec{\xi}_1}{2^{\frac{1}{2}}}\right)\right]\left[T_q^{(1)}[F_2|X_\tau]\left(\frac{y}{2^{\frac{1}{2}}}, \frac{\vec{\xi}_2 - \vec{\xi}_1}{2^{\frac{1}{2}}}\right)\right]\end{aligned}$$

and

$$\begin{aligned}&T_q^{(2)}[[(F_1 * F_2)_q|X_\tau](\cdot, \vec{\xi}_1)|X_\tau](y, \vec{\xi}_2)\\ = &\left[T_q^{(1)}[F_1|X_\tau]\left(\frac{y}{2^{\frac{1}{2}}}, \frac{\vec{\xi}_2 + \vec{\xi}_1}{2^{\frac{1}{2}}}\right)\right]\left[T_q^{(2)}[F_2|X_\tau]\left(\frac{y}{2^{\frac{1}{2}}}, \frac{\vec{\xi}_2 - \vec{\xi}_1}{2^{\frac{1}{2}}}\right)\right]\end{aligned}$$

for s-a.e. $y \in C_0(\mathbb{B})$.

Proof. The results follow from Corollary 10, Theorems 6, 11 and 12. □

5. Stability theories

Let $\mathcal{H}$ be an infinite dimensional separable real Hilbert space and let $\Delta_n = \{(s_1, s_2, \ldots, s_n) \in [0,T]^n : 0 = s_0 < s_1 < s_2 < \cdots < s_n \le T\}$ for any fixed $n \in \mathbb{N}$.

Let $\mathcal{M}_n'' = \mathcal{M}_n''(\Delta_n \times \mathcal{H}^n)$ be the class of all complex Borel measures on $\Delta_n \times \mathcal{H}^n$ and let $\|\mu\| = var\,\mu$, the total variation of μ in $\mathcal{M}_n''$. Let $\mathcal{S}_{n,\mathbb{B}}'' = \mathcal{S}_{n,\mathbb{B}}''(\Delta_n \times \mathcal{H}^n)$ be the space of functions of the form

$$F_n(x) = \int_{\Delta_n \times \mathcal{H}^n} \exp\left\{ i \sum_{j=1}^n (h_j, x(s_j))^\sim \right\} d\mu_{F_n}(\vec{s}, \vec{h}) \tag{11}$$

for s-a.e. $x \in C_0(\mathbb{B})$, where $\vec{s} = (s_1, \ldots, s_n)$, $\vec{h} = (h_1, \ldots, h_n)$ and $\mu_{F_n} \in \mathcal{M}_n''$. Here we take $\|F_n\|_n'' = \inf\{\|\mu_{F_n}\|\}$, where the infimum is taken over all μ_{F_n}'s so that F_n and μ_{F_n} are related by (11).

Let $\mathcal{M}'' = \mathcal{M}''(\sum \Delta_n \times \mathcal{H}^n)$ be the class of all sequences $\{\mu_n\}$ of measures such that each $\mu_n \in \mathcal{M}_n''$ and $\sum_{n=1}^\infty \|\mu_n\| < \infty$. Let $\mathcal{S}_{\mathbb{B}}'' = \mathcal{S}_{\mathbb{B}}''(\sum \Delta_n \times \mathcal{H}^n)$ be the space of functions of the form

$$F(x) = \sum_{n=1}^\infty F_n(x), \tag{12}$$

s-a.e. $x \in C_0(\mathbb{B})$, where each $F_n \in \mathcal{S}''_{n,\mathbb{B}}$ and $\sum_{n=1}^{\infty} \|F_n\|''_n < \infty$. The norm of F is defined by $\|F\|'' = \inf\{\sum_{n=1}^{\infty} \|F_n\|''_n\}$, where the infimum is taken over all representations of F given by (12).

Note that if n and l are positive integers then $\mathcal{S}''_{n,\mathbb{B}} \subset \mathcal{S}''_{n+l,\mathbb{B}}$ and $\mathcal{S}''_{n,\mathbb{B}} \subset \mathcal{S}''_{\mathbb{B}}$ for all $n \in \mathbb{N}$. Moreover we can show that $\mathcal{S}''_{n,\mathbb{B}}$ is a Banach space and $\mathcal{S}''_{\mathbb{B}}$ is a Banach algebra.

THEOREM 14. *Let $F_n \in \mathcal{S}''_{n,\mathbb{B}}$ be given by (11) and let X_τ be given as in Lemma 3. Let $q \in \mathbb{R} - \{0\}$ and $1 \le p \le \infty$. Then, for s-a.e. $\vec{\xi}$ in $\mathbb{B}^k$, $T_q^{(p)}[F_n|X_\tau](y, \vec{\xi})$ exists for s-a.e. $y \in C_0(\mathbb{B})$ and it is given by*

$$(13)\quad \begin{aligned} &T_q^{(p)}[F_n|X_\tau](y, \vec{\xi}) \\ &= \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k} \times \mathcal{H}^n} H_n(y, \vec{\xi}, \vec{s}, \vec{h}) G_n(-iq, \vec{\tau}, \vec{s}, \vec{h}) d\mu_{F_n}(\vec{s}, \vec{h}) \end{aligned}$$

where for $j_1 + \cdots + j_k = n$

$$(14)\quad \vec{s} = (s_{1,1}, \ldots, s_{1,j_1}, s_{2,1}, \ldots, s_{2,j_2}, \ldots, s_{k,1}, \ldots, s_{k,j_k});$$

$$(15)\quad \begin{aligned} &\Delta'_{n;j_1,\ldots,j_k} \\ &= \{\vec{s} : 0 = s_{1,0} < s_{1,1} < \cdots < s_{1,j_1} \le t_1 < s_{2,1} < \cdots < s_{2,j_2} \\ &\quad \le t_2 < \cdots \le t_{k-1} < s_{k,1} < \cdots < s_{k,j_k} \le t_k = T\}, \end{aligned}$$

$$(16)\quad \vec{h} = (h_{1,1}, \ldots, h_{1,j_1}, h_{2,1}, \ldots, h_{2,j_2}, \ldots, h_{k,1}, \ldots, h_{k,j_k});$$

$$(17)\quad H_n(y, \vec{\xi}, \vec{s}, \vec{h}) = \exp\left\{ i \sum_{\alpha=1}^{k} \sum_{\beta=1}^{j_\alpha} (h_{\alpha,\beta}, y(s_{\alpha,\beta}) + [\vec{\xi}](s_{\alpha,\beta}))^{\sim} \right\},$$

