

ON A SUBCLASS OF CERTAIN STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. A certain subclass $T_{\Omega}(n, p, \lambda, \alpha)$ of starlike functions in the unit disk is introduced. The object of the present paper is to derive several interesting properties of functions belonging to the class $T_{\Omega}(n, p, \lambda, \alpha)$. Coefficient inequalities, distortion theorems and closure theorems of functions belonging to the class $T_{\Omega}(n, p, \lambda, \alpha)$ are determined. Also we obtain radii of convexity for the class $T_{\Omega}(n, p, \lambda, \alpha)$. Furthermore, integral operators and modified Hadamard products of several functions belonging to the class $T_{\Omega}(n, p, \lambda, \alpha)$ are studied here.

1. Introduction

Let A be class of functions $f(z)$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the open unit disk $U = \{z : |z| < 1\}$. For $f(z)$ belong to A , Salagean [5] has introduced the following operator called the Salagean operator:

$$\begin{aligned} D^0 f(z) &= f(z), D^1 f(z) = Df(z) = zf'(z), \\ D^n f(z) &= D(D^{n-1}f(z)) \quad (n \in \mathbb{N} = \{1, 2, 3, \dots\}). \end{aligned}$$

Note that $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$, $n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Let $T(n, p)$ denote the class of functions $f(z)$ of the form:

$$(1.1) \quad \begin{aligned} f(z) &= z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \\ &\quad (a_{k+p} \geq 0; p \in \mathbb{N} := \{1, 2, 3, \dots\}; n \in \mathbb{N}), \end{aligned}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

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A function $f(z) \in T(n, p)$ is said to be in the class $T(n, p, \lambda, \alpha)$ if it satisfies the inequality:

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \right\} > \alpha$$

for some α ($0 \leq \alpha < 1$) and λ ($0 \leq \lambda \leq 1$), and for all $z \in U$ [2].

We can write the following equalities for the functions $f(z)$ belong to the class $T(n, p)$

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z) = z[pz^{p-1} - \sum_{k=n}^{\infty} (k+p)a_{k+p}z^{k+p-1}] \\ &= pz^p - \sum_{k=n}^{\infty} (k+p)a_{k+p}z^{k+p}, \\ D^2 f(z) &= D(Df(z)) = p^2 z^p - \sum_{k=n}^{\infty} (k+p)^2 a_{k+p} z^{k+p}, \\ &\vdots \\ D^\Omega f(z) &= D(D^{\Omega-1} f(z)) = p^\Omega z^p - \sum_{k=n}^{\infty} (k+p)^\Omega a_{k+p} z^{k+p}. \end{aligned}$$

A function $f(z) \in T(n, p)$ is said to be in the class $T_\Omega(n, p, \lambda, \alpha)$ if it satisfies the inequality:

$$(1.3) \quad \operatorname{Re} \left\{ \frac{(1-\lambda)z(D^\Omega f(z))' + \lambda z(D^{\Omega+1} f(z))'}{(1-\lambda)D^\Omega f(z) + \lambda D^{\Omega+1} f(z)} \right\} > \alpha \quad (\Omega \in \mathbb{N}_0)$$

for some α ($0 \leq \alpha < 1$) and λ ($0 \leq \lambda \leq 1$), and for all $z \in U$ [2].

We note that

$$\begin{aligned} T_0(n, p, \lambda, \alpha) &\equiv T(n, p, \lambda, \alpha), \\ T_0(n, 1, 0, \alpha) &\equiv T_\alpha(n), \\ T_0(n, 1, 1, \alpha) &\equiv C_\alpha(n), \\ T_0(1, 1, 0, \alpha) &\equiv T^*(\alpha), \\ T_0(1, 1, 1, \alpha) &\equiv C(\alpha), \\ T_0(n, 1, \lambda, \alpha) &\equiv P(n, \lambda, \alpha), \end{aligned}$$

and

$$T_1(n, 1, \lambda, \alpha) \equiv C(n, \lambda, \alpha).$$

The classes $T_\alpha(n)$ and $C_\alpha(n)$ were studied earlier by Srivastava et al. [8], the classes $T^*(\alpha) \equiv T_\alpha(1)$ and $C(\alpha) \equiv C_\alpha(1)$ were studied by Silverman [7], the class $P(n, \lambda, \alpha)$ was studied by Altıntaş [1], the class $T(n, p, \lambda, \alpha)$ were studied by Altıntaş et al. [2], and the class $C(n, \lambda, \alpha)$ were studied by Kamali and Akbulut [4].

2. A theorem on coefficient bounds

We begin by proving some sharp coefficient inequalities contained in the following theorem.

THEOREM 1. *A function $f(z) \in T(n, p)$ is in the class $T_\Omega(n, p, \lambda, \alpha)$ if and only if*

$$(2.1) \quad \sum_{k=n}^{\infty} (k+p)^\Omega (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1) a_{k+p} \leq p^\Omega (p-\alpha)(1+\lambda p - \lambda)$$

$$(0 \leq \alpha < 1; 0 \leq \lambda \leq 1; p \leq p^\Omega (p-\alpha)(1+\lambda p - \lambda) (p \neq 1); \\ p \in \mathbb{N}; n \in \mathbb{N}; \Omega \in \mathbb{N}_0).$$

The result is sharp.

