Robust Controller Design Method for Systems with Parametric Uncertainties

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Abstract: This paper presents iterative schemes and continuation schemes for designing robust controllers which stabilize dynamic systems having bounded parametric uncertainties. Utilizing results of the cheap control problem, some existence conditions of the robust controller are obtained, which are different from the matching conditions. Continuation schemes are used to overcome the divergence problem of iterative schemes. The robust controller design method is extended to nonlinear system and easily implementable series solution is also obtained. Results are illustrated with simple examples.

Keywords: optimal control, robust control, linear system, cheap control, nonlinear system, series solution

I. Introduction

There has been great interest recently in the influence of uncertainty in model parameters on the robustness of various process control strategies. There are many cases where parameters may be changing during operations and their exact values are difficult to measure on-line. In order to preserve closed-loop stability under parameter variations robust controllers can be designed if bounds on the parameter variations are known.

Many methods to obtain robust controllers have been reported. They are classified into three categories(Siljak, 1989): frequency domain methods, algebraic methods by the analysis of characteristic polynomials, and methods by the Lyapunov stability theorem. Frequency domain methods extend the classical methods for SISO systems based on gain and phase margins. They are now well-established and very powerful for linear multivariable uncertain systems (Dorato, 1987). Algebraic methods extract the robustness from the characteristic polynomial; recent progress in mathematical programming methods and computer graphics make the methods more attractive and promising (Siljak, 1989). Lyapunov methods have long been used in analyzing stability and robustness of control system since the work of Kalman and Bertram (1960). They directly use the state equations and unlike the other two approaches mentioned above, they can easily be applied to nonlinear systems.

Using the Lyapunov method, Leitmann (1979) obtained a robust controller for a class of uncertain systems which satisfy some conditions known as the matching conditions. This class of uncertain systems can be stabilized robustly for an arbitrarily large uncertainty bound. Other researchers have characterized and enlarged this class of uncertain systems (see references in Corless and Leitmann, 1988). Some results for nonlinear systems have been also reported, for example, a cone type of nonlinear perturbation (Noldus, 1982) and some nonlinear systems which are linearizable by the differential geometry method (Ha and Gilbert, 1987; Kravaris and Palanki, 1988). The Lyapunov method is very powerful but is highly dependent on the clever choice of a Lyapunov function which may sometimes be difficult to construct (Vannelli and Vidyasger, 1985). Chang and Peng (1972) proposed a robust controller design method known as guaranteed-cost control. It is based on the optimal control method, and uses a suboptimal cost as the Lyapunov function. In this method, an upper bound of the cost is obtained for bounded structured uncertainties (Rissanen, 1966). For linear systems, they obtained a robust controller by solving Riccati equations iteratively. Kosmidou and Bertrand (1989) proposed a different iterative method whose convergence is guaranteed. Peterson and Hollot (1986) proposed a robust controller design method which was also based on the Riccati equation but was non-iterative.

In this paper we further exploit the Kosmidou and Bertrand method and the Chang and Peng method. In section II, we have obtained a new convergent iterative scheme for the Kosmidou and Bertrand method and have shown that existence of the linear robust control law is closely related to the cheap control problem. We also propose a continuation scheme to find the maximum allowable perturbation. In section III, nonlinear uncertain systems have been considered and a nonlinear robust control law, which is in series form, has been obtained. In section IV, for a low gain robust control law, the Chang and Peng method has been treated.

II. Linear uncertain system

Consider the linear dynamic system:

$$\dot{x}(t) = (A + qD)x(t) + Bu(t) \tag{1}$$

where $x \in R^n$, $u \in R^m$, q is an unknown constant and A, D and B are known matrices with appropriate dimensions. The problem is to find a feedback control law which stabilizes the system (1) for any $q \in [-q_M, q_M]$ while not significantly degrading the cost functional:

$$J(x(t_0), u(\cdot)) = \int_{t_0}^{\infty} (x^T Qx + u^T Ru)dt$$

where $R > 0$ and $Q \geq 0$.

Chang and Peng (1972) formulated this problem as the guaranteed cost control problem and obtained a robust control law:

$$u(x) = -R^{-1}B^T Px. \tag{2}$$

Here P is the positive definite solution of

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$$PA + A^T P - PBR^{-1}B^T P + Q + \Phi(P, D) = 0 \quad (3)$$

where for any $q \in [-q_M, q_M]$,

$$q(PD + D^T P) \le \Phi(P, D).$$

Three candidates for $\Phi(P,D)$ have been proposed:

(a) (Chang and Peng, 1972)

$$\Phi(P,D) = q_M S^{-1} diag\{|\lambda_i(PD + D^T P)|\}S \quad (4)$$

where S is a diagonalization transformation matrix such that

$$PD + D^T P = S^{-1} diag\{\lambda_i(PD + D^T P)\}S$$

(b) (Peterson and Hollot, 1986)

$$\Phi(P,D) = \delta PVV^T P + \frac{q_M^2}{\delta} W^T W \quad (5)$$

where $D=VW$ and for some positive constant δ.