$$(18)\quad \begin{aligned} t_0 = s_{1,0} = 0, \qquad &t_\alpha = s_{\alpha+1,0} = s_{\alpha,j_\alpha+1} \ (\alpha = 1, \ldots, k-1), \\ &s_{k,j_k+1} = t_k = T; \end{aligned}$$

$$(19)\quad l_{\alpha,v} = s_{\alpha,v} - s_{\alpha,v-1}, \text{ for } \alpha = 1, \ldots, k; \text{ for } v = 1, \ldots, j_\alpha + 1,$$

$$(20)\quad \vec{\tau} = (t_1, \ldots, t_k);$$

$$
\begin{aligned}
(21) \qquad & G_n(\lambda, \vec{\tau}, \vec{s}, \vec{h}) \\
= & \exp\bigg\{ -\frac{1}{2\lambda} \sum_{\alpha=1}^{k} \sum_{v=1}^{j_\alpha+1} l_{\alpha,v} \bigg| \sum_{\beta=1}^{v-1} \frac{t_{\alpha-1} - s_{\alpha,\beta}}{t_\alpha - t_{\alpha-1}} h_{\alpha,\beta} \\
& + \sum_{\beta=v}^{j_\alpha} \frac{t_\alpha - s_{\alpha,\beta}}{t_\alpha - t_{\alpha-1}} h_{\alpha,\beta} \bigg|^2 \bigg\}
\end{aligned}
$$

with $\lambda \in \mathbb{C}_+^\sim$.

Proof. For $\lambda > 0$, for s-a.e. $\vec{\xi}$ in $\mathbb{B}^k$ and for s-a.e. y in $C_0(\mathbb{B})$, we have

$$
\begin{aligned}
& E[F_n(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])] \\
= & \int_{\Delta_n \times \mathcal{H}^n} \exp\bigg\{ i \sum_{j=1}^{n} (h_j, y(s_j) + [\vec{\xi}](s_j))^\sim \bigg\} \int_{C_0(\mathbb{B})} \exp\bigg\{ i\lambda^{-\frac{1}{2}} \sum_{j=1}^{n} (h_j, \\
& x(s_j) - [x](s_j))^\sim \bigg\} dm_{\mathbb{B}}(x) d\mu_{F_n}(\vec{s}, \vec{h})
\end{aligned}
$$

by Fubini theorem where $\vec{s} = (s_1, \ldots, s_n)$ and $\vec{h} = (h_1, \ldots, h_n)$. Let $\vec{s}$, $\Delta'_{n;j_1,\ldots,j_k}$, $\vec{h}$ and H_n be given by (14), (15), (16) and (17), respectively. Then, by Lemma 2, we have

$$
\begin{aligned}
& E[F_n(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])] \\
= & \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k} \times \mathcal{H}^n} H_n(y, \vec{\xi}, \vec{s}, \vec{h}) \prod_{\alpha=1}^{k} \bigg[\int_{\mathbb{B}^{j_\alpha+1}} \exp\bigg\{ i\lambda^{-\frac{1}{2}} \sum_{\beta=1}^{j_\alpha} \\
& \bigg(h_{\alpha,\beta}, \sum_{v=1}^{\beta} \sqrt{l_{\alpha,v}} x_{\alpha,v} - \frac{s_{\alpha,\beta} - t_{\alpha-1}}{t_\alpha - t_{\alpha-1}} \sum_{v=1}^{j_\alpha+1} \sqrt{l_{\alpha,v}} x_{\alpha,v} \bigg)^\sim \bigg\} \\
& dm^{j_\alpha+1}(\vec{x}_\alpha) \bigg] d\mu_{F_n}(\vec{s}, \vec{h})
\end{aligned}
$$

by Wiener integration theorem (Theorem 1) where $\vec{x}_\alpha = (x_{\alpha,1}, \ldots, x_{\alpha,j_\alpha+1})$ and $l_{\alpha,v}$ is given by (19) with (18). Let $\vec{\tau}$ and G_n be given by (20) and (21), respectively. Then we have

$$
\begin{aligned}
& E[F_n(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])] \\
= & \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k} \times \mathcal{H}^n} H_n(y, \vec{\xi}, \vec{s}, \vec{h}) \prod_{\alpha=1}^{k} \bigg[\int_{\mathbb{B}^{j_\alpha+1}} \exp\bigg\{ i\lambda^{-\frac{1}{2}} \sum_{v=1}^{j_\alpha+1}
\end{aligned}
$$

$$\left(\sqrt{l_{\alpha,v}}\left(\sum_{\beta=1}^{v-1}\frac{t_{\alpha-1}-s_{\alpha,\beta}}{t_\alpha-t_{\alpha-1}}h_{\alpha,\beta}+\sum_{\beta=v}^{j_\alpha}\frac{t_\alpha-s_{\alpha,\beta}}{t_\alpha-t_{\alpha-1}}h_{\alpha,\beta}\right),x_{\alpha,v}\right)^{\sim}\Bigg\}$$

$$dm^{j_\alpha+1}(\vec{x}_\alpha)\Bigg]d\mu_{F_n}(\vec{s},\vec{h})$$

$$=\sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\dots,j_k}\times\mathcal{H}^n}H_n(y,\vec{\xi},\vec{s},\vec{h})G_n(\lambda,\vec{\tau},\vec{s},\vec{h})d\mu_{F_n}(\vec{s},\vec{h})$$

since $(h,\cdot)^{\sim}$ is normally distributed with mean 0 and variance $|h|^2(h\neq 0)$. By Morera's theorem, we have

$$T_\lambda[F_n|X_\tau](y,\vec{\xi})=\sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\dots,j_k}\times\mathcal{H}^n}H_n(y,\vec{\xi},\vec{s},\vec{h})G_n(\lambda,\vec{\tau},\vec{s},\vec{h})d\mu_{F_n}(\vec{s},\vec{h})$$

for $\lambda\in\mathbb{C}_+$. For $1\le p\le\infty$ let $T_q^{(p)}[F_n|X_\tau](y,\vec{\xi})$ be given by (13). When $p=1$ we have

$$|T_\lambda[F_n|X_\tau](y,\vec{\xi})-T_q^{(1)}[F_n|X_\tau](y,\vec{\xi})|$$
$$\le\sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\dots,j_k}\times\mathcal{H}^n}|G_n(\lambda,\vec{\tau},\vec{s},\vec{h})-G_n(-iq,\vec{\tau},\vec{s},\vec{h})|\,d\|\mu_{F_n}\|(\vec{s},\vec{h})$$

and when $1<p\le\infty$, for $\frac{1}{p}+\frac{1}{p'}=1$ and $\gamma>0$, we have

$$\int_{C_0(\mathbb{B})}|T_\lambda[F_n|X_\tau](\gamma y,\vec{\xi})-T_q^{(p)}[F_n|X_\tau](\gamma y,\vec{\xi})|^{p'}dm_{\mathbb{B}}(y)$$
$$\le\Bigg[\sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\dots,j_k}\times\mathcal{H}^n}|G_n(\lambda,\vec{\tau},\vec{s},\vec{h})-G_n(-iq,\vec{\tau},\vec{s},\vec{h})|\,d\|\mu_{F_n}\|(\vec{s},\vec{h})\Bigg]^{p'}.$$