Proof. Suppose that $f(z) \in T_\Omega(n, p, \lambda, \alpha)$. Then we find from (1.3) that

$$Re \left\{ \frac{(1 + \lambda p - \lambda) p^{\Omega+1} z^p - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda) (k + p)^{\Omega+1} a_{k+p} z^{k+p}}{(1 + \lambda p - \lambda) p^\Omega z^p - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda) (k + p)^\Omega a_{k+p} z^{k+p}} \right\} > \alpha$$

$$(0 \leq \alpha < 1; 0 \leq \lambda \leq 1; p \leq p^\Omega (p-\alpha)(1+\lambda p - \lambda) (p \neq 1); \\ p \in \mathbb{N}; n \in \mathbb{N}; \Omega \in \mathbb{N}_0; z \in U).$$

If we choose z to be real and let $z \rightarrow 1^-$, we get

$$\left\{ \frac{(1 + \lambda p - \lambda) p^{\Omega+1} - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda) (k + p)^{\Omega+1} a_{k+p}}{(1 + \lambda p - \lambda) p^\Omega - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda) (k + p)^\Omega a_{k+p}} \right\} \geq \alpha$$

$$(0 \leq \alpha < 1; 0 \leq \lambda \leq 1; p \leq p^\Omega (p-\alpha)(1+\lambda p - \lambda) (p \neq 1); \\ p \in \mathbb{N}; n \in \mathbb{N}; \Omega \in \mathbb{N}_0)$$

or, equivalently,

$$\begin{aligned}
& \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^{\Omega+1} a_{k+p} \\
& - \alpha \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^{\Omega} a_{k+p} \\
& \leq (1 + \lambda p - \lambda)p^{\Omega+1} - \alpha(1 + \lambda p - \lambda)p^{\Omega} \\
& \quad (0 \leq \alpha < 1; 0 \leq \lambda \leq 1; p \leq p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)(p \neq 1); \\
& \quad p \in \mathbb{N}; n \in \mathbb{N}; \Omega \in \mathbb{N}_0).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
& \sum_{k=n}^{\infty} (k + p - \alpha)(k + p)^{\Omega} (\lambda k + \lambda p + 1 - \lambda) a_{k+p} \\
& \leq (p - \alpha)p^{\Omega}(1 + \lambda p - \lambda) \\
& \quad (0 \leq \alpha < 1; 0 \leq \lambda \leq 1; p \leq p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)(p \neq 1); \\
& \quad p \in \mathbb{N}; n \in \mathbb{N}; \Omega \in \mathbb{N}_0).
\end{aligned}$$

Conversely, suppose that the inequality (2.1) holds true and let

$$z \in \partial U = \{z : z \in \mathbb{C} \mid |z| = 1\}.$$

Then we find from the definition (1.1) that

$$\begin{aligned}
& \left| \frac{(1 - \lambda)z(D^{\Omega}f(z))' + \lambda z(D^{\Omega+1}f(z))'}{(1 - \lambda)D^{\Omega}f(z) + \lambda D^{\Omega+1}f(z)} - p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda) \right| \\
& = \left| \frac{(1 + \lambda p - \lambda)p^{\Omega+1}z^p - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^{\Omega+1} a_{k+p} z^{k+p}}{(1 + \lambda p - \lambda)p^{\Omega}z^p - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^{\Omega} a_{k+p} z^{k+p}} \right. \\
& \quad \left. - p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda) \right| \\
& \leq \frac{\left| -(1 + \lambda p - \lambda)p^{\Omega} \{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda) - p\} z^p \right|}{\left| (1 + \lambda p - \lambda)p^{\Omega}z^p \right| - \left| \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^{\Omega} a_{k+p} z^{k+p} \right|} \\
& \quad + \frac{\left| \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^{\Omega} \{k + p - p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)\} a_{k+p} z^{k+p} \right|}{\left| (1 + \lambda p - \lambda)p^{\Omega}z^p \right| - \left| \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^{\Omega} a_{k+p} z^{k+p} \right|} \\
& = \frac{(1 + \lambda p - \lambda)p^{\Omega} \{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda) - p\} + \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^{\Omega} \{k + p - p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)\} a_{k+p} |z|^k}{(1 + \lambda p - \lambda)p^{\Omega} - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^{\Omega} a_{k+p} |z|^k}
\end{aligned}$$

