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THE m-TH ROOT FINSLER METRICS ADMITTING (α, β)-TYPES

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ABSTRACT. The theory of m-th root metric has been developed by H. Shimada [8], and applied to the biology [1] as an ecological metric. The purpose of this paper is to introduce the m-th root Finsler metrics which admit (α, β)-types. Especially in cases of $m = 3, 4$, we give the condition for Finsler spaces with such metrics to be locally Minkowski spaces.

1. Introduction

Let $F^n = (M^n, L)$ be n-dimensional Finsler space with a fundamental metric function $L(x, y)$. The m-th root Finsler metric $L(x, y)$ of a differentiable manifold M^n is first defined by H. Shimada [8] as

$$L(x, y)^m = a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}, \tag{1.1}$$

where the coefficients $a_{i_1 i_2 \ldots i_m}(x)$ are components of a symmetric tensor field covariant of order m, depending on the position x alone. It is regarded as a direct generalization of Riemannian metric in a sense. Of course, the second root metric is a Riemannian metric. Now we shall restrict $m \geq 3$ throughout the paper. The third and fourth metrics are called the *cubic metric* and *quartic metric* respectively. A Finsler space with a cubic metric (resp. quartic metric) is called the *cubic Finsler space* (resp. *quartic Finsler space*).

A Finsler metric $L(\alpha, \beta)$ is called an (α, β)-metric if it is a positively homogeneous function of α and β of degree 1, where $\alpha^2 = a_{ij}(x) y^i y^j$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on M^n. Throughout this paper our discussion is restricted to such a domain of M^n that the

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β does not vanish. The interesting examples of an (α,β)-metric [5] are the Randers metric, Kropina metric and Matsumoto metric. The (α,β)-metric has been sometimes treated in theoretical physics ([1], [5]), and studied by some authors ([2], [4], [7]).

Let F^n be a Finsler space with a cubic metric $L(x,y)$. In the previous paper [3], authors dealt with the cubic metric, which admits an (α,β)-metric. In case of $n > 2$, if L is an (α,β)-metric where α is non-degenerate, then L^3 can be written in the form $L^3 = a\alpha^2\beta + b\beta^3$ with constants a and b. Therefore we can consider what is a general form of m-th root metric ($m \geq 3$) with (α,β)-metric.

Paying attention to the homogeneity of $L(\alpha,\beta)$, from (1.1) we obtain

$$
(1.2)\qquad
\begin{aligned}
&a)\ L^3 = c_1\alpha^2\beta + c_2\beta^3,\\
&b)\ L^4 = c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4,\\
&\qquad\vdots\\
&c)\ L^m = \sum_{r=0}^{s} c_{m-2r}\alpha^{2r}\beta^{m-2r},\ s \leq \frac{m}{2},
\end{aligned}
$$

where c's are arbitrary constants and s is an integer.

Thus we have

PROPOSITION 1.1. *Let L^m be the m-th root Finsler metric which admits an (α,β)-metric. Then the fundamental function L^m is characterized by the equation* (1.2)c).

On the other hand, if $\alpha^2 \equiv 0(mod\beta)$, that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. Hence in this paper, we assume that $b^2 \neq 0$ and $n \geq 3$.

In the section 3 and section 4, the Matsumoto's method of [4] will now be applied to find the condition that F^n be a locally Minkowski space.

2. The Berwald connection and locally Minkowski space

A Finsler space is called a *Berwald* space, if the connection coefficients $G_{j\,k}^{\,i}$ of $B\Gamma$ is function of position x^i alone, in any coordinate system. If a Finsler space has a covering of coordinate neighborhoods in which g_{ij} does not depend on x, then it is called *locally Minkowski*

[1]. A Finsler space is a locally Minkowski, if and only if it is a Berwald space and h-curvature tensor H^2 of $B\Gamma$ vanishes.

For the Berwald connection $B\Gamma = (G_{j\,k}^{\,i}, G^i_k, 0)$, the covariant derivative of a vector $X^i(x,y)$ is given by

$$X^i{}_{|j} = \partial_j X^i - \dot{\partial}_a X^i G^a{}_j + G_{a\,j}^{\,i} X^a,$$

where $\partial_j = \partial/\partial x^j$ and $\dot{\partial}_r = \partial/\partial y^r$.

