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ON CHOQUET INTEGRALS OF MEASURABLE FUZZY NUMBER-VALUED FUNCTIONS

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Abstract. In this paper, we consider fuzzy number-valued functions and fuzzy number-valued Choquet integrals. And we also discuss positively homogeneous and monotonicity of Choquet integrals of fuzzy number-valued functions(simply, fuzzy number-valued Choquet integrals). Furthermore, we prove convergence theorems for fuzzy number-valued Choquet integrals.

1. Introduction

It is well-known that closed set-valued functions had been used repeatedly in [1, 2, 4, 20]. We studied closed set-valued Choquet integrals in [5, 6] and convergence theorems under some sufficient conditions in [7, 8], for examples ; (i) convergence theorems for monotone convergent sequences of Choquet integrably bounded closed set-valued functions(see [7]), (ii) covergence theorems for the upper limit and the lower limit of a sequence of Choquet integrably bounded closed set-valued functions (see [8]).

In this paper, we consider fuzzy number-valued functions in [19]-[22] and will define Choquet integrals of fuzzy number-valued functions. But these concepts of fuzzy number-valued Choquet integrals are all based on the corresponding results of interval-valued Choquet integrals in [5]-[8]. We also discuss their properties which are positively homogeneous and monotonicity of fuzzy number-valued Choquet integrals. They will be used in the following applications : (1) Subjectively probability and expectation utility without additivity associated with fuzzy

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events as in Choquet integrable fuzzy number-valued functions, (2) Capacity measure which are presented by comonotonically additive fuzzy number-valued functionals, and (3) Ambiguity measure related with fuzzy number-valued fuzzy inference.

In section 2, we consider interval number-valued functions and discuss some characterizations of interval number-valued Choquet integrals. In section 3, we define Choquet integrals of fuzzy number-valued functions and discuss their properties on a suitable class of fuzzy number-valued functions. In section 4, we will prove convergence theorems for fuzzy number-valued Choquet integrals.

2. Interval number-valued Choquet integrals

A fuzzy measure on a measurable space $(X, \mathcal{A})$ is an extended real-valued function $\mu : \mathcal{A} \longrightarrow [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$

(ii) $\mu(A) \leq \mu(B)$, whenever $A, B \in \mathcal{A},\ A \subset B$.

A fuzzy measure μ is said to be autocontinuous from above[resp., below] if $\mu(A \cup B_n) \to \mu(A)$ [resp., $\mu(A \sim B_n) \to \mu(A)$] whenever $A \in \mathcal{A}$, $\{B_n\} \subset \mathcal{A}$ and $\mu(B_n) \to 0$. If μ is autocontinuous both from above and from below, it is said to be autocontinuous (see [8, 13]).

DEFINITION 2.1. ([3, 10, 11, 12, 13]) (1) The Choquet integral of a measurable function f with respect to a fuzzy measure μ is defined by

$$(C) \int f d\mu = \int_0^\infty \mu_f(r) dr$$

where $\mu_f(r) = \mu(\{x | f(x) > r\})$ and the integral on the right-hand side is an ordinary one.

(2) A measurable function f is called integrable if the Choquet integral of f can be defined and its value is finite.

Throughout this paper, R^+ will denote the interval $[0, \infty)$, $I(R^+) = \{[a, b] \mid a, b \in R^+ \text{ and } a \leq b\}$. Then an element in $I(R^+)$ is called an interval number. On the interval number set, we define; for each pair $[a, b], [c, d] \in I(R^+)$ and $k \in R^+$,

$[a, b] + [c, d] = [a + c, b + d]$,

$[a, b] \cdot [c, d] = [a \cdot c, b \cdot d]$,

$k[a, b] = [ka, kb]$,

$[a, b] < [c, d]$ if and only if either $a < c$ or ($a = c$ and $b < d$),

$[a, b] \leq [c, d]$ if and only if $[a, b] < [c, d]$ or $[a, b] = [c, d]$.