Letting $\lambda\to -iq$ through $\mathbb{C}_+$, we have the results by the dominated convergence theorem. □

THEOREM 15. *Let $F\in\mathcal{S}''_{\mathbb{B}}$ be given by (12) and let X_τ be given as in Lemma 3. Let $q\in\mathbb{R}-\{0\}$ and $1\le p\le\infty$. Then, for s-a.e. $\vec{\xi}$ in $\mathbb{B}^k$, $T_q^{(p)}[F|X_\tau](y,\vec{\xi})$ exists for s-a.e. $y\in C_0(\mathbb{B})$ and it is given by*

$$T_q^{(p)}[F|X_\tau](y,\vec{\xi})=\sum_{n=1}^{\infty}T_q^{(p)}[F_n|X_\tau](y,\vec{\xi})\tag{22}$$

where $T_q^{(p)}[F_n|X_\tau]$ is given by (13) *in Theorem 14.*

Proof. Without loss of generality, we can assume $\sum_{n=1}^\infty \|\mu_{F_n}\| < \infty$ where F_n and μ_{F_n} are related by (11). For $\lambda > 0$, for s-a.e. $\vec{\xi} \in \mathbb{B}^k$ and for s-a.e. $y \in C_0(\mathbb{B})$, we have

$$T_\lambda[F|X_\tau](y,\vec{\xi}) = \sum_{n=1}^\infty T_\lambda[F_n|X_\tau](y,\vec{\xi}) \tag{23}$$

by Theorem 14 and the dominated convergence theorem. Since $\sum_{n=1}^\infty T_\lambda[F_n|X_\tau](y,\vec{\xi})$ converges uniformly on $\mathbb{C}_+$ by Theorem 14, we have (23) for all $\lambda \in \mathbb{C}_+$. For $1 \le p \le \infty$ let $T_q^{(p)}[F|X_\tau](y,\vec{\xi})$ be given by (22). When $p = 1$ we have

$$\begin{aligned}
&|T_\lambda[F|X_\tau](y,\vec{\xi}) - T_q^{(1)}[F|X_\tau](y,\vec{\xi})| \\
\le\ & \sum_{n=1}^\infty \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k}\times\mathcal{H}^n} |G_n(\lambda,\vec{\tau},\vec{s},\vec{h}) - G_n(-iq,\vec{\tau},\vec{s},\vec{h})| \\
& d\|\mu_{F_n}\|(\vec{s},\vec{h})
\end{aligned}$$

and when $1 < p \le \infty$, for $\frac{1}{p}+\frac{1}{p'} = 1$ and $\gamma > 0$, we have

$$\begin{aligned}
&\int_{C_0(\mathbb{B})} |T_\lambda[F|X_\tau](\gamma y,\vec{\xi}) - T_q^{(p)}[F|X_\tau](\gamma y,\vec{\xi})|^{p'} dm_{\mathbb{B}}(y) \\
\le\ & \Bigg[\sum_{n=1}^\infty \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k}\times\mathcal{H}^n} |G_n(\lambda,\vec{\tau},\vec{s},\vec{h}) - G_n(-iq,\vec{\tau},\vec{s},\vec{h})| \\
& d\|\mu_{F_n}\|(\vec{s},\vec{h})\Bigg]^{p'}.
\end{aligned}$$

Letting $\lambda \to -iq$ through $\mathbb{C}_+$, we have the results by the dominated convergence theorem. □

Let $\mathcal{M}(\mathcal{H})$ be the class of all complex Borel measures on $\mathcal{H}$ and let $\mathcal{G}$ be the set of all $\mathbb{C}$-valued functions θ on $[0,T]\times\mathbb{B}$ which have the following form

$$\theta(s,y) = \int_{\mathcal{H}} \exp\{i(h,y)^\sim\} d\sigma_s(h) \tag{24}$$

where $\{\sigma_s : s \in [0,T]\}$ is the family from $\mathcal{M}(\mathcal{H})$ satisfying the following conditions;

1. for each Borel subset E of $\mathcal{H}$, $\sigma_s(E)$ is a Borel measurable function of s on $[0,T]$,

2. $\|\sigma_s\| \in L_1([0,T])$.

Let $\theta \in \mathcal{G}$ be given by (24) and for s-a.e. x in $C_0(\mathbb{B})$ let

$$F_n(x) = \left[\int_0^T \theta(s, x(s))ds\right]^n \tag{25}$$

and

$$F(x) = \exp\left\{\int_0^T \theta(s, x(s))ds\right\} \tag{26}$$

where n is any fixed natural number. For any Borel measurable subset E of $(0,T] \times \mathcal{H}$, let

$$\mu(E) = \int_0^T \sigma_s(E^{(s)})ds,$$

where $E^{(s)} = \{h \in \mathcal{H} : (s,h) \in E\}$. By unsymmetric Fubini theorem ([8]), we have, for s-a.e. x in $C_0(\mathbb{B})$,

$$\begin{aligned} F_1(x) &= \int_0^T \theta(s, x(s))ds = \int_0^T \int_{\mathcal{H}} \exp\{i(h, x(s))^\sim\}d\sigma_s(h)ds \\ &= \int_{(0,T]\times\mathcal{H}} \exp\{i(h, x(s))^\sim\}d\mu(s,h), \end{aligned}$$

and $\|\mu\| \leq \int_0^T \|\sigma_s\|ds$, so that $F_1 \in \mathcal{S}''_{1,\mathbb{B}} \subset \mathcal{S}''_{\mathbb{B}}$. Moreover, we can show that $F_n \in \mathcal{S}''_{n,\mathbb{B}}$ for each $n \in \mathbb{N}$. Thus we have the following theorems by Theorems 14 and 15.