(c) (Kosmidou and Bertrand, 1988)

$$\Phi(P,D) = \delta P + \frac{q_M^2}{\delta} D^T PD. \quad (6)$$

Equation (4)was originally proposed by Chang and Peng (1972). They obtained a robust control law by solving the resulting Riccati equation iteratively. The convergence of the iterative scheme was not proven due to the nonlinearity of equation (4) about P. Equation (5) results in

$$PA + A^T P - P(BR^{-1}B^T - \delta VV^T)P + Q + \frac{q_M^2}{\delta} W^T W = 0.$$

If $BR^{-1}B^T - \delta VV^T > 0$, it has a positive definite solution and the control law (2) becomes a robust control law. More elaborate results were given by Peterson and Hollot (1986). Equation (6) was used by Kosmidou and Bertrand (1988). They proposed a convergent iterative scheme for the resulting Riccati equation. Their iterative scheme is simple but it is somewhat hard to find an initial value for which convergence is guaranteed.

Here we propose a new convergent iterative scheme for the Riccati equation (3) with (6):

$$PA + A^T P - PBR^{-1}B^T P + Q + \delta P + \frac{q_M^2}{\delta} D^T PD = 0 \quad (7)$$

and by applying the cheap control theorem to the iterative scheme, existence conditions of a positive definite solution of (7) are exploited.

1 Iterative scheme

To solve equation (7), we propose an iterative scheme:

$$\begin{aligned} &P_{k+1}A_\delta + A_\delta^T P_{k+1} - P_{k+1}BR^{-1}B^T P_{k+1} \\ &+ Q + \frac{q_M^2}{\delta} D^T P_k D = 0 \\ &P_0 = 0 \end{aligned} \quad (8)$$

where $A_\delta = (A + \frac{\delta}{2} I)$.

Lemma 1: Assume that (A_δ, B) is stabilizable. The sequence P_k is non-decreasing.

Proof: Since $D^T P_a D \le D^T P_b D$ for $P_a \le P_b$, the above lemma follows from Theorem 1 of Wimmer (1985) ■

Theorem 1: Assume that (A_δ, B) is stabilizable. The iterative scheme (8) converges if and only if a positive definite solution of equation (7) exists.

Proof: If a positive definite solution P exist, the sequence P_k is bounded above by the solution as $0 \le P_1 \le \cdots \le P$. Hence it converges to a positive definite solution. Necessary part is self-evident. ■

The above theorem shows that we can obtain a positive definite solution of (7) if such a solution exists. That is, we can obtain a stabilizing controller (Plubelle et al., 1986) by the iterative scheme (8). Here some existence results for equation (7) are sought. That is, using results on the cheap control problem (Sannuti, 1983; Saberi and Sannuti, 1989), a bound of the sequence P_k is exploited. Since P_k is non-decreasing, an upper bound guarantees its convergence.

Theorem 2: Let $D=VW$. The iterative scheme (8) is convergent for $q_M \in (0, \alpha_1)$ where

$$\alpha_1 = 1\Big/\sqrt{\sigma_M(V^T P_\varepsilon V)} \quad (9)$$

and P_ε is the positive definite solution of

$$\begin{aligned} &P_\varepsilon A_\delta + A_\delta^T P_\varepsilon - \frac{1}{\varepsilon^2} P_\varepsilon BR^{-1}B^T P_\varepsilon \\ &+ \varepsilon^2 Q + \frac{1}{\delta} W^T W = 0 \end{aligned} \quad (10)$$

where, the value of ε means some positive constant.

Proof: Let $P_\varepsilon = \varepsilon^2 P$. Then equation (10) becomes

$$PA_\delta + A_\delta^T P - PBR^{-1}B^T P + Q + \frac{1}{\delta\varepsilon^2} W^T W = 0$$

The equality (9) means that $q_M^2 D^T PD \le \frac{1}{\varepsilon^2} W^T W$. If $q_M^2 D^T P_k D \le \frac{1}{\varepsilon^2} W^T W$, $P_{k+1} \le P$ and $q_M^2 D^T P_{k+1} D \le \frac{1}{\varepsilon^2} W^T W$. Since $q_M^2 D^T P_0 D \le \frac{1}{\varepsilon^2} W^T W$, $P_k \le P$ for all k. That is, the sequence P_k is bounded by $\varepsilon^{-2} P_\varepsilon$. Convergence follows. ■

Corollary 1: Assume that the dynamic system

$$\begin{aligned} \dot{x} &= A_\delta x + Bu \\ y &= Wx \end{aligned} \quad (11)$$

is stabilizable and detectable, and it has no transmission zeros with zero real parts. Then

$$\alpha_1 = 1\Big/\sqrt{\sigma_M(V^T \overline{P}_\varepsilon V + o(\varepsilon^0))}$$

where $\overline{P}_\varepsilon$ is the asymptotic solution of equation (10) as ε goes to zero.