Then $(I(R^+), d_H)$ is a metric space, where d_H is the Hausdorff metric defined by

$$d_H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}|x-y|, \sup_{y\in B}\inf_{x\in A}|x-y|\}$$

for all $A, B \in I(R^+)$. By the definition of the Hausdorff metric, it is easily to show that for each pair $[a,b], [c,d] \in I(R^+)$,

$$d_H([a,b],[c,d]) = \max\{|a-c|, |b-d|\}.$$

We note that $\leq$ is called an order of interval numbers and that $[a,b] \subset [c,d]$ means $[a,b]$ is a subset of $[c,d]$.

A fuzzy number is a fuzzy set u on R^+, satisfying the following conditions(see [18, 20, 21, 22]);

(i) (normality) $u(x) = 0$ for some, $x \in R^+$,
(ii) (fuzzy convexity) for every $\lambda \in (0,1]$, $[u]^\lambda = \{x \in R^+ | u(x) \geq \lambda\} \in I(R^+)$, and
(iii) $[u]^0 = \{x \in R^+ | \bar{u}(x) > 0\} \in I(R^+)$.

Let $F(R^+)$ denote the set of fuzzy numbers. We define (see [18, 21, 22]); for each pair $u, v \in F(R^+)$ and $k \in R^+$,

$[u+v]^\lambda = [u]^\lambda + [v]^\lambda$,
$[ku]^\lambda = k[u]^\lambda$,
$u \leq v$ if and only if $[u]^\lambda \leq [v]^\lambda$ for all $\lambda \in [0,1]$,
$u < v$ if and only if $u \leq v$ and $u \neq v$,
$u \subset v$ if and only if $[u]^\lambda \subset [v]^\lambda$ for all $\lambda \in [0,1]$.
$D : F(R^+) \times F(R^+) \to [0,\infty]$ on $F(R^+)$ by

$$D(u,v) = \sup\{d_H([u]^\lambda, [v]^\lambda) | \lambda \in (0,1]\}.$$

Let $C(R^+)$ be the class of closed subsets of R^+. Throughout this paper, we consider a closed set-valued function $F : X \to C(R^+)\backslash\{\emptyset\}$ and an interval number-valued function $F : X \to I(R^+)\backslash\{\emptyset\}$. We denote that $d_H - \lim_{n\to\infty} A_n = A$ if and only if $\lim_{n\to\infty} d_H(A_n, A) = 0$, where $A \in I(R^+)$ and $\{A_n\} \subset I(R^+)$. We also denote that $D-\lim_{n\to\infty} A_n = A$ if and only if $D - \lim_{n\to\infty} D(u_n, u) = 0$, where $u \in F(R^+)$ and $\{u_n\} \subset F(R^+)$.

Definition 2.3. ([1, 2]) A closed set-valued function F is said to be measurable if for each open set $O \subset R^+$,

$$F^{-1}(O) = \{x \in X | F(x) \cap O \neq \emptyset\} \in \mathcal{A}.$$

DEFINITION 2.4. ([1, 2]) Let F be a closed set-valued function. A measurable function $f : X \to R^+$ satisfying

$$f(x) \in F(x) \text{ for all } x \in X$$

is called a measurable selection of F.

We say $f : X \to R^+$ is in $L_c^1(\mu)$ if and only if f is measurable and $(C)\int f d\mu < \infty$. We note that "$x \in X$ $\mu - a.e.$" stands for " $x \in X$ μ-almost everywhere". The property $p(x)$ holds for $x \in X$ $\mu - a.e.$ means that there is a measurable set A such that $\mu(A) = 0$ and the property $p(x)$ holds for all $x \in A^c$, where A^c is the complement of A.