Let $\gamma_j{}^i{}_k(x)$ be Christoffel symbols of the Riemannian metric α and $(;)$ be the covariant differentiation with respect to $\gamma_j{}^i{}_k$. To find the Berwald connection $B\Gamma$, we put $2G^i (= G^i{}_0) = \gamma_0{}^i{}_0 + 2B^i$, where the subscript 0 means a contraction by y^i. Then we have

$$\begin{aligned} G^i{}_j &= \gamma_0{}^i{}_j + B^i{}_j, \\ G_j{}^i{}_k &= \gamma_j{}^i{}_k + B_j{}^i{}_k, \end{aligned} \tag{2.1}$$

where $B^i{}_j = \dot{\partial}_j B^i$ and $B_j{}^i{}_k = \dot{\partial}_j B^i{}_k$. Putting $L_\alpha = \partial L/\partial\alpha$ and $L_\beta = \partial L/\partial\beta$, on account of [4], $B_j{}^i{}_k$ is determined by

$$L_\alpha P_{i00} = \alpha L_\beta Q_{i0}, \tag{2.2}$$

where $P_{i00} = B_j{}^k{}_i y^j y_k$, $Q_{i0} = (b_{j;i} - B_j{}^k{}_i b_k) y^j$ and $y_k = a_{rk} y^r$.

It is obvious that a Finsler space with $L(\alpha, \beta)$ is a Berwald space if and only if $B_j{}^k{}_i$ given by (2.2) is a function of x alone. We denote by $R_h{}^i{}_{jk}$ a Riemannian curvature tensor with respect to the $\gamma_j{}^i{}_k$. Then h-curvature tensor H^2 of $B\Gamma$ is given by [4]

$$H_h{}^i{}_{jk} = R_h{}^i{}_{jk} + \mathcal{U}_{(jk)}\{B_h{}^i{}_{j;k} - B_0{}^r{}_k \dot{\partial}_r B_h{}^i{}_j + B_h{}^r{}_j B_r{}^i{}_k\}, \tag{2.3}$$

where $\mathcal{U}_{(jk)}$ denotes the terms obtained from the preceding terms by interchanging indices j and k. If F^n is locally Minkowski, then it is a Berwald with $H^2 = 0$. From (2.3), consequently we have

THEOREM 2.1 [4]. *A $F^n = (M^n, L(\alpha, \beta))$ is a locally Minkowski if and only if $B_j{}^k{}_i$ is a function of x alone and $R_h{}^i{}_{jk}$ of the Riemannian α is written as:*

$$R_h{}^i{}_{jk} = -\mathcal{U}_{(jk)}\{B_h{}^i{}_{j;k} + B_h{}^r{}_j B_r{}^i{}_k\}. \tag{2.4}$$

If (2.2) gives $P_{i00} = Q_{i0} = 0$ necessarily, then we have $B_j{}^k{}_i = 0$ and $b_{j;i} = 0$, and (2.4) shows $R_h{}^i{}_{jk} = 0$.

3. A locally Minkowski space in case of $m = 3$

We consider a cubic metric which admits an (α, β)-metric (1.2)a). Let $F^n = (M^n, L)$ be an n-dimensional Finsler space ($n \geq 3$) whose metric function is given by (1.2)a).

From (1.2)a), the equation (2.2) gives

$$2c_1\beta P_{i00} = (c_1\alpha^2 + 3c_2\beta^2)Q_{i0}. \tag{3.1}$$

Now we assume that F^n is a Berwald space, that is, $B_j{}^i{}_k$ is a function of position only. Then above equation is a polynomial of three order in y and shows the existence of function $f_i(x)$ satisfying

$$a)\ P_{i00} = (c_1\alpha^2 + 3c_2\beta^2)f_i,\ \ b)\ Q_{i0} = 2c_1\beta f_i. \tag{3.2}$$

Differentiating (3.2)a) with respect to y and using the Christoffel process, we obtain

$$B_j{}^k{}_i a_{kh} + B_h{}^k{}_i a_{kj} = 2\psi_{jh} f_i,$$

from which

$$B_j{}^k{}_i = \psi_j^k f_i + \psi_i^k f_j - \psi_{ji} f^k, \tag{3.3}$$

where $\psi_j^k = c_1\delta_j^k + 3c_2 b^k b_j$ and $\psi_{ji} = a_{ki}\psi_j^k$. The equation (3.2)b) is written in the form $b_{j;i} = B_j{}^k{}_i b_k + 2c_1 b_j f_i$. From this and (3.3), we get

$$b_{j;i} = 3(c_1 + c_2 b^2)b_j f_i + (c_1 + 3c_2 b^2)b_i f_j - \sigma(c_1 a_{ji} + 3c_2 b_j b_i), \tag{3.4}$$

where $b^2 = a^{ij}b_i b_j$ and $\sigma = f^k b_k$. From (2.1) and (3.4), in the similar way as the Kropina space [4], we have