Proof: see Sannute(1983). ■

The equation for $\overline{P}_\varepsilon$ is available in Sannuti (1983). It depends only on the system matrices (A_δ, B, W) of (11), so computation of $\overline{P}_\varepsilon$ is straightforward. However, its analytical description is rather complicated because it requires some magnitude scaling and transformation of states. The above corollary shows that, if $V^T \overline{P}_\varepsilon V = 0$, α_1 grows to infinity as ε goes to zero. The system with such uncertainties can be stabilized for arbitrarily large uncertainty bounds like uncertainties satisfying the matching conditions. For some systems, $V^T \bar{P}_\varepsilon V$ is zero regardless of V.

Corollary 2: If the system (11) is stabilizable, detectable and of minimum phase, then α_1 grows to infinity as ε goes to zero.

Proof: Under the condition, $\bar{P}_\varepsilon = 0$ (Sannuti, 1983). ■

Example 1: Consider a system

$$\dot{x} = \begin{bmatrix} -1.25 & 0 \\ 0 & -1.25 \end{bmatrix} x + q \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$Q=1$ and $R=1$. If one chooses δ between 0 and 2.5, the system satisfies the condition of Corollary 2. Hence the iterative scheme (8) converges regardless of the magnitude of q_M.

The system matrix used in Corollaries 1 and 2 is A_δ. Sometimes its structure is much different from A and is inadequate for the above analysis. Results which can be applied directly to the system matrix A are now sought.

Theorem 3: The iterative scheme (8) is convergent for $q_M \in (0, \alpha_2]$ and $\delta = \mu / \sigma_M(P_\mu)$ where

$$\alpha_2 = \sqrt{\mu / (\sigma_M(P_\mu) \sigma_M(V^T P_\mu V))} \tag{12}$$

and P_μ is the solution of

$$P_\mu A + A^T P_\mu - \frac{1}{\mu^2} P_\mu B R^{-1} B^T P_\mu + \mu^2 Q + \mu I + W^T W = 0$$

Proof: The equality (12) means that $\delta P_\mu + \frac{q_M^2}{\delta} D^T P_\mu D \le \mu I + W^T W$. Hence, as in Theorem 2, $\mu^{-2} P_\mu$ bounds the sequence P_k and convergence follows.

Example 2: Consider a system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x + q \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$R=1$ and $Q = \mathrm{diag}(1,0)$. Then we have

$$P_\mu = \begin{bmatrix} 0(\mu^{1/2}) & 0(\mu^{3/2}) \\ 0(\mu^{3/2}) & 0(\mu^1) \end{bmatrix}.$$

Hence α_2 grows to infinity as μ goes to zero. That is, a solution of (7) can be found by the iterative scheme (8) for an arbitrary large q_M, whenever δ is chosen as in Theorem 3.

Corollary 3: Assume that the system

$$\dot{x} = A_\delta x + Bu$$
$$y = Wx$$

is stabilizable, detectable and of minimum phase. If WB is non-singular, then α_2 grows to infinity as μ goes to zero.

Proof: Under the conditions, $P_\mu = O(\mu)$ (Sannuti, 1983).

As in Peterson and Hollot (1986), in consideration of the above analysis, the part of uncertainty matrix D which satisfies the matching conditions can be removed. That is, if D can be decomposed as $D = BW_a + VW$, then we can choose

$$\Phi(P,D) = \tfrac{1}{2} PBR^{-1}B^T P + 2q_M^2 W_a R W_a + \delta P + \frac{q_M^2}{\delta} W^T V^T P V W$$

and the resulting Riccati equation has the same structure as that of equation (7) for V and W. The above asymptotic analysis can be applied to the part VW of D.

For some cone-bounded nonlinear perturbation such as

$$\dot{x} = As + q f_D(x) + Bu \tag{13}$$

the above results can be extended and a robust linear controller can be obtained.

Theorem 4: If the nonlinear function $f_D(x)$ is bounded as, for any positive definite P,

$$f_D(x)^T P f_D(x) \le x^T D^T P D x \tag{14}$$

and the solution of equation (7) exists, ;then the control law (2) stabilizes the system (13) robustly for any $q \in [-q_M, q_M]$.

Proof: Under the condition (14), $x^T P x$ becomes a Lyapunov function for the system (13) for any $q \in [-q_M, q_M]$. ■

2. Continuation scheme

Sometimes there does not exist a solution P of (7) for the required q_M. In this case we should choose another method or reduce q_M for a robust controller. Here, as a method illustrating the latter approach, a continuation method (Allgower and Georg, 1980) is adopted to find a maximum possible q_M as well as a solution. The continuation method has been used as a globally convergent method for nonlinear equations which are very hard to solve. For example, it has been applied to solve the Riccati equation (Jamshidi et al., 1970) and a root-locus problem (Pan and Chao, 1978).