$$\begin{aligned}
 & (1 + \lambda p - \lambda)p^\Omega \{p^\Omega(p - \alpha)(1 + \lambda p - \lambda) - p\} \\
 & + \{p - p^\Omega(p - \alpha)(1 + \lambda p - \lambda)\} \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^\Omega a_{k+p} \\
 \leq & \frac{\hspace{10em}}{(1 + \lambda p - \lambda)p^\Omega - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^\Omega a_{k+p}} \\
 & + \frac{\sum_{k=n}^{\infty} k(\lambda k + \lambda p + 1 - \lambda)(k + p)^\Omega a_{k+p}}{(1 + \lambda p - \lambda)p^\Omega - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^\Omega a_{k+p}} \\
 \leq & \frac{\{p^\Omega(p - \alpha)(1 + \lambda p - \lambda) - p\} \{ (1 + \lambda p - \lambda)p^\Omega - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^\Omega a_{k+p} \}}{(1 + \lambda p - \lambda)p^\Omega - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^\Omega a_{k+p}} \\
 & + \frac{p^\Omega(p - \alpha)(1 + \lambda p - \lambda) - \sum_{k=n}^{\infty} (p - \alpha)(\lambda k + \lambda p + 1 - \lambda)(k + p)^\Omega a_{k+p}}{(1 + \lambda p - \lambda)p^\Omega - \sum_{k=n}^{\infty} (\lambda k + \lambda p + 1 - \lambda)(k + p)^\Omega a_{k+p}} \\
 = & p^\Omega(p - \alpha)(1 + \lambda p - \lambda) - p + p - \alpha \\
 = & p^\Omega(p - \alpha)(1 + \lambda p - \lambda) - \alpha \\
 & (0 \leq \alpha < 1; 0 \leq \lambda \leq 1; p \leq p^\Omega(p - \alpha)(1 + \lambda p - \lambda)(p \neq 1); \\
 & p \in \mathbb{N}; n \in \mathbb{N}; \Omega \in \mathbb{N}_0),
 \end{aligned}$$

provided that the inequality (2.1) is satisfied. Hence, by the maximum modulus theorem, we have

$$f(z) \in T_\Omega(n, p, \lambda, \alpha).$$

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being

$$\begin{aligned}
 (2.2) \quad f(z) &= z^p - \frac{p^\Omega(p - \alpha)(1 + \lambda p - \lambda)}{(k + p)^\Omega(k + p - \alpha)(\lambda k + \lambda p + 1 - \lambda)} z^{k+p} \\
 & \quad (k \geq n; p, n \in \mathbb{N}, \Omega \in \mathbb{N}_0).
 \end{aligned}$$

COROLLARY 1. *Let the function $f(z)$ defined by (1.1) be in the class $T_\Omega(n, p, \lambda, \alpha)$. Then*

$$(2.3) \quad a_{k+p} \leq \frac{p^\Omega(p - \alpha)(1 + \lambda p - \lambda)}{(k + p)^\Omega(k + p - \alpha)(\lambda k + \lambda p + 1 - \lambda)} \quad (k \geq n).$$

The equality in (2.3) is attained for the function $f(z)$ given by (2.2).

3. Distortion theorems

THEOREM 2. *Let the function $f(z)$ defined by (1.1) be in the class $T_{\Omega}(n, p, \lambda, \alpha)$. Then we have*

$$(3.1) \quad |f(z)| \leq |z|^p + \frac{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)}{(n + p)^{\Omega}(n + p - \alpha)(\lambda p + \lambda n + 1 - \lambda)} |z|^{p+n}$$

and

$$(3.2) \quad |f(z)| \geq |z|^p - \frac{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)}{(n + p)^{\Omega}(n + p - \alpha)(\lambda p + \lambda n + 1 - \lambda)} |z|^{p+n}$$

for $z \in U$. Then equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$(3.3) \quad f(z) = z^p - \frac{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)}{(n + p)^{\Omega}(n + p - \alpha)(\lambda p + \lambda n + 1 - \lambda)} z^{p+n}.$$

Proof. Note that

$$\begin{aligned} & (n + p)^{\Omega}(n + p - \alpha)(\lambda p + \lambda n + 1 - \lambda) \sum_{k=n}^{\infty} a_{k+p} \\ & \leq \sum_{k=n}^{\infty} (k + p)^{\Omega}(k + p - \alpha)(\lambda k + \lambda p + 1 - \lambda) a_{k+p} \\ & \leq p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda), \end{aligned}$$

this last inequality following from Theorem 1. Thus

$$\begin{aligned} |f(z)| & \leq |z|^p + \sum_{k=n}^{\infty} |a_{k+p}| |z|^{k+p} \\ & \leq |z|^p + |z|^{n+p} \sum_{k=n}^{\infty} a_{k+p} \\ & \leq |z|^p + \frac{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)}{(n + p)^{\Omega}(n + p - \alpha)(\lambda p + \lambda n + 1 - \lambda)} |z|^{p+n}. \end{aligned}$$

Similarly,

$$\begin{aligned} |f(z)| & \geq |z|^p - |z|^{n+p} \sum_{k=n}^{\infty} a_{k+p} \\ & \geq |z|^p - \frac{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)}{(n + p)^{\Omega}(n + p - \alpha)(\lambda p + \lambda n + 1 - \lambda)} |z|^{p+n}. \end{aligned}$$

THEOREM 3. *If $f(z) \in T_{\Omega}(n, p, \lambda, \alpha)$, then*

$$\begin{aligned} & |z|^{p-1} \left\{ 1 - \frac{p^{\Omega-1}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)} |z|^n \right\} \\ & \leq \frac{1}{p} |f'(z)| \\ & \leq |z|^{p-1} \left\{ 1 + \frac{p^{\Omega-1}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)} |z|^n \right\}. \end{aligned}$$

Proof. We have

$$\begin{aligned} (3.4) \quad |f'(z)| & \leq p|z|^{p-1} + \sum_{k=n}^{\infty} (k+p)a_{k+p} |z|^{k+p-1} \\ & \leq p|z|^{p-1} + |z|^{n+p-1} \sum_{k=n}^{\infty} (k+p)a_{k+p}. \end{aligned}$$

