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On the Evaluation of a Vortex-Related Definite Trigonometric Integral

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ABSTRACT: *Using the solution to the contour integral of the complex logarithmic function* $\oint_C \ln(z-z_0)dz$ *, the following definite integral, derived from the formula to calculate the forces exerted to a circular cylinder by the discrete vortices shed from it, has been evaluated* $\int_0^\pi \left[\arcsin\frac{\sin\alpha}{\sqrt{1+a^2-2a\cos\alpha}}\right]\sin\alpha\, d\alpha = \begin{cases}\pi+\frac{1}{2}\pi a & when\ \ |a|<1\\ \frac{\pi}{2|a|} & when\ \ |a|>1\end{cases}$

1. Introduction

When there is a relative motion between a solid body and water, vortex shedding from the body always occurs. The lift and drag, resulting from this shedding of the vortex are common, but extremely important, effects observed on offshore structures. If the vortex shedding is simulated by the discrete vortex method, these forces can be conveniently calculated from Sarpkaya's formula (Sarpkaya, 1963). This force formula demands some mathematical insight and judgment, and as such, has attracted further research interest (Lee, 1990).

The discrete vortex method is a substitute for the actual viscous rotational flow by the inviscid flow with imbedded vortices. Due to the cylinder boundary condition, these outside vortices have their images within the cylinder. Thus, vortices exist not only in the actual flow field, but also inside the cylinder. In this theoretical model, the solid cylinder is replaced by the fluid, endowing its boundary with a special meaning of zero normal velocity. Under this situation, the force formula can be derived with the help of the extended Blasius theorem (Milne-Thomson, 1960).

While using the Milne-Thomson's extension of the Blasius theorem to calculate forces exerted on a circular cylinder that is placed in a flow of ideal fluid with imbedded vortices, it becomes necessary to undertake the integration of a complex logarithm around the unit circle C:

$$\oint_C \ln(z-z_0)dz \tag{1}$$

z_0 representing the position of a vortex. In dealing with this integral of a complex logarithmic function, two nonzero real integrals are present:

$$\int_0^\pi [\ln\sqrt{1+a^2-2a\cos\alpha}]\cos\alpha\, d\alpha$$

and

$$I \equiv \int_0^\pi \left[\arcsin\frac{\sin\alpha}{\sqrt{1+a^2-2a\cos\alpha}}\right]\sin\alpha\, d\alpha \tag{2}$$

The first of these integrals is known and can be found in integral tables, but the second one has not yet been investigated. In the present study, using the result of integral (1), this second integral was evaluated, for some given real number a $\neq$ 1 and for the values of arcsin interpreted in equation (3), as detailed below.

2. Integration

Let $z=e^{i\theta}$ and $z_0=ae^{i\theta_0}$ with $\alpha=\theta-\theta_0$, and let r and β be those denoted in Figures 1-4. Then we have

$$z-z_0=\begin{cases}re^{i(\theta_0+\beta)} & : a<1\\ re^{i(\theta_0+\pi-\beta)} & : a>1,\end{cases}$$

$$r\sin\beta=\sin\alpha,$$

and $r^2=1+a^2-2a\cos\alpha$
which implies that

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$$\beta = \arcsin \frac{\sin \alpha}{\sqrt{1+a^2-2a\cos\alpha}} \tag{3}$$

where $\beta \in \begin{cases} (-\pi, \pi) & : |a| < 1 \\ (-\frac{\pi}{2}, \frac{\pi}{2}) & : |a| > 1 \end{cases}$

for any $\alpha \in (-\pi, \pi)$

When $|a| > 1$, the contour integral (1) obviously vanishes, since there is no singular point within C. To evaluate the integral when $|a| < 1$, take a branch cut at $\theta = \theta_0 + \pi$. Now,

$$\oint_C \ln(z-z_0)dz \tag{4}$$

$$= [(e^{i\theta}-z_0)\ln(e^{i\theta}-z_0)-(e^{i\theta}-z_0)]_{\theta=\theta_0-\pi}^{\theta=\theta_0+\pi}$$

$$= [(e^{i\theta}-z_0)(\ln r(\theta)+i(\beta(\theta)+\theta_0+2k\pi)) - (e^{i\theta}-z_0)]_{\theta=\theta_0-\pi}^{\theta=\theta_0+\pi}$$

$$= -2\pi(1+a)ie^{i\theta_0}$$

because $\ln(re^{i\theta}) = \ln r + i(\theta+2k\pi)$ where k is an arbitrary integer and, by examination of Fig. 1 and Fig 2,

$$r(\theta_0+\pi) = r(\theta_0-\pi)$$

and $\beta(\theta_0+\pi) = -\beta(\theta_0-\pi) = \pi$.