Assume that P and q_M are functions of a dummy parameter τ and let

$$\Psi(P(\tau), q_M(\tau)) = PA + A^T P - PBR^{-1}B^T P + Q + \delta P + \frac{q_M^2}{\delta} D^T P D \tag{15}$$

We track P satisfying as q_M varies. To do this, we differentiate equation (15) about the dummy parameter τ and equate as

$$\dot{\Psi}(P(\tau), q_M(\tau)) = \dot{P}(A_\delta - BR^{-1}B^T P) + (A_\delta - BR^{-1}B^T P)^T \dot{P} + \frac{q_M^2}{\delta} D^T \dot{P} D + \frac{2 q_M \dot{q}_M}{\delta} D^T P D = -\Psi(P, q_M). \tag{16}$$

Since $\Psi(P(\tau), q_M(\tau)) = e^{-\tau} \Psi(P(0), q_M(0))$, $P(\tau)$ becomes the solution of (7) for $q_M(\tau)$ whenever $P(0)$ is the solution of (7) for $q_M(0)$. By applying the Euler integration rule to the equation (16), we can obtain

$$P_{k+1}(A_\delta - BR^{-1}B^T P_k) + (A_\delta - BR^{-1}B^T P_k)^T P_{k+1} + \frac{q_{M,k}^2}{\delta} D^T P_{k+1} D = -(P_k BR^{-1}B^T P_k + Q + \frac{q_{M,k}\dot{q}_M}{\delta} D^T P_k D) \tag{17a}$$

$$q_{M,k+1} = q_{M,k} + \dot{q}_M. \tag{17b}$$

We solve equation (17a) iteratively with a standard Lyapunov equation solver, that is,

$$P_{k+1,i+1}(A_\delta - BR^{-1}B^T P_k) + (A_\delta - BR^{-1}B^T P_k)^T P_{k+1,i+1} = -(P_k BR^{-1}B^T P_k + Q + \frac{2q_{M,k}\dot{q}_M}{\delta} D^T P_k D) - \frac{2q_{M,k}^2}{\delta} D^T P_{k+1,i} D$$

$$P_{k+1,0} = P_k. \tag{18}$$

Theorem 5: Assume that P_k is positive definite and $(A_\delta - BR^{-1}P_k)$ is stable. The iterative scheme (18) is convergent if and only if equation (17a) has a positive definite solution.

Proof of Theorem 5 is very similar to that of Theorem 1 and omitted here. According to the Theorem 5, we increase q_M to the point just before the iterative scheme (18) diverges

III. Extension to nonlinear systems

In this section, the method of section II is extended to nonlinear systems. Consider a nonlinear system described by

$$\dot{x}(t) = f(x(t)) + qf_D(x(t)) + g(x(t))u(t) \tag{19}$$

with a cost functional:

$$J(x(t_0), u(\bullet)) = \int_{t0}^{\infty} (m(x) + u^T Ru)dt$$

Here $f(0) = f_D(0) = 0$. It is assumed that the above system is controllable and observable for any $q \in [-q_M, q_M]$ (Moylan and Anderson, 1973) and all nonlinear functions are analytic.

A guaranteed cost control law is

$$u(x) = -\tfrac{1}{2}R^{-1}g(x)^T \nabla\phi(x) \tag{20}$$

where $\phi(x)$ is the positive definite solution of following Hamilton-Jacobi equation:

$$\begin{aligned}&\nabla\phi(x)^T f(x) - \tfrac{1}{4}\nabla\phi(x)^T g(x)R^{-1}g(x)^T\nabla\phi(x)\\ &+ m(x) + \Phi(\phi(x), f_D(x)) = 0.\end{aligned} \tag{21}$$

Theorem 6: Assume that the Hamilton-Jacobi equation (21) has a positive definite solution around the origin. Let Ω be a level set of the function $\phi(x)$ such that, for some positive constant v_0, $0 \le \phi(x) \le v_0$, and for any $q \in [-q_M, q_M]$,

$$q\nabla\phi(x)^T f_D(x) \le \Phi(\phi(x), f_D(x)) \tag{22}$$

Then the system (19) with the control law (20) is asymptotically stable in the region Ω such that every trajectory starting in Ω remains in Ω and converges to the origin.

Proof: With in Ω, $\phi(x)$ becomes a Lyapunov function for the system (19) with control law (20). This theorem is followed from the Lyapunov theorem (Chang and Peng, 1972; Hahn, 1967). ■

It is remarked that system (19) with control law (20) remains stable in the case of a 50% reduction in controller gain (Glad, 1987) and hence the true attraction region may be larger than Ω. We can obtain a robust control law by choosing a specific $\Phi(\phi(x), f_D(x))$ and sonving the Hamilton-Jacobi equation (21). If $f_D(x)$ is of the form $f_D(x) = g(x)r(x)$, then we can choose

$$\begin{aligned}&\Phi(\phi(x), f_D(x))\\ &= \frac{\delta}{4}\nabla\phi(x)^T g(x)R^{-1}g(x)^T\nabla\phi(x) + \frac{1}{\delta}r(x)^T Rr(x)\end{aligned}$$

and we have

$$\begin{aligned}&\nabla\phi(x)^T f(x) - \tfrac{1}{4}(1-\delta)\nabla\phi(x)^T g(x)R^{-1}g(x)^T\nabla\phi(x)^T\\ &+ m(x) + \frac{1}{\delta}r(x)^T Rr(x) = 0.\end{aligned} \tag{23}$$