THEOREM 16. *Let F_n be given by (25) and let X_τ be given as in Lemma 3. Let $q \in \mathbb{R} - \{0\}$ and $1 \leq p \leq \infty$. For any Borel subset E of $\Delta_n \times \mathcal{H}^n$ let*

$$\mu'_{F_n}(E) = n! \int_{\Delta_n} \int_{\mathcal{H}^n} \chi_E(\vec{s}, \vec{h})d\left(\prod_{j=1}^n \sigma_{s_j}\right)(\vec{h})d\vec{s}$$

where $\vec{s} = (s_1, \ldots, s_n)$. Then, for s-a.e. $\vec{\xi}$ in $\mathbb{B}^k$, $T_q^{(p)}[F_n|X_\tau](y, \vec{\xi})$ exists for s-a.e. $y \in C_0(\mathbb{B})$ and it is given by (13) with replacing μ_{F_n} by μ'_{F_n}.

THEOREM 17. *Let F be given by (26) and let X_τ be given as in Lemma 3. Let $q \in \mathbb{R} - \{0\}$ and $1 \leq p \leq \infty$. Then, for s-a.e. $\vec{\xi}$ in $\mathbb{B}^k$, $T_q^{(p)}[F|X_\tau](y, \vec{\xi})$ exists for s-a.e. $y \in C_0(\mathbb{B})$ and it is given by*

$$T_q^{(p)}[F|X_\tau](y, \vec{\xi}) = 1 + \sum_{n=1}^\infty \frac{1}{n!} T_q^{(p)}[F_n|X_\tau](y, \vec{\xi})$$

where $T_q^{(p)}[F_n|X_\tau]$ is given as in Theorem 16.

Let $\mathcal{F}(\mathbb{B})$ be the class of all functions of the form

$$\phi(x_1) = \int_{\mathcal{H}} \exp\{i(h, x_1)^{\sim}\} d\nu(h) \tag{27}$$

for s-a.e. x_1 in $\mathbb{B}$ where $\nu \in \mathcal{M}(\mathcal{H})$. For s-a.e. x in $C_0(\mathbb{B})$ let

$$K_n(x) = F_n(x)\phi(x(T)) \text{ and } K(x) = F(x)\phi(x(T)) \tag{28}$$

where F_n and F are given by (25) and (26), respectively. Then for $\lambda > 0$, $x, y \in C_0(\mathbb{B})$ and $\vec{\xi} = (\xi_1, \ldots, \xi_k) \in \mathbb{B}^k$, we have

$$\begin{aligned} & |\phi(y(T) + \lambda^{-\frac{1}{2}}(x(T) - [x](T)) + [\vec{\xi}](T))| \\ = \;& |\phi(y(T) + \xi_k)| \leq \|\nu\|. \end{aligned} \tag{29}$$

Thus we have the following theorem.

THEOREM 18. *Let K_n, K be given by* (28) *and let X_τ be given as in Lemma 3. Let $q \in \mathbb{R} - \{0\}$ and $1 \leq p \leq \infty$. Then, for s-a.e. $\vec{\xi}$ in $\mathbb{B}^k$, both $T_q^{(p)}[K_n|X_\tau](y, \vec{\xi})$ and $T_q^{(p)}[K|X_\tau](y, \vec{\xi})$ exist for s-a.e. $y \in C_0(\mathbb{B})$, and they are given by*

$$T_q^{(p)}[K_n|X_\tau](y, \vec{\xi}) = T_q^{(p)}[F_n|X_\tau](y, \vec{\xi})\phi(y(T) + \xi_k)$$

and

$$\begin{aligned} T_q^{(p)}[K|X_\tau](y, \vec{\xi}) &= T_q^{(p)}[F|X_\tau](y, \vec{\xi})\phi(y(T) + \xi_k) \\ &= \phi(y(T) + \xi_k) + \sum_{n=1}^{\infty} \frac{1}{n!} T_q^{(p)}[K_n|X_\tau](y, \vec{\xi}), \end{aligned}$$

where $T_q^{(p)}[F_n|X_\tau]$ and $T_q^{(p)}[F|X_\tau]$ are given as in Theorems 16 and 17, respectively.

Let ζ be a complex-valued Borel measure on $[0, T]$. Then $\zeta = \mu + \nu_d$ can be decomposed uniquely into the sum of a continuous measure μ and a discrete measure ν_d([6, p.142]). Let δ_{τ_j} denote the Dirac measure with total mass one concentrated at τ_j.

Let $\mathcal{G}^*$ be the set of all $\mathbb{C}$-valued functions θ on $[0, T] \times \mathbb{B}$ which have the form (24) where $\{\sigma_s : s \in [0, T]\}$ is the family from $\mathcal{M}(\mathcal{H})$ satisfying the following conditions;

1. for each Borel subset E of $\mathcal{H}$, $\sigma_s(E)$ is a Borel measurable function of s on $[0, T]$,
2. $\|\sigma_s\| \in L_1([0, T], \mathcal{B}([0, T]), \|\zeta\|)$.

In convenience, let

$$\zeta = \mu + \sum_{j=1}^{r} w_j \delta_{\tau_j} \tag{30}$$

where $0 \le \tau_1 < \cdots < \tau_r \le T$ and the w_j's are in $\mathbb{C}$ for $j = 1, \ldots, r (\in \mathbb{N})$, and let $\theta \in \mathcal{G}^*$ be given by (24). For s-a.e. x in $C_0(\mathbb{B})$ let

$$F_n(x) = \left[\int_0^T \theta(s, x(s)) d\zeta(s) \right]^n \tag{31}$$

and

$$F(x) = \exp\left\{ \int_0^T \theta(s, x(s)) d\zeta(s) \right\}. \tag{32}$$

THEOREM 19. *Let F_n be given by (31) and let X_τ be given as in Lemma 3. Let $q \in \mathbb{R} - \{0\}$ and $1 \le p \le \infty$. By reordering τ_j's, t_j's and renaming τ_j's by $\tau_{\alpha,u}$'s ($\alpha = 1, \ldots, k; u = 1, \ldots, r_\alpha$) let $0 \le \tau_{1,1} < \tau_{1,2} < \cdots < \tau_{1,r_1} \le t_1 < \tau_{2,1} < \cdots < \tau_{2,r_2} \le t_2 < \cdots \le t_{k-1} < \tau_{k,1} < \cdots < \tau_{k,r_k} \le t_k = T$, where $r_1 + \cdots + r_k = r$. Then, for s-a.e. $\vec{\xi}$ in $\mathbb{B}^k$, $T_q^{(p)}[F_n|X_\tau](y, \vec{\xi})$ exists for s-a.e. $y \in C_0(\mathbb{B})$ and it is given by*