Also, for $a < 1$

$$\oint_C \ln(z-z_0)dz$$

$$= \int_{-\pi}^{\pi} [\ln r + i(\beta+\theta_0+2k\pi)]ie^{i\theta_0}e^{i\alpha}d\alpha \tag{5a}$$

$$= e^{i\theta_0}\int_{-\pi}^{\pi} (i[(\ln r)\cos\alpha - \beta\sin\alpha] - [(\ln r)\sin\alpha + \cos\alpha])d\alpha,$$

and for $a > 1$

$$\oint_C \ln(z-z_0)dz$$

$$= \int_{-\pi}^{\pi} [\ln r + i(\theta_0+\pi-\beta+2k\pi)]ie^{i\theta_0}e^{i\alpha}d\alpha \tag{5b}$$

$$= e^{i\theta_0}\int_{-\pi}^{\pi} (i[(\ln r)\cos\alpha + \beta\sin\alpha] - [(\ln r)\sin\alpha - \beta\cos\alpha])d\alpha.$$

Examining each term of these integrals individually, we find that, since $r(\alpha)$ is an even function of α,

$$\int_{-\pi}^{\pi} (\ln r)\cos\alpha\, d\alpha$$

$$= 2\int_0^{\pi} (\ln r)\cos\alpha\, d\alpha$$

$$= \int_0^{\pi} \ln(1+a^2-2a\cos\alpha)\cos\alpha\, d\alpha$$

$$= \begin{cases} -a\pi & : |a| < 1 \\ -\pi/a & : |a| > 1 \end{cases}$$

after use has been made of the result 4.397(6) of Gradshteyn et al(Gradshteyn, et al., 1994). Also, making use of the odd nature of the function β with respect to α, we find

$$\int_{-\pi}^{\pi} \beta \sin\alpha\, d\alpha = 2\int_0^{\pi} \beta\sin\alpha\, d\alpha$$

$$= 2I$$

where I was defined previously in equation (2). Similarly, due to the odd integrands

$$\int_{-\pi}^{\pi} (\ln r)\sin\alpha\, d\alpha = 0,$$

and $\int_{-\pi}^{\pi} \beta\cos\alpha\, d\alpha = 0.$

So, returning to equation (5) and equating it when $|a| < 1$ to equation (4), and when $|a| > 1$ to zero, we arrive at the results

$$-ie^{i\theta_0}(a\pi+2I) = -2\pi i(1+a)e^{i\theta_0} \quad : |a| < 1$$

$$-ie^{i\theta_0}(-\frac{\pi}{a}+2I) = 0 \quad : a < -1$$

$$-ie^{i\theta_0}(\frac{\pi}{a}-2I) = 0 \quad : a > 1$$

giving

$$I = \begin{cases} \pi + \frac{1}{2}a\pi & : |a| < 1 \\ \frac{\pi}{2|a|} & : |a| > 1. \end{cases}$$

That is

$$\int_0^{\pi} [\arcsin \frac{\sin\alpha}{\sqrt{1+a^2-2a\cos\alpha}}]\sin\alpha\, d\alpha$$

$$= \begin{cases} \pi + \frac{1}{2}\pi a & : |a| < 1 \\ \frac{\pi}{2|a|} & : |a| > 1 \end{cases}$$

3. Further Discussion

It is to be noted that the proposed integral is discontinuous at $a=1$, which is a rather unfortunate, but inevitable, aspect of the problem, as the integral is indeterminate that point. The reason is that the integrand(i.e., β in the present notation), *itself*, is discontinuous at $a=1$, which can be recognized with help of the Fig. 1-4.

In the present integration, the branch cut was located at $a=\pi$ as explicitly specified in the equation (3) and shown in the Fig. 1-4. Based on these figures, the range of β is dependent upon that of a, which, in turn, makes the integral dependent upon the employed branch cut. More specifically, the integral has different values from the present one if the branch cut was chosen at $a=2\pi$. This, again, is an unfortunate aspect of the problem, but such is the nature of an integral of any multivalued function. However, the branch cut, employed in the present study, is the preferred choice for most users.

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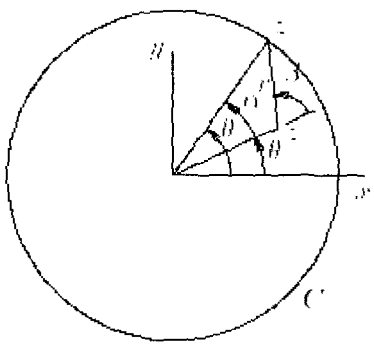

Fig. 1 Notations when $0 < a < 1$

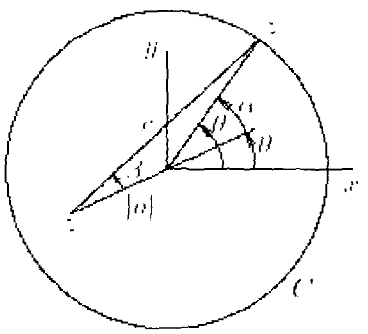

Fig. 2 Notations when $-1 < a < 0$

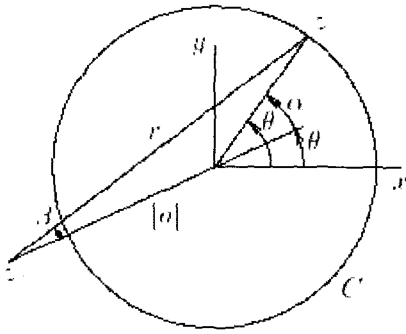

Fig. 3 Notations when $a < -1$

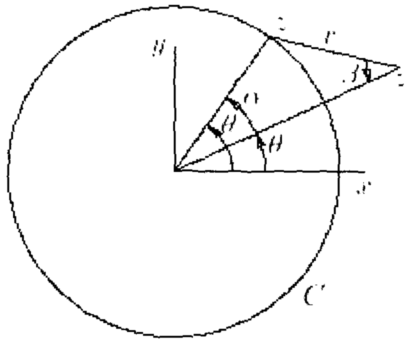

Fig. 4 Notations when $a > 1$

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