The above equation satisfies condition (22) automatically and may have a positive definite solution around the origin. Furthermore it can be solved via the standard series solution method for the nonlinear optimal regulation problem (Lukes, 1969). This uncertainty is included in the matching conditions.

For general uncertainties, we choose

$$\Phi(\phi(x), f_D(x)) = \tfrac{1}{2}[\delta x^T\nabla\phi(x) + \frac{1}{\delta}f_D(x)^T\nabla\phi(f_D(x))]$$

which is an extension of the linear quadratic problem of section II. Then equation (21) becomes

$$\begin{aligned}&\nabla\phi(x)^T(f(x) + \frac{\delta}{2}x) - \tfrac{1}{4}\nabla\phi(x)^T g(x)R^{-1}g(x)^T\nabla\phi(x)\\ &+ m(x) + \frac{1}{2\delta}f_D(x)^T\nabla\phi(f_D(x)) = 0.\end{aligned} \tag{24}$$

We solve equation (24) by the power series method. For this, we assume that all nonlinear functions are expanded as

$$f(x) = Ax + f^{(2)}(x) + \cdots$$
$$f_D(x) = Dx + f_D^{(2)}(x) + \cdots$$
$$g(x) = B + g^{(1)}(x) + \cdots$$
$$m(x) = x^T Qx + m^{(3)}(x) + \cdots$$

and the linear quadratic problem of equation (7) has a solution P. Under the assumptions that the linear quadratic solution exists, the local power series solution of equation (24) may exist. But its proof would require more elaborate study as in the nonlinear optimal regulation problem (Lukes, 1969).

Assume that the solution of equation (24) is represented as

$$\phi(x) = x^T Px + \phi^{(3)}(x) + \cdots.$$

Then we have the equation (7) for P and

$$\begin{aligned}&\nabla\phi^{(k)}(x)^T(A_\delta - BR^{-1}B^T P)x + \frac{1}{2\delta}x^T D^T\nabla\phi^{(k)}(Dx)\\ &= h^{(k)}(x),\ k = 3,4,\cdots\end{aligned}$$

there $h^{(k)}(x)$ is a function of all known and previously calculated variables.

Theorem 7 : Define $D^{[j-1]}$ such that

$$D^{[j-1]}x^{[j-1]} = (Dx)^{[j-1]}$$

where $x^{[j-1]}$ is a lexio-graphic listing vector for (*j-1*)th order polynomials of x (Brockett, 1976). If $(A_\delta - BR^{-1}B^T P)$ is stable and $\lambda_i(D(A_\delta - BR^{-1}B^T P)^{-1})\lambda_l(D^{[j-1]}) \ne -2\delta$, for all i and l, then the solution of (25)for j exists.

Proof: Let $\nabla\phi^{(j)}(x) = P^{[j-1]}x^{[j-1]}$ (Yoshida and Loparo, 1989). Equation (25) becomes

$$x^T(A_\delta - BR^{-1}B^TP)^TP^{[j-1]}x^{[j-1]} + \frac{1}{2\delta}x^TD^TP^{[J-1]}D^{[J-1]}x^{[J-1]}$$
$$= x^TH^{[j-1]}x^{[j-1]}$$

where $x^TH^{[j-1]}x^{[j-1]} = h^{(j)}(x)$. Hence we have

$$(A_\delta - BR^{-1}B^TP)^TP^{[j-1]} + \frac{1}{2\delta}D^TP^{[J-1]}D^{[J-1]} = H^{[j-1]}.$$

Under the above condition $P^{[j-1]}$ can be calculated (Bartell and stewart, 1972). ■

The above condition is easy to check because it depends only on the linearized terms. Higher order terms can be calculated with some symbolic operation of multivariable polynomials and solving systems of linear equations (Lee, 1990). To use the series of $\phi(x)$, it is truncated at some order and the region of attraction of the truncated control law should be determined.

IV. Reducing control costand magnitude of feedback gain

High feedback gain is not recommended when unmodeled high frequency parasitic dynamic exists. So it would be important to reduce the controller gain while maintaining the robustness. Peterson and Barmish (1987) studied properties of the lowest gain robust controller. But such a controller is very hard to obtain. The robust controller design method of Chang and Peng (1972) sometimes gives a much lower controller gain than that of section II. Here we reformulate their scheme with the matrix square root to obtain some analytical results and more attractive numerical methods.