In view of Theorem 1, we have

$$\begin{aligned} & \sum_{k=n}^{\infty} (k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)a_{k+p} \\ & \leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda) \end{aligned}$$

and then

$$\begin{aligned} & (n+p-\alpha)(\lambda n+\lambda p-\lambda+1)(n+p)^{\Omega-1} \sum_{k=n}^{\infty} (k+p)a_{k+p} \\ & \leq \sum_{k=n}^{\infty} (k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)a_{k+p} \\ & \leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda) \end{aligned}$$

or

$$\begin{aligned} (3.5) \quad & \sum_{k=n}^{\infty} (k+p)a_{k+p} \\ & \leq \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)}. \end{aligned}$$

A substitution of (3.5) in to (3.4) yields the right-hand inequality.

On the other hand,

$$\begin{aligned}
|f'(z)| &\geq p|z|^{p-1} - \sum_{k=n}^{\infty} (k+p)a_{k+p}|z|^{k+p-1} \\
&\geq p|z|^{p-1} - |z|^{n+p-1} \sum_{k=n}^{\infty} (k+p)a_{k+p} \\
&\geq p|z|^{p-1} - |z|^{n+p-1} \sum_{k=n}^{\infty} (k+p)a_{k+p} \\
&\geq p|z|^{p-1} - |z|^{n+p-1} \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)}
\end{aligned}$$

or

$$\frac{|f'(z)|}{p} \geq |z|^{p-1} \left\{ 1 - \frac{p^{\Omega-1}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}(n+p-\alpha)(\lambda p+\lambda n-\lambda+1)} |z|^n \right\}.$$

COROLLARY 2. *Let the function $f(z)$ defined by (1.1) be in the class $T_{\Omega}(n, p, \lambda, \alpha)$. Then the unit disk U is mapped onto a domain that contains the disk*

$$(3.6) \quad |w| < 1 - \left(\frac{p}{n+p} \right)^{\Omega} \left(\frac{p-\alpha}{n+p-\alpha} \right) \left(\frac{1+\lambda p-\lambda}{\lambda p+\lambda n+1-\lambda} \right).$$

The result is sharp with the extremal function given by (3.3).

4. Closure theorems

Let the functions $f_j(z)$ be defined, for $j = 1, 2, \dots, m$ by

$$(4.1) \quad f_j(z) = z^p - \sum_{k=n}^{\infty} a_{k+p,j} z^{k+p} (a_{k+p,j} \geq 0)$$

for $z \in U$ [3].

We shall prove the following results for the closure of functions in the class $T_{\Omega}(n, p, \lambda, \alpha)$.

THEOREM 4. *Let the functions $f_j(z)$ defined by (4.1) be in the class $T_{\Omega}(n, p, \lambda, \alpha)$ for every $j = 1, 2, \dots, m$. Then the functions $h(z)$ defined by*

$$h(z) = \sum_{j=1}^m c_j f_j(z) (c_j \geq 0)$$

is also in the same class $T_{\Omega}(n, p, \lambda, \alpha)$, where

$$\sum_{j=1}^m c_j = 1.$$

Proof. According to the definition of $h(z)$, we can write

$$\begin{aligned} h(z) &= \sum_{j=1}^m c_j \left[z^p - \sum_{k=n}^{\infty} a_{k+p,j} z^{k+p} \right] \\ &= \left(\sum_{j=1}^m c_j \right) z^p - \sum_{k=n}^{\infty} \left(\sum_{j=1}^m c_j a_{k+p,j} \right) z^{k+p} \\ &= z^p - \sum_{k=n}^{\infty} \left(\sum_{j=1}^m c_j a_{k+p,j} \right) z^{k+p}. \end{aligned}$$

Further, since $f_j(z)$ are in $T_{\Omega}(n, p, \lambda, \alpha)$ for every $j = 1, 2, \dots, m$ we get

$$\begin{aligned} &\sum_{k=n}^{\infty} (k+p)^{\Omega} (k+p-\alpha) (\lambda k + \lambda p - \lambda + 1) a_{k+p,j} \\ &\leq p^{\Omega} (p-\alpha) (1 + \lambda p - \lambda) \end{aligned}$$

for every $j = 1, 2, \dots, m$. Hence we can see that

$$\begin{aligned} &\sum_{k=n}^{\infty} (k+p)^{\Omega} (k+p-\alpha) (\lambda k + \lambda p - \lambda + 1) \left(\sum_{j=1}^m c_j a_{k+p,j} \right) \\ &= \sum_{k=n}^{\infty} (k+p)^{\Omega} (k+p-\alpha) (\lambda k + \lambda p - \lambda + 1) \\ &\quad \times (c_1 a_{k+p,1} + c_2 a_{k+p,2} + \dots + c_m a_{k+p,m}) \\ &= c_1 \sum_{k=n}^{\infty} (k+p)^{\Omega} (k+p-\alpha) (\lambda k + \lambda p - \lambda + 1) a_{k+p,1} \\ &\quad + c_2 \sum_{k=n}^{\infty} (k+p)^{\Omega} (k+p-\alpha) (\lambda k + \lambda p - \lambda + 1) a_{k+p,2} + \dots \\ &\quad + c_m \sum_{k=n}^{\infty} (k+p)^{\Omega} (k+p-\alpha) (\lambda k + \lambda p - \lambda + 1) a_{k+p,m} \end{aligned}$$