DEFINITION 2.5. ([10]-[13]) Let f, g be measurable nonnegative functions. We say that f and g are comonotonic, in symbols $f \sim g$ if and only if

$$f(x) < f(x') \implies g(x) \leq g(x') \text{ for all } x, x' \in X.$$

THEOREM 2.6. ([10]-[13]) *Let f, g, h be measurable functions. Then we have*

(1) $f \sim f$,
(2) $f \sim g \Longrightarrow g \sim f$,
(3) $f \sim a$ *for all* $a \in R^+$,
(4) $f \sim g$ *and* $f \sim h \Longrightarrow f \sim (g + h)$.

THEOREM 2.7. ([10]-[13]) *Let f, g be nonnegative measurable functions.*

(1) *If* $f \leq g$, *then* $(C)\int f d\mu \leq (C)\int g d\mu$.
(2) *If* $f \sim g$ *and* $a, b \in R^+$, *then*

$$(C)\int (af + bg) d\mu = a(C)\int f d\mu + b(C)\int g d\mu.$$

DEFINITION 2.8. ([5]-[8]) (1) Let F be a closed set-valued function and $A \in \mathcal{A}$. The Choquet integral of F on A is defined by

$$(C)\int_A F d\mu = \{(C)\int_A f d\mu | \, f \in S_c(F)\}$$

where $S_c(F)$ is the family of $\mu - a.e.$ Choquet integrable selections of F, that is,

$$S_c(F) = \{f \in L_c^1(\mu) | \, f(x) \in F(x) \; x \in X \; \mu - a.e.\}.$$

(2) A closed set-valued function F is said to be Choquet integrable if $(C)\int F d\mu \neq \emptyset$.

(3) A closed set-valued function F is said to be Choquet integrably bounded if there is a function $g \in L^1_c(\mu)$ such that

$$\|F(x)\| = \sup_{r\in F(x)} |r| \leq g(x) \text{ for all x} \in \text{X}.$$

Instead of $(C)\int_X Fd\mu$, we will write $(C)\int Fd\mu$. Let us discuss some properties of interval-valued Choquet integrals which mean Choquet integrals of measurable interval number-valued functions.

ASSUMPTION (A). For each pair $f, g \in S_c(F)$, there exists $h \in S_c(F)$ such that $f \sim h$ and $(C)\int gd\mu = (C)\int hd\mu$.

We consider the following classes of interval number-valued functions (see [8]);

$$\Im = \{F|\ F : X \longrightarrow I(R^+) \text{ is measurable and Choquet integrablybounded}\}$$

and

$$\Im_1 = \{F \in \Im|\ F \text{ satisfies the assumption(A)}\}.$$

EXAMPLE 2.9. We will take an example of the class $\Im_1$. Let m be the Lebesgue measure on $X = [0,1]$ and $\mu = m^2$. It is easily to show that μ is a fuzzy measure. We define a set-valued function $F : X \longrightarrow C(R^+)$ by $F(x) = [0,1]$ for every $x \in X$. Then, it is easily to show that $F : X \longrightarrow C(R^+)$ is measurable, Choquet integrably bounded, and convex. Furthermore, for each $f, g \in S_c(F)$, we can put $d = (C)\int gd\mu$. If we define a function $h : X \longrightarrow R^+$ by $h(x) = d$ for every $x \in X$, then $h \in S_c(F)$ and $(C)\int hd\mu = d = (C)\int gd\mu$. We also have $f \sim h$. Thus, the set-valued function F satisfied the Assumption (A). That is, $F \in \Im_1$.

THEOREM 2.10. (([5], Proposition 3.3) and ([8], Theorem 3.3)) *If $F \in \Im_1$, then we have*

(1) $cF \in \Im_1$ *for all* $c \in R^+$,
(2) $F \leq G \implies (C)\int Fd\mu \leq (C)\int Gd\mu$,
(3) $A \subset B(A, B \in \mathcal{A}) \implies (C)\int_A Fd\mu \subset (C)\int_B Fd\mu$,
(4) $A \leq B(A, B \in \mathcal{A}) \implies (C)\int_A Fd\mu \leq (C)\int_B Fd\mu$,
(5) $(C)\int Fd\mu = [(C)\int f_* d\mu,\ (C)\int f^* d\mu]$,
where $f^*(x) = \sup\{r|r \in F(x)\}$ *and* $f_*(x) = \inf\{r|r \in F(x)\}$.