Equation (3) combined with (4) can be written as

$$PA + A^TP - PBR^{-1}B^TP + Q + U = 0 \qquad (26a)$$

$$U^2 - q_M^2(PD + D^TP)^2 = 0 \qquad (26b)$$

where U is symmetric and positive semi-definite. We solve the equation (26) iteratively as

$$P_{k+1}A + A^TP_{k+1} - P_{k+1}BR^{-1}B^TP_{k+1} + Q$$
$$+ q_M[(P_kD + D^TP_k)^2]^{1/2} = 0,\ P_0 = 0$$

where $[\bullet]^{1/2}$ means the principal matrix square root (see Appendix for calculation by the transformation method). Equation (26b) is much easier to manipulate than the original one of Chang and Peng (1972). But it is still difficult to prove the convergence of its iterative scheme (27) except for a limited class of D. For example, if D is symmetric and orthogonal (that is, $D^2 = I$ and $D = D^T$), then the sequence P_k is non-decreasing. So it will converge if a positive definite solution of (26) exists. Utilizing the solution of equation (7), following bounds for the sequence P_k of iterative scheme (27) are obtained.

Theorem 8: Assume that D is symmetric. If there exist δ such that

$$q_M\sigma_M(D)\sqrt{(\sigma_M^2(P_\delta) + \sigma_m^2(P_1))/\sigma_m^2(P_1)} \le \delta \qquad (28)$$

there P_δ is the solution of (7) and P_1 is the first iteration of (27), then the sequence P_k of (27) is bounded between P_1 and P_δ.

Proof: For any P between $P_1 \le P \le P_\delta$, inequality (28) implies that

$$q_M^2(PD^2P + DP^2D) \le \delta^2P^2 + \frac{q_M^2}{\delta^2}DPD^2PD.$$

Hence we have $q_M^2(PD + D^TP)^2 \le (\delta P + \frac{q_M^2}{\delta}D^TPD)^2$. Therefore, from the inequality about the matrix square root (Bellman, 1965), we have

$$q_M[(PD + D^TP)^2]^{1/2} \le (\delta P + \frac{q_M^2}{\delta}D^TPD).$$

Theorem 8 follows ■

Theorem 8 shows that, for some uncertainties, iteration of (27) is bounded and the solution of (26) is not greater than some P_δ, a solution of (7). Since $\sigma_m(P_1)$ is independent on q_M and $\sigma_M(P_\delta)$ goes to some constant as q_M decreases, we may always find δ which satisfies the condition (28) for sufficiently small q_M. Because we can differentate equation (26b) with respect to q_M and P, we can also apply the continuation method to the equation (26) as

$$P_{k+1}(A_\delta - BR^{-1}B^TP_k) + (A_\delta - BR^{-1}B^TP_k)^TP_{k+1}$$
$$+ P_kBR^{-1}B^TP_k + Q + U_{k+1} = 0$$
$$U_{k+1}U_k + U_kU_{k+1} - q_{M,k}^2[P_{k+1}D + D^TP_{k+1})(P_kD + D^TP_k)$$
$$+ (P_kD + D^TP_k)(P_{k+1}D + D^TP_{k+1})]$$
$$- 2q_{M,k}\dot{q}_M(P_kD + D^TP_k)^2 = 0$$
$$q_{M,k+1} = q_{M,k} + \dot{q}_M.$$

Continuation will be used to prevent convergence problems of the iterative scheme (27) and to find the maximum possible q_M. The method can be also extended to treat some nonlinear perturbations. We will illustrate it with a simple example in the following section.

V. Examples

Here we compare schemes by applying them to the well-known Van der Pol oscillator problem.

Example 3: Consider the dynamical system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_2 - qx_1^2x_2 + u$$

with a cost functional

$$J(x(t_0), u(\bullet)) = \int_{t_0}^{\infty}(x_1^2 + u^2)dt.$$

Let $q_M = 1$. With restricting x_1 as $|x_1| \le 1$, we can obtain three linear robust control laws as

(from equation (4))

$$P = \begin{bmatrix} 5.1535 & .42957 \\ .42957 & 4.2044 \end{bmatrix}$$
$$u = -(.42957x_1 + 4.2044x_2)$$

(from equation (5))

$$P = \begin{bmatrix} 5.2480 & .42705 \\ .42705 & 4.2300 \end{bmatrix}$$

$$u = -(.42705x_1 + 4.2300x_2),\ for\ \delta = 0.2$$

(from equation (6))

$$P = \begin{bmatrix} 12.219 & 2.0547 \\ 2.0547 & 5.0762 \end{bmatrix}$$

$$u = -(2.0547x_1 + 5.0762x_2),\ for\ \delta = 0.6 \cdot$$

For this problem the cost and feedback control gain obtained from the equation (6) is larger than those of the other two.