$$\begin{aligned}
&\leq c_1[p^\Omega(p-\alpha)(1+\lambda p-\lambda)] + c_2[p^\Omega(p-\alpha)(1+\lambda p-\lambda)] + \dots \\
&\quad + c_m[p^\Omega(p-\alpha)(1+\lambda p-\lambda)] \\
&= (c_1 + c_2 + \dots + c_m)[p^\Omega(p-\alpha)(1+\lambda p-\lambda)] \\
&= \left(\sum_{j=1}^m c_j\right)p^\Omega(p-\alpha)(1+\lambda p-\lambda) \\
&= p^\Omega(p-\alpha)(1+\lambda p-\lambda)
\end{aligned}$$

which implies that $h(z)$ in $T_\Omega(n, p, \lambda, \alpha)$. Thus we have the theorem.

COROLLARY 3. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by*

$$g(z) = z^p - \sum_{k=n}^{\infty} b_{k+p} z^{k+p} \quad (b_{k+p} \geq p \in \mathbb{N}, n \in \mathbb{N})$$

be in the same class $T_\Omega(n, p, \lambda, \alpha)$. Then the function $h(z)$ defined by

$$\begin{aligned}
h(z) &= (1-\gamma)f(z) + \gamma g(z) \\
&= z^p - \sum_{k=n}^{\infty} c_{k+p} z^{k+p} \\
&\quad (c_{k+p} \geq 0; 0 \leq \gamma \leq 1; p \in \mathbb{N}; n \in \mathbb{N})
\end{aligned}$$

is also in the class $T_\Omega(n, p, \lambda, \alpha)$.

Proof. Suppose that each of the functions $f(z)$ and $g(z)$ is in the class $T_\Omega(n, p, \lambda, \alpha)$. Then making use of (2.1), we see that

$$\begin{aligned}
&\sum_{k=n}^{\infty} (k+p)^\Omega(k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)c_{k+p} \\
&= (1-\gamma) \sum_{k=n}^{\infty} (k+p)^\Omega(k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)a_{k+p} \\
&\quad + \gamma \sum_{k=n}^{\infty} (k+p)^\Omega(k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)b_{k+p} \\
&\leq (1-\gamma)p^\Omega(p-\alpha)(1+\lambda p-\lambda) + \gamma(p-\alpha)p^\Omega(1+\lambda p-\lambda) \\
&= p^\Omega(p-\alpha)(1+\lambda p-\lambda) \\
&\quad (0 \leq \alpha < 1, 0 \leq \lambda \leq 1, p \leq p^\Omega(p-\alpha)(1+\lambda p-\lambda)(p \neq 1); \\
&\quad p \in \mathbb{N}, n \in \mathbb{N}; \Omega \in \mathbb{N}_0),
\end{aligned}$$

which of the completes the proof of Corollary 3. □

As a consequence of Corollary 3, there exists the extreme points of the class $T_{\Omega}(n, p, \lambda, \alpha)$.

THEOREM 5. Let $f_{n-1}(z) = z^p$ and

$$f_k(z) = z^p - \frac{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)}{(k + p)^{\Omega}(k + p - \alpha)(\lambda k + \lambda p - \lambda + 1)} z^{k+p}, \quad (k \geq n)$$

for $0 \leq \alpha < 1, 0 \leq \lambda \leq 1$ and $n \in \mathbb{N}$. Then $f(z)$ is in the class $T_{\Omega}(n, p, \lambda, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=n-1}^{\infty} \eta_k f_k(z)$$

where $\eta_k \geq 0, (k \geq n - 1)$ and $\sum_{k=n-1}^{\infty} \eta_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=n-1}^{\infty} \eta_k f_k(z).$$

Then

$$\begin{aligned} f(z) &= \sum_{k=n-1}^{\infty} \eta_k f_k(z) = \eta_{n-1} f_{n-1}(z) + \sum_{k=n}^{\infty} \eta_k f_k(z) \\ &= \eta_{n-1} z^p + \sum_{k=n}^{\infty} \eta_k \left[z^p - \frac{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)}{(k + p)^{\Omega}(p + k - \alpha)(\lambda k + \lambda p - \lambda + 1)} \right] z^{k+p} \\ &= \left(\sum_{k=n-1}^{\infty} \eta_k \right) z^p - \sum_{k=n}^{\infty} \eta_k \frac{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)}{(k + p)^{\Omega}(p + k - \alpha)(\lambda k + \lambda p - \lambda + 1)} z^{k+p} \\ &= z^p - \sum_{k=n}^{\infty} \frac{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)}{(k + p)^{\Omega}(p + k - \alpha)(\lambda k + \lambda p - \lambda + 1)} \eta_k z^{k+p}. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{k=n}^{\infty} \eta_k \left[\frac{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)}{(k + p)^{\Omega}(p + k - \alpha)(\lambda k + \lambda p - \lambda + 1)} \right] \\ &\quad \times \left[\frac{(k + p)^{\Omega}(k + p - \alpha)(\lambda k + \lambda p - \lambda + 1)}{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)} \right] \\ &= \sum_{k=n}^{\infty} \eta_k = \sum_{k=n-1}^{\infty} \eta_k - \eta_{n-1} = 1 - \eta_{n-1} \leq 1, \end{aligned}$$

so by Theorem 1, $f(z) \in T_{\Omega}(n, p, \lambda, \alpha)$.