In the paper [8], we proved that f^*, f_* are Choquet integrable selections of F. Now, we consider a function $\triangle_S$ on $\mathcal{F}_1$ defined by

$$\triangle_S(F,G) = \sup_{x\in X} d_H(F(x), G(x))$$

for all $F, G \in \mathcal{F}_1$. Then, it is easily to show that $\triangle_S$ is a metric on $\mathcal{F}_1$.

DEFINITION 2.11. Let $F \in \Im_1$. A sequence $\{F_n\} \subset \Im_1$ converges to F in the metric $\triangle_S$, in symbols, $F_n \rightarrow_{\triangle_S} F$ if

$$\lim_{n\to\infty} \triangle_S(F_n, F) = 0.$$

THEOREM 2.12. ([8], Theorem 3.5) *Let $F, G, H \in \Im_1$ and $\{F_n\}$ be a sequence in $\Im_1$. If a fuzzy measure μ is autocontinuous and if $F_n \rightarrow_{\triangle_S} F$ and $G \leq F_n \leq H$, then we have*

$$d_H - \lim_{n\to\infty} (C) \int F_n d\mu = (C) \int F d\mu.$$

3. Fuzzy number-valued Choquet integrals

In this section, we will define various concepts associated with fuzzy number-valued Choquet integral which is the Choquet integral of a fuzzy number-valued function.

DEFINITION 3.1. (1) A fuzzy number-valued function F is said to be measurable if for each $\lambda \in [0,1]$, the interval number-valued function $F^\lambda : X \longrightarrow F(R^+) \setminus \{\emptyset\}$, defined by $F^\lambda(x) = [F(x)]^\lambda$ is measurable.

(2) F is called Choquet integrably bounded if F^0 is Choquet integrably bounded.

DEFINITION 3.2. (1) Let F be a fuzzy number-valued function and $A \in \mathcal{A}$. The Choquet integral of F on A is defined by for each $\lambda \in [0,1]$,

$$\begin{aligned} [(C) \int_A F d\mu]^\lambda &= (C) \int_A F^\lambda d\mu \\ &= \left\{ (C) \int_A f d\mu | f \in S_c(F^\lambda) \right\}. \end{aligned}$$

(2) If there exists $u \in F(R^+)$ such that $[u]^\lambda = \int_A F^\lambda d\mu,\ \lambda \in [0,1]$, then F is called Choquet integrable on A. Write $u = (C) \int_A F d\mu$.

LEMMA 3.3. ([18], Lemma 2.1) *If $\{[a^\lambda, b^\lambda] | \lambda \in [0,1]\}$ is a given family of nonempty interval numbers of. If (1) for all $0 \leq \lambda_1 \leq \lambda_2$, $[a^{\lambda_1}, b^{\lambda_1}] \supset [a^{\lambda_2}, b^{\lambda_2}]$ and (2) for any nonincreasing sequence $\{\lambda_k\}$ in $[0,1]$ in converging to λ, $[a^\lambda, b^\lambda] = \cap_{k=1}^\infty [a^{\lambda_k}, b^{\lambda_k}]$. Then there exists a unique fuzzy number $u \in F(R^+)$ such that the family $[a^\lambda, b^\lambda]$ represents the λ-level sets of u.*

Conversely, if $[a^\lambda, b^\lambda]$ are the λ-level sets of a fuzzy number $u \in F(R^+)$, then the conditions (1) and (2) are satisfied.

We denote the following classes of fuzzy number-valued functions :

$$\bar{\Im} = \{F : X \to F(R^+) \mid F^\lambda \in \Im, \lambda \in [0,1]\}$$

and

$$\bar{\Im}_1 = \{F \in \bar{\Im} \mid F^\lambda \in \Im_1, \lambda \in [0,1]\}.$$

THEOREM 3.4. *If* $F : X \longrightarrow F(R^+)$ *is measurable and Choquet integrably bounded,* $F \in \bar{\Im}$.