THEOREM 3.1. *Let $F^n = (M^n, L)$ be an n-dimensional Finsler space ($n \geq 3$) with the metric* (1.2)a). *The F^n is a Berwald space if and only if there exists $f_i(x)$ satisfying* (3.4), *and then the Berwald connection is written as*

$$B\Gamma = (\gamma_j{}^k{}_i + B_j{}^k{}_i, \gamma_0{}^k{}_i + B_0{}^k{}_i, 0),$$

where $B_j{}^k{}_i$ is given by (3.3).

Further, contraction of (3.4) by b^j yields

$$b^j b_{j;i} = 3b^2(c_1 + c_2 b^2)f_i. \tag{3.5}$$

Since $(b^2)_{;i} = 2b^j b_{j;i}$, (3.5) leads us to

$$\begin{aligned} f_i &= (b^2)_{;i}/6b^2(c_1 + c_2 b^2) \\ &= (6c_1)^{-1}\partial_i\{log\ b^2/(c_1 + c_2 b^2)\}. \end{aligned} \tag{3.6}$$

From (3.6) we can see that $f_i(x)$ is a gradient vector. Consequently we have

LEMMA 3.1. *The vector field $f_i(x)$ in (3.2) is a gradient vector, which is given by $f_i = (6c_1)^{-1}\partial_i\{log\ b^2/(c_1 + c_2b^2)\}$.*

Further, a locally Mikowski space is characterized as a Berwald space with the vanishing h-curvature tensor H^2 of $B\Gamma$. From Theorem 3.1 and Lemma 3.1, we have

THEOREM 3.2. *Let $F^n = (M^n, L)$ be an n-dimensional Finsler space $(n \geq 3)$ with the metric (1.2)a). It is a locally Minkowski space if and only if $b_{j;i}$ and R_{hijk} are written in the forms (3.4) and (2.4) respectively, where $f_i(x) = (6c_1)^{-1}\partial_i\{log\ b^2/(c_1 + c_2b^2)\}$ and $B_j{}^k{}_i$ is given by (3.3).*

4. A locally Minkowski space in case of $m = 4$

Next we consider quartic metric form (1.2)b):

$$L^4 = c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4, \tag{4.1}$$

where c_1, c_2 and c_3 are non-zero constants.

If $D = c_2^2 - 4c_1c_3 = 0$, then (4.1) is reduced to $L^2 = a\alpha^2 + b\beta^2$ for arbitrary constants a and b. In this case the metric $L(\alpha, \beta)$ is a Riemannian metric. Hence we shall treat the non-Riemannian space afterward and assume that $D \neq 0$.

From (4.1), the equation (2.2) gives

$$(2c_1P_{i00} - c_2\beta Q_{i0})\alpha^2 + (c_2P_{i00} - 2c_3\beta Q_{i0})\beta^2 = 0. \tag{4.2}$$

Assuming that F^n be a Berwald space, then there exists the covariant vector $\lambda_i(x)$ such that

$$(4.3) \qquad \begin{aligned} &a)\ c_2P_{i00} - 2c_3\beta Q_{i0} = \alpha^2\lambda_i, \\ &b)\ 2c_1P_{i00} - c_2\beta Q_{i0} = -\beta^2\lambda_i. \end{aligned}$$

From (4.3) we have

$$(4.4) \quad a)\ P_{i00} = \lambda_i(c_2\alpha^2 + 2c_3\beta^2)/D, \ b)\ \beta Q_{i0} = \lambda_i(2c_1\alpha^2 + c_2\beta^2)/D.$$

Differentiating (4.4)a) by y and using the Christoffel processing, we get

$$B_j{}^k{}_i = (\phi_j^k\lambda_i + \phi_i^k\lambda_j - \phi_{ji}\lambda^k)/D, \tag{4.5}$$

where $\phi_j^k = c_2\delta_j^k + 2c_3 b^k b_j$ and $\phi_{ji} = a_{ir}\phi_j^r$.