Conversely, suppose $f(z) \in T_{\Omega}(n, p, \lambda, \alpha)$. Since

$$a_{k+p} \leq \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)} \quad (k = n, n+1, \dots),$$

we may set

$$\eta_k = \frac{(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)}{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)} a_{k+p}$$

and

$$\eta_{n-1} = 1 - \sum_{k=n}^{\infty} \eta_k.$$

Then

$$\begin{aligned} f(z) &= z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \\ &= z^p - \sum_{k=n}^{\infty} \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^{\Omega}(p+k-\alpha)(\lambda k+\lambda p-\lambda+1)} \eta_k z^{k+p} \\ &= z^p - \sum_{k=n}^{\infty} \eta_k [z^p - f_k(z)] \\ &= z^p - \sum_{k=n}^{\infty} \eta_k z^p + \sum_{k=n}^{\infty} \eta_k f_k(z) \\ &= \left(1 - \sum_{k=n}^{\infty} \eta_k\right) z^p + \sum_{k=n}^{\infty} \eta_k f_k(z) \\ &= \eta_{n-1} z^p + \sum_{k=n}^{\infty} \eta_k f_k(z) \\ &= \eta_{n-1} f_{n-1}(z) + \sum_{k=n}^{\infty} \eta_k f_k(z) \\ &= \sum_{k=n-1}^{\infty} \eta_k f_k(z). \end{aligned}$$

This completes the proof. □

5. Integral operators

THEOREM 6. *Let the function $f(z)$ defined by (1.1) be in the class $T_{\Omega}(n, p, \lambda, \alpha)$ and let c be real number such that $c > -p$. Then the function $F(z)$ defined by*

$$(5.1) \quad F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt$$

also belongs to the class $T_{\Omega}(n, p, \lambda, \alpha)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=n}^{\infty} b_{k+p} z^{k+p},$$

where

$$b_{k+p} = \left(\frac{c+p}{c+p+k} \right) a_{k+p}.$$

Therefore,

$$\begin{aligned} & \sum_{k=n}^{\infty} (k+p)^{\Omega} (k+p-\alpha) (\lambda k + \lambda p - \lambda + 1) b_{k+p} \\ &= \sum_{k=n}^{\infty} (k+p)^{\Omega} (k+p-\alpha) (\lambda k + \lambda p - \lambda + 1) \left(\frac{c+p}{c+p+k} \right) a_{k+p} \\ &\leq \sum_{k=n}^{\infty} (k+p)^{\Omega} (k+p-\alpha) (\lambda k + \lambda p - \lambda + 1) a_{k+p} \\ &\leq p^{\Omega} (p-\alpha) (1 + \lambda p - \lambda), \end{aligned}$$

since $f(z) \in T_{\Omega}(n, p, \lambda, \alpha)$. Hence, by Theorem 1, $F(z) \in T_{\Omega}(n, p, \lambda, \alpha)$.

THEOREM 7. *Let c be real number such that $c > -p$. If $F(z) \in T_{\Omega}(n, p, \lambda, \alpha)$, then the function $f(z)$ defined by (5.1) is p -valent in $|z| < R_p^*$, where*

$$(5.2) \quad R_p^* = \inf_k \left\{ \left(\frac{k+p}{p} \right)^{\Omega-1} \left(\frac{c+p}{c+p+k} \right) \left(\frac{k+p-\alpha}{p-\alpha} \right) \left(\frac{\lambda k + \lambda p - \lambda + 1}{\lambda p - \lambda + 1} \right) \right\}^{\frac{1}{k}} \quad (k \geq n).$$

The result is sharp.

Proof. Let

$$F(z) = z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (a_{k+p} \geq 0).$$

It follow from (5.1) that

$$\begin{aligned} f(z) &= \frac{z^{1-c}(z^c F(z))'}{(c+p)} \quad (c > -p) \\ &= z^p - \sum_{k=n}^{\infty} \left(\frac{c+k+p}{c+p} \right) a_{k+p} z^{k+p}. \end{aligned}$$

To prove the result, it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p \text{ for } |z| < R_p^*.$$

Now

$$\begin{aligned} \left| \frac{f'(z)}{z^{p-1}} - p \right| &= \left| \frac{pz^{p-1} - \sum_{k=n}^{\infty} (k+p) \left(\frac{c+k+p}{c+p} \right) a_{k+p} z^{k+p-1}}{z^{p-1}} - p \right| \\ &= \left| - \sum_{k=n}^{\infty} (k+p) \left(\frac{c+p+k}{c+p} \right) a_{k+p} z^k \right| \leq \sum_{k=n}^{\infty} (k+p) \left(\frac{c+p+k}{c+p} \right) a_{k+p} |z|^k. \end{aligned}$$