Proof. By measurability of F, $F^\lambda : X \longrightarrow I(R^+)$ is measurable. Since F is Choquet integrably bounded, F^0 is Choquet integrably bounded. Thus there exists $h \in L^1_c(\mu)$ such that

$$||F^0(x)|| \leq h(x), \ \forall x \in X.$$

We note that $[F(x)]^\lambda \subset [F(x)]^0$ for all $\lambda \in [0,1]$. So, we have

$$||F^\lambda(x)|| \leq h(x), \ \forall x \in X.$$

That is, F^λ is Choquet integrably bounded for all $\lambda \in [0,1]$. Thus we have $F^\lambda \in \Im$ and hence $F \in \bar{\Im}$. □

THEOREM 3.5. *Let* $F \in \bar{\Im}_1$ *and* $F^{\lambda *}(x) = \sup\{r | r \in F^\lambda(x)\}$ *and* $f^\lambda_*(x) = \inf\{r | r \in F^\lambda(x)\}$ *for all* $x \in X$. *If we denote*

$$u^\lambda = [(C)\int f^\lambda_* d\mu, (C)\int f^{\lambda *} d\mu] \text{ for all } \lambda \in [0,1],$$

then we have

(1) *for all* λ_1, λ_2 *with* $0 \leq \lambda_1 \leq \lambda_2$, $u^{\lambda_1} \supset u^{\lambda_2}$ *and*

(2) *for any nondecreasing sequence* $\{\lambda_k\}$ *in* $[0,1]$ *converging to* λ, $u^\lambda = \cap_{k=1}^\infty u^{\lambda_k}$.

Proof. (1) Let λ_1, λ_2 with $0 \leq \lambda_1 \leq \lambda_2$. Since $F^{\lambda_1}(x) \supset F^{\lambda_2}(x)$ for all $x \in X$, $f^{\lambda_1}_*(x) \leq f^{\lambda_2}_*(x)$ and $f^{\lambda_1 *}(x) \geq f^{\lambda_2 *}(x)$ for all $x \in X$. By Theorem 2.7 (1),

$$(C)\int f^{\lambda_1}_* d\mu \leq (C)\int f^{\lambda_2}_* d\mu \text{ and } (C)\int f^{\lambda_1 *} d\mu \geq (C)\int f^{\lambda_2 *} d\mu.$$

Thus we have

$$\begin{aligned} u^{\lambda_1} &= [(C)\int f^{\lambda_1}_* d\mu, (C)\int f^{\lambda_1 *} d\mu] \\ &\supset [(C)\int f^{\lambda_2}_* d\mu, (C)\int f^{\lambda_2 *} d\mu] = u^{\lambda_2}. \end{aligned}$$

(2) Let $\{\lambda_k\}$ be any nonincreasing sequence in $[0,1]$ converging to λ. So, we have $f^{\lambda_k *} \searrow f^{\lambda *}$ and $f_*^{\lambda_k} \nearrow f_*^{\lambda}$. Thus, by Proposition 3.2 [10],

$$\lim_{k\to\infty}(C)\int f^{\lambda_k *}d\mu = (C)\int f^{\lambda *} \text{ and } \lim_{k\to\infty}(C)\int f_*^{\lambda_k}d\mu = (C)\int f_*^{\lambda}d\mu.$$

Therefore, we have

$$\begin{aligned}
\cap_{k=1}^{\infty} u^{\lambda_k} &= \cap_{k=1}^{\infty}[(C)\int f_*^{\lambda_k}d\mu, (C)\int f^{\lambda_k *}d\mu] \\
&= [\sup_k (C)\int f_*^{\lambda_k}d\mu, \inf_k (C)\int f^{\lambda_k *}d\mu] \\
&= [\lim_{k\to\infty}(C)\int f_*^{\lambda_k}d\mu, \lim_{k\to\infty}(C)\int f^{\lambda_k *}d\mu] \\
&= [(C)\int f_*^{\lambda}d\mu, (C)\int f^{\lambda *}d\mu] \\
&= u^{\lambda}.
\end{aligned}$$