Next, differentiating (4.4)b) by y we have

$$b_h Q_{ij} + b_j Q_{ih} = 2\lambda_i(2c_1 a_{hj} + c_2 b_h b_j)/D,$$

which is written as

$$(4.6)\quad b_h b_{j;i} + b_j b_{h;i} + B_j{}^k{}_i b_k b_h + B_h{}^k{}_i b_k b_j = 2\lambda_i(2c_1 a_{hj} + c_2 b_h b_j)/D.$$

Substituting (4.5) into (4.6) and contracting this by $b^h b^j$, we obtain

$$(4.7)\qquad b^h b_{h;i} = 2\lambda_i(c_1 - c_3 b^4)/D.$$

Similarly, contracting (4.6) a^{hj} we have $b^h b_{h;i} = 2\lambda_i(nc_1 - c_3 b^4)/D$. Combining this and (4.7) we have $(n-1)c_1\lambda_i = 0$, which implies $\lambda_i = 0$. Hence, from (4.5) and (4.6) we get $B_j{}^k{}_i = 0$ and $b_{j;i} = 0$.

Conversely if $b_{i;j} = 0$, then F^n with (α, β)-metric is a Berwald space. From (2.1), consequently we have

THEOREM 4.1. *Let F^n be an n-dimensional Finsler space ($n \geq 3$) with the metric* (1.2)b). *It is a Berwald space if and only if $b_{j;i} = 0$, and then $B\Gamma = (\gamma_j{}^k{}_i, \gamma_0{}^k{}_i, 0)$.*

In the case of $B_j{}^k{}_i = 0$, from (2.4) we obtain $R_h{}^i{}_{jk} = 0$. Summarizing up the above results and using Theorem 2.1, we have

THEOREM 4.2. *Let F^n be an n-dimensional Finsler space ($n \geq 3$) with the metric* (1.2)b). *It is a locally Minkowski space if and only if $R_h{}^i{}_{jk} = 0$ and $b_{j;i} = 0$.*

On the other hand, for a function $\sigma(x)$ a conformal change [1] of (α, β)-metric is expressed as $(\alpha, \beta) \to (\bar{\alpha}, \bar{\beta})$ where $\bar{\alpha} = e^\sigma \alpha$, $\bar{\beta} = e^\sigma \beta$. A Finsler space is called *conformally flat*, if it is conformal to a locally Minkowski space. In previous papers ([2], [4], [7]), the authors dealt with conformally flat spaces.

For an (α, β)-metric, a conformally invariant symmetric linear connection $M_j{}^i{}_k$ is defined by [2]

$$M_j{}^i{}_k = \gamma_j{}^i{}_k + \delta_j^i M_k + \delta_k^i M_j - M^i a_{jk},$$

where $M_j = \{b_{j;k}b^k - b^k{}_{;k} b_j/(n-1)\}/b^2, M^i = a^{ij}M_j$.

We denote by $\overset{m}{\nabla}$ and $M_h{}^i{}_{jk}$ the covariant differentiation with respect to $M_j{}^i{}_k$ and the curvature tensor of this connection respectively. A Finsler space with an (α, β)-metric is called *flat-parallel*, if $R_h{}^i{}_{jk} = 0$ and $b_{i;j} = 0$.

THEOREM 4.3 [4]. *A Finsler space with (α, β)-metric is conformal to a flat-parallel Minkowski space if and only if the condition*

$$M_h{}^i{}_{jk} = 0, \overset{m}{\nabla}_j M_i = \overset{m}{\nabla}_i M_j, \overset{m}{\nabla}_j b_i = -b_i M_j \tag{4.8}$$

is satisfied.

In an (α, β)-metric, a conformal change preserves the type of metric invariant. From Theorem 4.2, we can see that F^n with the metric (1.2)b) is flat-parallel. Thus these conditions are also applicable to the metric (1.2)b). Consequently, from Theorem 4.3 we have

THEOREM 4.4. *Let F^n be an n-dimensional Finsler space ($n \geq 3$) with the metric* (1.2)b). *It is conformally flat if and only if the condition* (4.8) *is satisfied.*

Whenever we find the condition that $F^n = (M^n, L(\alpha, \beta))$ be a locally Minkowski space, we have necessarily two types. One is a Randers type, that is, flat-parallel, and the other is a Kropina type ([1], [4]). Hence we can construct the following.

REMARK. We have obtained two interesting conditions: 1) F^n has a Kropina type in case of $m = 3$, and 2) F^n has a Randers type in case of $m = 4$. Further, on account of Theorem 3.2 it is observed that a F^n with the metric (1.2)a) is not necessarily conformal to a flat-parallel Minkowski space, even if it is conformally flat.

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