Thus $\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p$ if

$$(5.3) \quad \sum_{k=n}^{\infty} \left(\frac{k+p}{p} \right) \left(\frac{c+p+k}{c+p} \right) a_{k+p} |z|^k \leq 1.$$

But Theorem 1 confirms that

$$(5.4) \quad \sum_{k=n}^{\infty} \frac{(k+p)^\Omega (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)}{p^\Omega (p-\alpha)(1+\lambda p - \lambda)} a_{k+p} \leq 1.$$

Thus (5.3) will be satisfied if

$$\begin{aligned} & \left(\frac{k+p}{p} \right) \left(\frac{c+p+k}{c+p} \right) |z|^k \\ & \leq \frac{(k+p)^\Omega}{p^\Omega} \left(\frac{k+p-\alpha}{p-\alpha} \right) \left(\frac{\lambda k + \lambda p - \lambda + 1}{1 + \lambda p - \lambda} \right) \quad (k \geq n), \end{aligned}$$

or if

$$(5.5) \quad |z| \leq \left\{ \left(\frac{k+p}{p} \right)^{\Omega-1} \left(\frac{c+p}{c+p+k} \right) \left(\frac{k+p-\alpha}{p-\alpha} \right) \left(\frac{\lambda k + \lambda p - \lambda + 1}{\lambda p - \lambda + 1} \right) \right\}^{\frac{1}{k}} (k \geq n).$$

The required result follows now from (5.5). The result is sharp for the function

$$(5.6) \quad f(z) = z^p - \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^\Omega(k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)} \times \left(\frac{c+p+k}{c+p} \right) z^{k+p} \quad (k \geq n).$$

THEOREM 8. *Let the function $f(z)$ defined by (1.1) be in the class $T_\Omega(n, p, \lambda, \alpha)$. Then $f(z)$ is convex of order q ($0 \leq q < 1$) in $|z| < r$, where*

$$r = \inf_k \left\{ \left(\frac{k+p}{p} \right)^{\Omega-1} \left(\frac{p-q}{p-\alpha} \right) \left(\frac{k+p-\alpha}{k+p-q} \right) \left(\frac{\lambda k + \lambda p - \lambda + 1}{\lambda p - \lambda + 1} \right) \right\}^{\frac{1}{k}} (k \geq n).$$

Proof. We must show that

$$\left| \frac{z f''(z)}{f'(z)} + 1 - p \right| < p - q$$

($0 \leq q < 1$) for $|z| < r$. We have

$$\begin{aligned} & \left| \frac{z f''(z)}{f'(z)} + 1 - p \right| = \left| \frac{z f''(z) + (1-p)f'(z)}{f'(z)} \right| \\ &= \left| \frac{p(p-1)z^{p-1} - \sum_{k=n}^{\infty} (k+p)(k+p-1)a_{k+p}z^{k+p-1}}{pz^{p-1} - \sum_{k=n}^{\infty} (k+p)a_{k+p}z^{k+p-1}} \right. \\ & \quad \left. - \frac{(p-1)pz^{p-1} + \sum_{k=n}^{\infty} (p-1)(k+p)a_{k+p}z^{k+p-1}}{pz^{p-1} - \sum_{k=n}^{\infty} (k+p)a_{k+p}z^{k+p-1}} \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \frac{-\sum_{k=n}^{\infty} (k+p)ka_{k+p}z^{k+p-1}}{pz^{p-1} - \sum_{k=n}^{\infty} (k+p)a_{k+p}z^{k+p-1}} \right| \\
&\leq \frac{\sum_{k=n}^{\infty} k(k+p)a_{k+p}|z|^k}{p - \sum_{k=n}^{\infty} (k+p)a_{k+p}|z|^k}.
\end{aligned}$$

Thus $\left| \frac{zf''(z)}{f'(z)} + 1 - p \right| < p - q$ if

$$(5.7) \quad \sum_{k=n}^{\infty} \frac{(k+p)(k+p-q)}{p(p-q)} a_{k+p} |z|^k \leq 1.$$

But Theorem 1 confirms that

$$\sum_{k=n}^{\infty} \frac{(k+p)^{\Omega}(k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)}{p^{\Omega}(p-\alpha)(1+\lambda p - \lambda)} a_{k+p} \leq 1.$$

Hence (5.7) will be true if

$$\frac{(k+p)(k+p-q)}{p(p-q)} |z|^k \leq \frac{(k+p)^{\Omega}(k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)}{p^{\Omega}(p-\alpha)(1+\lambda p - \lambda)}$$

or if

$$\begin{aligned}
|z|^k &\leq \frac{(k+p)^{\Omega}(k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)p(p-q)}{p^{\Omega}(p-\alpha)(1+\lambda p - \lambda)(k+p-q)(k+p)}, \\
|z| &\leq \left\{ \left(\frac{k+p}{p} \right)^{\Omega-1} \left(\frac{p-q}{p-\alpha} \right) \left(\frac{k+p-\alpha}{k+p-q} \right) \right. \\
&\quad \left. \left(\frac{\lambda k + \lambda p - \lambda + 1}{1+\lambda p - \lambda} \right) \right\}^{\frac{1}{k}} (k \geq n).
\end{aligned}$$