Two nonlinear robust control laws are computed as (from equation (23))

$$\phi(x) = 4.9491x_1^2 + .89898x_1x_2 + 4.4079x_2^2 + .16600x_1^6 + 0.x_1^5x_2 + .40661x_1^4x_2^2 + .27786x_1^3x_2^3 + .12714x_1^2x_2^4 + .03614x_1x_2^5 + .00500x_2^6 + \cdots$$

$$u(x) = -(.11949x_1 + 4.4079x_2 + 0.x_1^5 + .40661x_1^4x_2 + .41679x_1^3x_2^2 + .25429x_1^2x_2^3 + .09034x_1x_2^4 + .01501x_2^5 + \cdots),\ for\ \delta = 0.5.$$

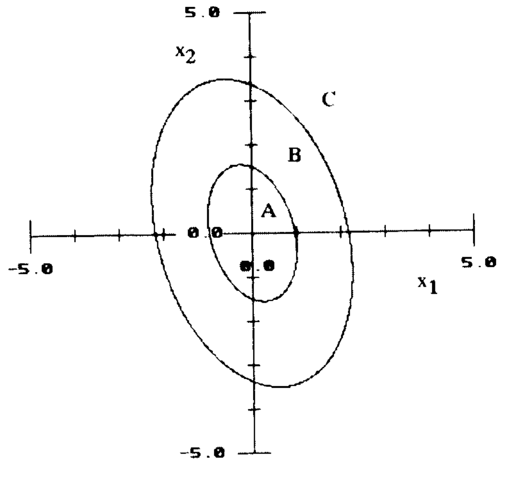

Fig. 1. Attraction regions for the system of Example 3. a) for the linear robust controller of equation (6) with 50% feedback gain reduction tolerance, b) for the linear robust controller of equation (6), c) for the nonlinear robust controller of equation (24) (truncated at the 5-th order).

(from equation (24))

$$\phi(x) = 5.1602x_1^2 + 2.2802x_1x_2 + 3.2103x_2^2 + 1.4458x_1^6 + 1.0133x_1^5x_2 + 1.8585x_1^4x_2^2 + .94377x_1^3x_2^3 + .30275x_1^2x_2^4 + .05689x_1x_2^5 + .00484x_2^6 + \cdots$$

$$u(x) = -(1.1401x_1 + 3.2103x_2 + .50667x_1^5 + 1.8585x_1^4x_2 + 1.4157x_1^3x_2^2 + .60550x_1^2x_2^3 + .14223x_1x_2^4 + .01451x_2^5 + \cdots),\ for\ \delta = 0.6 .$$

Attraction regions for the linear controller of equation (6) and the nonlinear controller of equation (24) (truncated at the fifth order) are shown in Fig. 1. For this problem, the nonlinear controller has much larger attraction region and gives lower control cost than that of the linear controller.

Example 4: Consider the dynamical system (Glad, 1987)

$$\dot{x}_1 = x_2 + qx_1^2$$
$$\dot{x}_2 = -x_2 + u$$

with a cost functional

$$J(x(t_0), u(\bullet)) = \int_{t_0}^{\infty} (x_1^2 + u^2)dt \cdot$$

Let $q_M = 1$. Restricting x_1 as $|x_1| \leq 1$, we can obtain a linear robust control law from equation (6) as

$$P = \begin{bmatrix} 19.725 & 6.36 \\ 6.36 & 3.1014 \end{bmatrix}$$

$$u = -(6.36x_1 + 3.1014x_2),\ for\ \delta = 1.0.$$

The nonlinear robust control law is computed from equation (24) as

$$\phi(x) = 6.4641x_1^2 + 7.4641x_1x_2 + 2.73205x_2^2 + 8.71474x_1^4 + 10.2065x_1^3x_2 + 5.03661x_1^2x_2^2 + 1.17648x_1^1x_2^3 + 0.107655x_2^4 + \cdots .$$

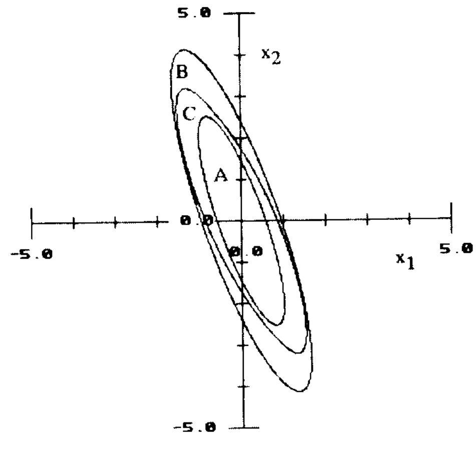

Fig. 2. Attraction regions for the system of Example 4. a) for the linear robust controller of equation (6) with 50% feedback gain reduction tolerance, b) for the linear robust controller of equation (6), c) for the nonlinear robust controller of equation (24) (truncated at the 3-rd order).

$$u(x) = -(3.73205x_1 + 2.73205x_2 + 5.10323x_1^3 + 5.03661x_1^2x_2 + 1.76472x_1^1x_2^2 + .215311x_2^3 + \cdots),\ for\ \delta = 2.0.$$

Attraction regions for the above controllers are shown in Fig. 2. For this problem, the nonlinear controller has a smaller attraction region but gives a much lower control cost than that of the linear controller.