□

COROLLARY 3.6. *If $F \in \bar{\Im}_1$, then there exists a unique fuzzy number $u \in F(R^+)$ such that*

$$u^{\lambda} = (C)\int F^{\lambda}d\mu \text{ for all } \lambda \in [0,1].$$

Proof. Theorem 3.5 implies that assumptions (1) and (2) of Lemma 3.3 hold. Thus by Lemma 3.3, there exists a unique fuzzy number $u \in F(R^+)$ such that the family $(C)\int F^{\lambda}d\mu$ represents the λ-level sets of u, that is,

$$u^{\lambda} = [(C)\int Fd\mu]^{\lambda} = (C)\int F^{\lambda}d\mu.$$

□

We remark that for all $F \in \bar{\Im}_1$, F is Choquet integrable and that $F \leq G$ if and only if $F(x) \leq G(x)$ for all $x \in X$ (see [22]).

THEOREM 3.7. (1) *If $F \in \bar{\Im}_1$ and $c \in R^+$, then $cF \in \bar{\Im}_1$ and*

$$(C)\int cFd\mu = c(C)\int Fd\mu.$$

(2) *If $F, G \in \bar{\Im}_1$ and $F \leq G$, then $(C)\int Fd\mu \leq (C)\int Gd\mu$.*
(3) *If $F \in \bar{\Im}_1$ and $A \leq B(A, B \in \mathcal{A})$, then $(C)\int_A Fd\mu \leq (C)\int_B Fd\mu$.*
(4) *If $F \in \bar{\Im}_1$ and $A \subset B(A, B \in \mathcal{A})$, then $(C)\int_A Fd\mu \subset (C)\int_B Fd\mu$.*

Proof. (1) It is easily to show that $cF \in \bar{\Im}_1$. Theorems 2.7(2) and 2.10(5) imply that for each $\lambda \in [0, 1]$,

$$\begin{aligned}
[(C)\int cF d\mu]^\lambda &= (C)\int (cF)^\lambda d\mu \\
&= (C)\int cF^\lambda d\mu \\
&= [(C)\int cf_*^\lambda d\mu, (C)\int cf^{\lambda *} d\mu] \\
&= c[(C)\int f_*^\lambda d\mu, (C)\int f^{\lambda *} d\mu] \\
&= c[(C)\int F^\lambda d\mu] \\
&= c[(C)\int F d\mu]^\lambda.
\end{aligned}$$

Thus we have $(C)\int cF d\mu = c(C)\int F d\mu$.

(2) Since $F \leq G$, $F^\lambda \leq G^\lambda$ for all $\lambda \in [0, 1]$. By Theorem 2.10(2),

$$\begin{aligned}
[(C)\int F d\mu]^\lambda &= (C)\int F^\lambda d\mu \\
&\leq (C)\int G^\lambda d\mu \\
&= [(C)\int G d\mu]^\lambda
\end{aligned}$$

for all $\lambda \in [0, 1]$. Therefore, $(C)\int F d\mu \leq (C)\int G d\mu$.

(3) By Theorem 2.10(3) and (4), we have

$$\begin{aligned}
[(C)\int_A F d\mu]^\lambda &= (C)\int_A F^\lambda d\mu \\
&\leq (C)\int_B F^\lambda d\mu \\
&= [(C)\int_B F d\mu]^\lambda,
\end{aligned}$$

and

$$\begin{aligned}
[(C)\int_A F d\mu]^\lambda &= (C)\int_A F^\lambda d\mu \\
&\subset (C)\int_B F^\lambda d\mu
\end{aligned}$$

$$= [(C)\int_B F d\mu]^\lambda,$$

for all $\lambda \in [0, 1]$.