$$\begin{aligned} &T_q^{(p)}[F_n|X_\tau](y, \vec{\xi}) \\ &= n! \sum_{q_1+\cdots+q_k=n} \prod_{\alpha=1}^{k} \Bigg[\sum_{l_0+l_1+\cdots+l_{r_\alpha}=q_\alpha} L(q_\alpha) W(q_\alpha) \Bigg[\sum_{j_1+\cdots+j_{r_\alpha+1}=l_0} \\ &\int_{\Delta_{l_0;j_1,\ldots,j_{r_\alpha+1}}} \int_{\mathcal{H}^{q_\alpha}} H_{q_\alpha}(y, \vec{s}, \vec{h}, \vec{h}', \vec{\xi}, \vec{\tau}_\alpha) G_{q_\alpha}(-iq, \vec{s}, \vec{h}, \vec{h}', \vec{\tau}_\alpha) \\ &d(\sigma_{\vec{s}} \times \sigma_{\vec{\tau}_\alpha})(\vec{h}, \vec{h}') d\mu^{l_0}(\vec{s}) \Bigg] \Bigg] \end{aligned} \tag{33}$$

where for $q_1 + \cdots + q_k = n$ and for $\alpha = 1, \ldots, k$; for $l_0 + l_1 + \cdots + l_{r_\alpha} = q_\alpha$

$$L(q_\alpha) = \frac{1}{l_1! \cdots l_{r_\alpha}!} \tag{34}$$

and

$$w_{\alpha,u} = w_{r_1+\cdots+r_{\alpha-1}+u} \quad (u = 1, \ldots, r_\alpha); \tag{35}$$

$$W(q_\alpha) = \prod_{u=1}^{r_\alpha} w_{\alpha,u}^{l_u}; \tag{36}$$

for $j_1 + \cdots + j_{r_\alpha+1} = l_0$

(37) $\vec{s} = (s_{0,1}, \ldots, s_{0,j_1}, s_{1,1}, \ldots, s_{1,j_2}, \ldots, s_{r_\alpha,1}, \ldots, s_{r_\alpha,j_{r_\alpha+1}})$;

(38) $\Delta_{l_0;j_1,\ldots,j_{r_\alpha+1}} = \{\vec{s} : 0 \le t_{\alpha-1} \le s_{0,1} \le \cdots \le s_{0,j_1} \le \tau_{\alpha,1} < s_{1,1} < \cdots < s_{1,j_2} < \tau_{\alpha,2} < \cdots < \tau_{\alpha,r_\alpha} \le s_{r_\alpha,1} \le \cdots \le s_{r_\alpha,j_{r_\alpha+1}} \le t_\alpha \le T\}$,

(39) $\vec{h} = (h_{0,1}, \ldots, h_{0,j_1}, h_{1,1}, \ldots, h_{1,j_2}, \ldots, h_{r_\alpha,1}, \ldots, h_{r_\alpha,j_{r_\alpha+1}})$

and

$$\vec{\tau}_\alpha = (\tau_{\alpha,1}, \ldots, \tau_{\alpha,r_\alpha}); \tag{40}$$

$$\sigma_{\vec{s}} = \prod_{u=0}^{r_\alpha} \prod_{v=1}^{j_{u+1}} \sigma_{s_{u,v}}, \qquad \sigma_{\vec{\tau}_\alpha} = \prod_{u=1}^{r_\alpha} \prod_{1}^{l_u} \sigma_{\tau_{\alpha,u}} \tag{41}$$

and

(42) $\vec{h}' = (h'_{1,1}, \ldots, h'_{1,l_1}, h'_{2,1}, \ldots, h'_{2,l_2}, \ldots, h'_{r_\alpha,1}, \ldots, h'_{r_\alpha,l_{r_\alpha}})$;

for $u = 0, \ldots, r_\alpha - 1$

$$h_{u,j_{u+1}+1} = \sum_{\beta=1}^{l_{u+1}} h'_{u+1,\beta}, \quad s_{u,j_{u-1}+1} = \tau_{\alpha,u+1} = s_{u+1,0}, \tag{43}$$

$$h_{r_\alpha,j_{r_\alpha+1}+1} = 0 \text{ and } s_{r_\alpha,j_{r_\alpha+1}+1} = t_\alpha;$$

$$\begin{aligned} &H_{q_\alpha}(y, \vec{s}, \vec{h}, \vec{h}', \vec{\xi}, \vec{\tau}_\alpha) \\ = \ &\exp\left\{ i \sum_{u=0}^{r_\alpha} \sum_{v=1}^{j_{u+1}+1} (h_{u,v}, y(s_{u,v}) + [\vec{\xi}](s_{u,v}))^\sim \right\} \end{aligned} \tag{44}$$

and for $a = 0, 1, \ldots, r_\alpha$; *for* $b = 1, \ldots, j_{a+1} + 1$

$$\beta_{a,b} = s_{a,b} - s_{a,b-1} \text{ with } s_{0,0} = t_{\alpha-1}; \tag{45}$$

for $\lambda \in \mathbb{C}_+^{\sim}$

$$(46)\qquad G_{q_\alpha}(\lambda,\vec{s},\vec{h},\vec{h}',\vec{\tau}_\alpha)$$
$$= \exp\Big\{-\frac{1}{2\lambda}\sum_{u=0}^{r_\alpha}\sum_{v=1}^{j_{u+1}+1}\beta_{u,v}\Big|\sum_{a=0}^{u-1}\sum_{b=1}^{j_{a+1}+1}\frac{t_{\alpha-1}-s_{a,b}}{t_\alpha-t_{\alpha-1}}h_{a,b}$$
$$+\sum_{b=1}^{v-1}\frac{t_{\alpha-1}-s_{u,b}}{t_\alpha-t_{\alpha-1}}h_{u,b}+\sum_{b=v}^{j_{u+1}+1}\frac{t_\alpha-s_{u,b}}{t_\alpha-t_{\alpha-1}}h_{u,b}$$
$$+\sum_{a=u+1}^{r_\alpha}\sum_{b=1}^{j_{a+1}+1}\frac{t_\alpha-s_{a,b}}{t_\alpha-t_{\alpha-1}}h_{a,b}\Big|^2\Big\}.$$