VI. Conclusions

Guaranteed-cost control methods of Chang and Peng (1972) and Kosmidou and Bertrand (1988) have been re-examined for designing a robust controller. We obtained a new convergent iterative scheme modifying the iterative scheme of Kosmidou and Bertrand. By applying cheap control theorems to this scheme, we obtained some explicit results about existence of the robust controller. In particular we obtained systems which can be stabilized for an arbitrarily large uncertainty bound, which are different from those of the matching conditions. Since the cheap control theorems used are based on earlier papers (Sannuti, 1983), our results can be improved by more recent results about cheap control (Saberi and Sannuti, 1989). A continuation scheme was also proposed, which was especially useful for finding the maximum possible uncertainty bound as well as the robust controller. The method was extended to nonlinear systems and an easily implementable series solution was obtained.

Convergence of the iterative scheme of Chang and Peng (1972) is hard to prove. However, their method is still useful for some systems which contain unmodeled high frequency parasitic dynamics or suffer from stochastic noise, because it sometimes gives much lower feedback gains. The scheme was reformulated so that some analytical results and more reliable numerical methods including the continuation method could be applied.

The Lyapunov equations and the Riccati equations were solved by the Schur transformation method (Bartell and Stewart, 1972; Laub, 1979). These may also be solved by the reliable matrix sign function method (Roberts, 1980; Bierman, 1984; Shieh et al., 1987) without rather lengthy programs to find Schur vectors. A general program for the power series solution of the nonlinear problems was developed with subroutines using symbolic algebraic manipulation of multivariable polynomials (Lee, 1990). All programs were written in FORTRAN. For low dimensional systems, we obtained robust controllers very efficiently. However, a fast and reliable method for solving the equation

$$PA + A^T P + D^T PD + Q = 0$$

would be required to speed up the calculations for large dimensional systems.

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Notations

$\lambda_i(\bullet)$: eigenvalue of a matrix.

$\sigma_m(\bullet)$: minimum singular value (minimum absolute eigenvalue) of a matrix (Kailath, 1980).

$\sigma_M(\bullet)$: maximum singular value(maximum absolute eigenvalue) of a matrix

$O(\bullet)$: order of large O. For example, $P = O(\varepsilon)$ means that P/ε goes to some matrix as ε goes to zero

$o(\bullet)$: order of small o. For example, $P = o(\varepsilon)$ means that P/ε goes to zero as ε goes to zero

$diag(\bullet)$: diagonal matrix $P < Q(P \le Q)$: $P-Q$ is positive definite (semi-definite)

A_δ : $A + \delta/2I$

$\nabla\phi(x)$: gradient of $\phi(x)$ about x.

$h^{(j)}(x)$: j-th order terms in the power series expansion of an analytic function $h(x)$

$x^{[j]}$: lexico-graphic listing vector for multivariable polynomials of x (Brockett, 1976; Yoshida and Loparo, 1989)

$D^{[j]}$: a matrix such that $D^{[j]}x^{[j]} = (Dx)^{[j]}$.

$(\bullet)^{1/2}$: principal square root of a matrix (Bellman, 1960)

Appendix

(Principal square root of a positive semi-definite matrix)

We calculate the principal square root of a given positive semi-definite matrix A by the Schur transformation method. Since A and $B = A^{1/2}$ have the same eigen structure (Bellman, 1965), We have

$$\widetilde{B}^2 = \widetilde{A}$$

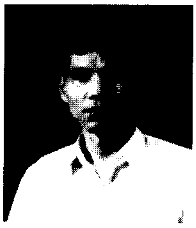
where $\widetilde{A} = SAS^{-1}$ and $\widetilde{B} = SBS^{-1}$. If a transformation S is chosen as Schur vectors, $\widetilde{A}$ and $\widetilde{B}$ become upper triangular matrices and the upper triangular matrix $\widetilde{B}$ can be calculated as

$$\widetilde{b}_{ii} = (\widetilde{a}_{ii})^{1/2},\ i = 1,2,\cdots,n$$

for $k=1$ to $n-1$

$$\widetilde{b}_{ii+k} = \left(\widetilde{a}_{ii+k} - \sum_{j=i+1}^{i+k-1}\widetilde{b}_{ij}\widetilde{b}_{ji+k}\right) \Big/ (\widetilde{b}_{ii} + \widetilde{b}_{i+k\,i+k}),\quad i = 1,2,\cdots,n-k.$$

Because B satisfies that $B^2 = A$ and the eigenvalues of B are all non-negative real values, B becomes the principal square root of A. All matrices used are real, because A is symmetric and positive semi-definite.



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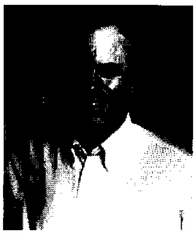

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