6. Modified Hadamard products

Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by

$$g(z) = z^p - \sum_{k=n}^{\infty} b_{k+p} z^{k+p} \quad (b_{k+p} \geq 0; p \in \mathbb{N}, n \in \mathbb{N})$$

be in the same class $T_{\Omega}(n, p, \lambda, \alpha)$. We define the modified Hadamard product of the functions $f(z)$ and $g(z)$ by

$$f * g(z) = z^p - \sum_{k=n}^{\infty} a_{k+p} b_{k+p} z^{k+p}.$$

THEOREM 9. *If each of the functions $f(z)$ and $g(z)$ is in the class $T_{\Omega}(n, p, \lambda, \alpha)$, then*

$$f * g(z) \in T_{\Omega}(n, p, \lambda, \delta),$$

where

$$(6.1)$$

$$\delta \leq p - n$$

$$\times \frac{p^{\Omega}(p - \alpha)^2(1 + \lambda p - \lambda)}{(n + p)^{\Omega}(n + p - \alpha)^2(\lambda n + \lambda p - \lambda + 1) - p^{\Omega}(p - \alpha)^2(1 + \lambda p - \lambda)}$$

$(p \in \mathbb{N}, n \in \mathbb{N}).$

The result is sharp for the functions $f(z)$ and $g(z)$ given by

$$\begin{aligned} f(z) &= g(z) \\ &= z^p - \frac{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)}{(n + p)^{\Omega}(n + p - \alpha)(\lambda n + \lambda p - \lambda + 1)} z^{n+p} \\ &\quad (p \in \mathbb{N}, n \in \mathbb{N}). \end{aligned}$$

Proof. Employing the technique used earlier by Schild and Silverman [6], we need to find the largest δ such that

$$\sum_{k=n}^{\infty} \frac{(k + p)^{\Omega}(k + p - \delta)(\lambda k + \lambda p - \lambda + 1)}{p^{\Omega}(p - \delta)(1 + \lambda p - \lambda)} a_{k+p} b_{k+p} \leq 1.$$

Since

$$\sum_{k=n}^{\infty} \frac{(k + p)^{\Omega}(k + p - \alpha)(\lambda k + \lambda p - \lambda + 1)}{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)} a_{k+p} \leq 1$$

and

$$\sum_{k=n}^{\infty} \frac{(k + p)^{\Omega}(k + p - \alpha)(\lambda k + \lambda p - \lambda + 1)}{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)} b_{k+p} \leq 1,$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=n}^{\infty} \frac{(k + p)^{\Omega}(k + p - \alpha)(\lambda k + \lambda p - \lambda + 1)}{p^{\Omega}(p - \alpha)(1 + \lambda p - \lambda)} \sqrt{a_{k+p} b_{k+p}} \leq 1.$$

Thus it is sufficient to show that

$$\begin{aligned} & \frac{(k+p)^\Omega(k+p-\delta)(\lambda k+\lambda p-\lambda+1)}{p^\Omega(p-\delta)(1+\lambda p-\lambda)} a_{k+p} b_{k+p} \\ & \leq \frac{(k+p)^\Omega(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)}{p^\Omega(p-\alpha)(1+\lambda p-\lambda)} \sqrt{a_{k+p} b_{k+p}}, \end{aligned}$$

that is, that

$$\sqrt{a_{k+p} b_{k+p}} \leq \frac{(k+p-\alpha)}{(k+p-\delta)} \cdot \frac{(p-\delta)}{(p-\alpha)}.$$

Not that

$$\sqrt{a_{k+p} b_{k+p}} \leq \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^\Omega(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)} \quad (k \geq n).$$

Consequently, we need only to prove that

$$\frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^\Omega(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)} \leq \frac{k+p-\alpha}{k+p-\delta} \frac{p-\delta}{p-\alpha} \quad (k \geq n),$$

or, equivalently, that

$$\begin{aligned} & \delta \leq p \\ & - \frac{p^\Omega(p-\alpha)^2(1+\lambda p-\lambda)}{(k+p)^\Omega(k+p-\alpha)^2(\lambda k+\lambda p-\lambda+1) - p^\Omega(p-\alpha)^2(1+\lambda p-\lambda)} k \\ & (k \geq n). \end{aligned}$$

Since

$$\begin{aligned} (6.2) \quad & \psi(k) \\ & = p - \frac{p^\Omega(p-\alpha)^2(1+\lambda p-\lambda)}{(k+p)^\Omega(k+p-\alpha)^2(\lambda k+\lambda p-\lambda+1) - p^\Omega(p-\alpha)^2(1+\lambda p-\lambda)} k \\ & (k \geq n). \end{aligned}$$

is an increasing function of k ($k \geq n$), letting $k = n$ (6.2), we obtain

$$\begin{aligned} & \delta \leq \psi(n) \\ & = p - \frac{p^\Omega(p-\alpha)^2(1+\lambda p-\lambda)}{(n+p)^\Omega(n+p-\alpha)^2(\lambda n+\lambda p-\lambda+1) - p^\Omega(p-\alpha)^2(1+\lambda p-\lambda)} n, \end{aligned}$$

which completes the proof Theorem 9. \square

Finally, by taking the function $f_j(z)$ given by

$$f_j(z) = z^p - \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^\Omega(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)} z^{n+p} \quad (j = 1, 2),$$

we can see that the result is sharp.

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