Proof. For $q_1+\cdots+q_k=n$ let

$$Q(n)=q_1!\cdots q_k!$$

and let $w_{\alpha,u}$ be given by (35). For $\lambda>0$, for s-a.e. $\vec{\xi}\in\mathbb{B}^k$ and for s-a.e. $y\in C_0(\mathbb{B})$, we have

$$E[F_n(y+\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}])]$$
$$=\int_{C_0(\mathbb{B})}\Big[\sum_{\alpha=1}^{k}\Big[\int_{t_{\alpha-1}}^{t_\alpha}\theta(s,y(s)+\lambda^{-\frac{1}{2}}(x(s)-[x](s))+[\vec{\xi}](s))d\mu(s)$$
$$+\sum_{u=1}^{r_\alpha}w_{\alpha,u}\theta(\tau_{\alpha,u},y(\tau_{\alpha,u})+\lambda^{-\frac{1}{2}}(x(\tau_{\alpha,u})-[x](\tau_{\alpha,u}))+[\vec{\xi}](\tau_{\alpha,u}))\Big]\Big]^n$$
$$dm_{\mathbb{B}}(x)$$
$$=\sum_{q_1+\cdots+q_k=n}\frac{n!}{Q(n)}\prod_{\alpha=1}^{k}\Big[\int_{C_0(\mathbb{B})}\Big[\int_{t_{\alpha-1}}^{t_\alpha}\theta(s,y(s)+\lambda^{-\frac{1}{2}}(x(s)-[x](s))$$
$$+[\vec{\xi}](s))d\mu(s)+\sum_{u=1}^{r_\alpha}w_{\alpha,u}\theta(\tau_{\alpha,u},y(\tau_{\alpha,u})+\lambda^{-\frac{1}{2}}(x(\tau_{\alpha,u})-[x](\tau_{\alpha,u}))$$
$$+[\vec{\xi}](\tau_{\alpha,u}))\Big]^{q_\alpha}dm_{\mathbb{B}}(x)\Big]$$

by binomial expansion and Lemma 2 where $t_0=0$. Let $L(q_\alpha)$ and $W(q_\alpha)$ be given by (34) and (36), respectively. For $q_1+\cdots+q_k=n$ and for $\alpha=1,\ldots,k$; for $l_0+l_1+\cdots+l_{r_\alpha}=q_\alpha$, let $\vec{s}_0=(s_1,s_2,\ldots,s_{l_0})$ and

$\Delta_{l_0} = \{\vec{s}_0 : t_{\alpha-1} < s_1 < \cdots < s_{l_0} \le t_\alpha\}$. Then, we have

$$
\begin{aligned}
&E[F_n(y+\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}])] \\
= \; & n! \sum_{q_1+\cdots+q_k=n} \prod_{\alpha=1}^{k} \Big[\sum_{l_0+l_1+\cdots+l_{r_\alpha}=q_\alpha} L(q_\alpha)W(q_\alpha) \int_{C_0(\mathbb{B})} \Big[\int_{\Delta_{l_0}} \prod_{j=1}^{l_0} \theta(s_j, \\
& y(s_j)+\lambda^{-\frac{1}{2}}(x(s_j)-[x](s_j))+[\vec{\xi}](s_j)) d\mu^{l_0}(\vec{s}_0) \Big] \Big[\prod_{u=1}^{r_\alpha} (\theta(\tau_{\alpha,u}, y(\tau_{\alpha,u}) \\
& +\lambda^{-\frac{1}{2}}(x(\tau_{\alpha,u})-[x](\tau_{\alpha,u}))+[\vec{\xi}](\tau_{\alpha,u})))^{l_u} \Big] dm_{\mathbb{B}}(x) \Big] \\
= \; & n! \sum_{q_1+\cdots+q_k=n} \prod_{\alpha=1}^{k} \Big[\sum_{l_0+l_1+\cdots+l_{r_\alpha}=q_\alpha} L(q_\alpha)W(q_\alpha) \int_{C_0(\mathbb{B})} \Big[\sum_{j_1+\cdots+j_{r_\alpha+1}=l_0} \\
& \int_{\Delta_{l_0;j_1,\ldots,j_{r_\alpha+1}}} \prod_{u=0}^{r_\alpha} \prod_{v=1}^{j_{u+1}} \theta(s_{u,v}, y(s_{u,v})+\lambda^{-\frac{1}{2}}(x(s_{u,v})-[x](s_{u,v}))+ \\
& [\vec{\xi}](s_{u,v})) d\mu^{l_0}(\vec{s}) \Big] \Big[\prod_{u=1}^{r_\alpha} (\theta(\tau_{\alpha,u}, y(\tau_{\alpha,u})+\lambda^{-\frac{1}{2}}(x(\tau_{\alpha,u})-[x](\tau_{\alpha,u}))+ \\
& [\vec{\xi}](\tau_{\alpha,u})))^{l_u} \Big] dm_{\mathbb{B}}(x) \Big]
\end{aligned}
$$

where $\vec{s}$ and $\Delta_{l_0;j_1,\ldots,j_{r_\alpha+1}}$ are given by (37) and (38), respectively. Let $\vec{h}$ and $\vec{\tau}_\alpha$ be given by (39) and (40), respectively. Then we have

$$
\begin{aligned}
&E[F_n(y+\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}])] \\
= \; & n! \sum_{q_1+\cdots+q_k=n} \prod_{\alpha=1}^{k} \Big[\sum_{l_0+l_1+\cdots+l_{r_\alpha}=q_\alpha} L(q_\alpha)W(q_\alpha) \int_{C_0(\mathbb{B})} \Big[\sum_{j_1+\cdots+j_{r_\alpha+1}=l_0} \\
& \int_{\Delta_{l_0;j_1,\ldots,j_{r_\alpha+1}}} \int_{\mathcal{H}^{l_0}} \exp\Big\{ i \sum_{u=0}^{r_\alpha} \sum_{v=1}^{j_{u+1}} (h_{u,v}, y(s_{u,v})+\lambda^{-\frac{1}{2}}(x(s_{u,v})- \\
& [x](s_{u,v}))+[\vec{\xi}](s_{u,v}))^{\sim} \Big\} d\sigma_{\vec{s}}(\vec{h}) d\mu^{l_0}(\vec{s}) \Big] \Big[\int_{\mathcal{H}^{l_1+\cdots+l_{r_\alpha}}} \exp\Big\{ i \sum_{u=1}^{r_\alpha} \sum_{\beta=1}^{l_u} \\
& (h'_{u,\beta}, y(\tau_{\alpha,u})+\lambda^{-\frac{1}{2}}(x(\tau_{\alpha,u})-[x](\tau_{\alpha,u}))+[\vec{\xi}](\tau_{\alpha,u}))^{\sim} \Big\} d\sigma_{\vec{\tau}_\alpha}(\vec{h}') \Big] \\
& dm_{\mathbb{B}}(x) \Big]
\end{aligned}
$$

where $\sigma_{\vec{s}}$, $\sigma_{\vec{\tau}_\alpha}$ are given by (41) and $\vec{h}'$ is given by (42). For $u = 0,\ldots,r_\alpha - 1$, let $h_{u,j_{u+1}+1}$ and $s_{u,j_{u+1}+1}$ be given by (43). Then we have