4. The convergence theorem for fuzzy number-valued Choquet integrals

In this section, our aim is to prove the convergence theorem for fuzzy number-valued Choquet integrals. To this end, we introduce a metric in $F(R^+)$ (see [9, 15, 16]) and define a metric in $\bar{\Im}_1$.

Let $u, v \in F(R^+)$, and set

$$D(u, v) = \sup_{\lambda \in (0,1]} d_H(u^\lambda, v^\lambda).$$

Then, by Proposition 4.1 ([15]), we have $(F(R^+), D)$ is a metric space. Using this definition, clearly, we can define a metric D_S in $\bar{\Im}_1$.

DEFINITION 4.1. A function $D_S : \bar{\Im}_1 \times \bar{\Im}_1 \to [0, \infty]$ is defined by

$$D_S(F, G) = \sup_{x \in X} D(F(x), G(x)),$$

for all $F, G \in \bar{\Im}_1$.

Then we have a relation between D_S and Δ_S.

THEOREM 4.2 *Let $F, G \in \bar{\Im}_1$, and D_S and Δ_S be as in the above. Then we have*

$$D_S(F, G) = \sup_{\lambda \in (0,1]} \Delta_S(F^\lambda, G^\lambda).$$

Proof.

$$\begin{aligned} D_S(F, G) &= \sup_{x \in X} D(F(x), G(x)) \\ &= \sup_{x \in X} \sup_{\lambda \in (0,1]} d_H(F^\lambda(x), G^\lambda(x)) \\ &= \sup_{\lambda \in (0,1]} \sup_{x \in X} d_H(F^\lambda(x), G^\lambda(x)) \\ &= \sup_{\lambda \in (0,1]} \Delta_S(F^\lambda, G^\lambda). \end{aligned}$$

□

DEFINITION 4.3. Let $F \in \bar{\Im}_1$. A sequence $\{F_n\} \subset \bar{\Im}_1$ converges to F in the metric D_S, in symbols $F_n \rightarrow_{D_S} F$ if

$$\lim_{n\to\infty} D_S(F_n, F) = 0.$$

Using Theorem 4.2 and Definition 4.3, we obtain the following convergence theorem for fuzzy number-valued functions.

THEOREM 4.4. *Let $F, G, H \in \bar{\Im}_1$ and $\{F_n\}$ be a sequence in $\bar{\Im}_1$. If a fuzzy measure μ is autocontinuous and if $F_n \rightarrow_{D_S} F$ and $G \le F_n \le H$, then we have*

$$D - \lim_{n\to\infty} (C)\int F_n d\mu = (C)\int F d\mu.$$

Proof. Since $F_n \rightarrow_{D_S} F$ and $G \le F_n \le H$, ${F_n}^{\lambda} \rightarrow_{\triangle_S} F^{\lambda}$ and $G^{\lambda} \le {F_n}^{\lambda} \le H^{\lambda}$ for all $\lambda \in ((0,1]$. By Theorem 2.12,

$$\lim_{n\to\infty} d_H((C)\int {F_n}^{\lambda} d\mu, (C)\int F^{\lambda} d\mu) = 0.$$

Thus, if $\varepsilon > 0$ is fixed, then there exists a natural number N_0 such that

$$d_H((C)\int {F_n}^{\lambda} d\mu, (C)\int F^{\lambda} d\mu) < \varepsilon,$$

for all $n \ge N_0$ and for all $\lambda \in (0,1]$. Thus,

$$\begin{aligned} & D((C)\int {F_n}^{\lambda} d\mu, (C)\int F^{\lambda} d\mu) \\ = & \sup_{\lambda\in(0,1]} d_H((C)\int {F_n}^{\lambda} d\mu, (C)\int F^{\lambda} d\mu) \\ < & \ \varepsilon, \end{aligned}$$

for all $n \ge N_0$. Therefore, the proof is complete. □

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