$$\begin{aligned}
&E[F_n(y+\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}])] \\
= \;& n! \sum_{q_1+\cdots+q_k=n} \prod_{\alpha=1}^{k} \Big[\sum_{l_0+l_1+\cdots+l_{r_\alpha}=q_\alpha} L(q_\alpha)W(q_\alpha) \int_{C_0(\mathbb{B})} \Big[\sum_{j_1+\cdots+j_{r_\alpha+1}=l_0} \\
& \int_{\Delta_{l_0;j_1,\ldots,j_{r_\alpha+1}}} \int_{\mathcal{H}^{q_\alpha}} \exp\Big\{ i \sum_{u=0}^{r_\alpha} \sum_{v=1}^{j_{u+1}+1} (h_{u,v}, y(s_{u,v}) + [\vec{\xi}](s_{u,v}))^\sim \Big\} \\
& \times \exp\Big\{ i\lambda^{-\frac{1}{2}} \sum_{u=0}^{r_\alpha} \sum_{v=1}^{j_{u+1}+1} (h_{u,v}, x(s_{u,v}) - [x](s_{u,v}))^\sim \Big\} \\
& d(\sigma_{\vec{s}} \times \sigma_{\vec{\tau}_\alpha})(\vec{h},\vec{h}')d\mu^{l_0}(\vec{s}) \Big] dm_{\mathbb{B}}(x) \Big]
\end{aligned}$$

where $h_{r_\alpha,j_{r_\alpha+1}+1} = 0$ and $s_{r_\alpha,j_{r_\alpha+1}+1} = t_\alpha$. For $a = 0,1,\ldots,r_\alpha$; for $b = 1,\ldots,j_{a+1}+1$ let $\beta_{a,b}$ be given by (45) and let H_{q_α} be given by (44). Then we have

$$\begin{aligned}
&E[F_n(y+\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}])] \\
= \;& n! \sum_{q_1+\cdots+q_k=n} \prod_{\alpha=1}^{k} \Big[\sum_{l_0+l_1+\cdots+l_{r_\alpha}=q_\alpha} L(q_\alpha)W(q_\alpha) \Big[\sum_{j_1+\cdots+j_{r_\alpha+1}=l_0} \\
& \int_{\Delta_{l_0;j_1,\ldots,j_{r_\alpha+1}}} \int_{\mathcal{H}^{q_\alpha}} H_{q_\alpha}(y,\vec{s},\vec{h},\vec{h}',\vec{\xi},\vec{\tau}_\alpha) \int_{\mathbb{B}^{l_0+r_\alpha+1}} \exp\Big\{ i\lambda^{-\frac{1}{2}} \sum_{u=0}^{r_\alpha} \\
& \sum_{v=1}^{j_{u+1}+1} \Big(h_{u,v}, \sum_{a=0}^{u-1} \sum_{b=1}^{j_{a+1}+1} \sqrt{\beta_{a,b}} y_{a,b} + \sum_{b=1}^{v} \sqrt{\beta_{u,b}} y_{u,b} - \frac{s_{u,v}-t_{\alpha-1}}{t_\alpha - t_{\alpha-1}} \sum_{a=0}^{r_\alpha} \\
& \sum_{b=1}^{j_{a+1}+1} \sqrt{\beta_{a,b}} y_{a,b} \Big)^\sim \Big\} dm^{l_0+r_\alpha+1}(\vec{y}) d(\sigma_{\vec{s}} \times \sigma_{\vec{\tau}_\alpha})(\vec{h},\vec{h}')d\mu^{l_0}(\vec{s}) \Big] \Big]
\end{aligned}$$

by Theorem 1(Wiener integration theorem) and Fubini theorem where $\vec{y} = (y_{0,1},\ldots,y_{0,j_1+1},y_{1,1},\ldots,\ y_{1,j_2+1},\ldots,y_{r_\alpha,1},\ldots,\ y_{r_\alpha,j_{r_\alpha+1}+1})$. Let

G_{q_α} be given by (46). Then we have

$$\begin{aligned}
&E[F_n(y+\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}])]\\
=\ & n!\sum_{q_1+\cdots+q_k=n}\prod_{\alpha=1}^{k}\Big[\sum_{l_0+l_1+\cdots+l_{r_\alpha}=q_\alpha}L(q_\alpha)W(q_\alpha)\Big[\sum_{j_1+\cdots+j_{r_\alpha+1}=l_0}\\
&\int_{\Delta_{l_0;j_1,\dots,j_{r_\alpha+1}}}\int_{\mathcal{H}^{q_\alpha}}H_{q_\alpha}(y,\vec{s},\vec{h},\vec{h}',\vec{\xi},\vec{\tau}_\alpha)\int_{\mathbb{B}^{l_0+r_\alpha+1}}\exp\Big\{i\lambda^{-\frac{1}{2}}\sum_{u=0}^{r_\alpha}\\
&\sum_{v=1}^{j_{u+1}+1}\Big(\sqrt{\beta_{u,v}}\Big(\sum_{a=0}^{u-1}\sum_{b=1}^{j_{a+1}+1}\frac{t_{\alpha-1}-s_{a,b}}{t_\alpha-t_{\alpha-1}}h_{a,b}+\sum_{b=1}^{v-1}\frac{t_{\alpha-1}-s_{u,b}}{t_\alpha-t_{\alpha-1}}h_{u,b}+\\
&\sum_{b=v}^{j_{u+1}+1}\frac{t_\alpha-s_{u,b}}{t_\alpha-t_{\alpha-1}}h_{u,b}+\sum_{a=u+1}^{r_\alpha}\sum_{b=1}^{j_{a+1}+1}\frac{t_\alpha-s_{a,b}}{t_\alpha-t_{\alpha-1}}h_{a,b}\Big),y_{u,v}\Big)^{\sim}\Big\}\\
&dm^{l_0+r_\alpha+1}(\vec{y})d(\sigma_{\vec{s}}\times\sigma_{\vec{\tau}_\alpha})(\vec{h},\vec{h}')d\mu^{l_0}(\vec{s})\Big]\Big]\\
=\ & n!\sum_{q_1+\cdots+q_k=n}\prod_{\alpha=1}^{k}\Big[\sum_{l_0+l_1+\cdots+l_{r_\alpha}=q_\alpha}L(q_\alpha)W(q_\alpha)\Big[\sum_{j_1+\cdots+j_{r_\alpha+1}=l_0}\\
&\int_{\Delta_{l_0;j_1,\dots,j_{r_\alpha+1}}}\int_{\mathcal{H}^{q_\alpha}}H_{q_\alpha}(y,\vec{s},\vec{h},\vec{h}',\vec{\xi},\vec{\tau}_\alpha)G_{q_\alpha}(\lambda,\vec{s},\vec{h},\vec{h}',\vec{\tau}_\alpha)\\
&d(\sigma_{\vec{s}}\times\sigma_{\vec{\tau}_\alpha})(\vec{h},\vec{h}')d\mu^{l_0}(\vec{s})\Big]\Big]
\end{aligned}$$

since $(h,\cdot)^{\sim}$ is normally distributed with mean 0 and variance $|h|^2(h\neq 0)$. By analytic extension, $T_\lambda[F_n|X_\tau](y,\vec{\xi})$ exists and it is given by the above result for $\lambda\in\mathbb{C}_+$. For $1\le p\le\infty$ let $T_q^{(p)}[F_n|X_\tau](y,\vec{\xi})$ be given by (33). When $p=1$ we have

$$|T_\lambda[F_n|X_\tau](y,\vec{\xi})-T_q^{(1)}[F_n|X_\tau](y,\vec{\xi})|\le 2\Big[\int_0^T\|\sigma_s\|d\|\zeta\|(s)\Big]^n$$

and when $1<p\le\infty$, for $\frac{1}{p}+\frac{1}{p'}=1$ and $\gamma>0$, we have

$$\begin{aligned}
&\int_{C_0(\mathbb{B})}|T_\lambda[F_n|X_\tau](\gamma y,\vec{\xi})-T_q^{(p)}[F_n|X_\tau](\gamma y,\vec{\xi})|^{p'}dm_{\mathbb{B}}(y)\\
\le\ &\Big[2\Big[\int_0^T\|\sigma_s\|d\|\zeta\|(s)\Big]^n\Big]^{p'}.
\end{aligned}$$

Letting $\lambda \to -iq$ through $\mathbb{C}_+$, we have the results by the dominated convergence theorem. $\square$

REMARK 3. Let $\zeta_1 = \mu + \sum_{j=1}^{\infty} w_j \delta_{\tau_j}$, where the τ_j's are in $[0, T]$ and the w_j's are in $\mathbb{C}$. Using the following version of the $\aleph_0$-nomial formula([9, p.41])

$$\left(\sum_{j=0}^{\infty} b_j\right)^n = \sum_{j=0}^{\infty} \sum_{q_0+\cdots+q_j=n, q_j \neq 0} \frac{n!}{q_0! \cdots q_j!} b_0^{q_0} \cdots b_j^{q_j},$$

we can show that the results in Theorem 19 hold with replacing ζ in (30), (31) by ζ_1.

THEOREM 20. *Let F be given by (32) and let X_τ be given as in Lemma 3. Let $q \in \mathbb{R} - \{0\}$ and $1 \le p \le \infty$. Then, for s-a.e. $\vec{\xi}$ in $\mathbb{B}^k$, $T_q^{(p)}[F|X_\tau](y, \vec{\xi})$ exists for s-a.e. $y \in C_0(\mathbb{B})$ and it is given by*

$$T_q^{(p)}[F|X_\tau](y, \vec{\xi}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} T_q^{(p)}[F_n|X_\tau](y, \vec{\xi}) \tag{47}$$

where $T_q^{(p)}[F_n|X_\tau]$ is given by (33) in Theorem 19.

Proof. For s-a.e. $x \in C_0(\mathbb{B})$, by Maclaurin series expansion, we have

$$F(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} F_n(x)$$

where F_n is given by (31), and for $\lambda \in \mathbb{C}_+$ we also have

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left| T_\lambda[F_n|X_\tau](y, \vec{\xi}) \right| \le \exp\left\{ \int_0^T \|\sigma_s\| d\|\zeta\|(s) \right\} < \infty$$

for s-a.e. $\vec{\xi} \in \mathbb{B}^k$ and for s-a.e. $y \in C_0(\mathbb{B})$. Hence we have

$$T_\lambda[F|X_\tau](y, \vec{\xi}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} T_\lambda[F_n|X_\tau](y, \vec{\xi})$$

for $\lambda \in \mathbb{C}_+$. For $1 \le p \le \infty$ let $T_q^{(p)}[F|X_\tau](y, \vec{\xi})$ be given by (47). When $p = 1$ we have

$$|T_\lambda[F|X_\tau](y, \vec{\xi}) - T_q^{(1)}[F|X_\tau](y, \vec{\xi})| \le 2\exp\left\{ \int_0^T \|\sigma_s\| d\|\zeta\|(s) \right\}$$

and when $1 < p \leq \infty$, for $\frac{1}{p} + \frac{1}{p'} = 1$ and $\gamma > 0$, we have

$$\begin{aligned} &\int_{C_0(\mathbb{B})} |T_\lambda[F_n|X_\tau](\gamma y, \vec{\xi}) - T_q^{(p)}[F_n|X_\tau](\gamma y, \vec{\xi})|^{p'} dm_{\mathbb{B}}(y) \\ \leq\ & \left[2\exp\left\{ \int_0^T \|\sigma_s\| d\|\zeta\|(s) \right\} \right]^{p'}. \end{aligned}$$

Letting $\lambda \to -iq$ through $\mathbb{C}_+$, we have the results by the dominated convergence theorem. □

For s-a.e. x in $C_0(\mathbb{B})$ let

(48) $$K_n(x) = F_n(x)\phi(x(T)) \text{ and } K(x) = F(x)\phi(x(T)),$$

where ϕ, F_n and F are given by (27), (31) and (32), respectively. By (29), we have the following theorem.

THEOREM 21. *Let K_n, K be given by* (48) *and let X_τ be given as in Lemma 3. Let $q \in \mathbb{R} - \{0\}$ and $1 \leq p \leq \infty$. Then, for s-a.e. $\vec{\xi}$ in $\mathbb{B}^k$, both $T_q^{(p)}[K_n|X_\tau](y, \vec{\xi})$ and $T_q^{(p)}[K|X_\tau](y, \vec{\xi})$ exist for s-a.e. $y \in C_0(\mathbb{B})$ and they are given by*

$$T_q^{(p)}[K_n|X_\tau](y, \vec{\xi}) = T_q^{(p)}[F_n|X_\tau](y, \vec{\xi})\phi(y(T) + \xi_k)$$

and

$$\begin{aligned} T_q^{(p)}[K|X_\tau](y, \vec{\xi}) &= T_q^{(p)}[F|X_\tau](y, \vec{\xi})\phi(y(T) + \xi_k) \\ &= \phi(y(T) + \xi_k) + \sum_{n=1}^{\infty} \frac{1}{n!} T_q^{(p)}[K_n|X_\tau](y, \vec{\xi}), \end{aligned}$$

where $T_q^{(p)}[F_n|X_\tau]$ and $T_q^{(p)}[F|X_\tau]$ are given by (33) *in Theorem 19 and* (47) *in Theorem 20, respectively.*

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