Robust Pole Assignment in a Specified Disk

Van Giap Nguyen, Tan Tien Nguyen, Gun You Lee, and Sang Bong Kim

Abstract: This paper presents a method to assign robustly the closed loop system's poles in a specified disk by a state feedback for a linear time invariant system with structured or unstructured uncertainties. The proposed robust design procedure includes two steps. Firstly, the perturbed closed loop matrix $A_{clp} = A_{cl} + \triangle A_{cl}$ is rearranged such that it is a function of the nominal closed loop matrix A_{cl}. Hence, we can control the positions of the perturbed closed loop poles by choosing A_{cl} appropriately. Secondly, the feedback control law F that assigns the closed loop poles of the perturbed system in a specified disk is determined from the equation $A_{cl}=A+BF$. A procedure for finding F is proposed based on partitioning every matrix of the equation $A_{cl}=A+BF$ in the horizontal direction.

Keywords: robust pole assignment, structured and unstructured perturbations, decomposition

I. Introduction

In the last few years, many efforts have been done to design a state feedback controller which assigns all the closed loop poles in a specified region for a linear time invariant system with perturbations. This type of a pole assignment problem arises frequently when a good model of the plant is available, but uncertainties exist with respect to the parameters which may be changed during operation or are unknown or difficult to measure. Friction coefficient, inertia, mass, spring constant, reaction rate, and aerodynamic coefficient etc. are common examples of such parameters. The controller must preserve the closed loop pole locations in a specified region for known ranges of parameter excursions.

Among different ways for realizing the above problem, one of the most popular is robust pole assignment in a specified region [1]-[6]. Furuta and Kim [1] proposed a design method for assigning the closed loop poles in a specified disk based on gain and phase margins which is named D-stability margin. They considered that case, when the perturbations are unknown gains as a diagonal form. Figueroa and Romagnoli [2] presented a method for designing controllers which attempt to place the roots of a characteristic polynomial of an uncertain system inside some prescribed regions. The analysis is based on a transfer function of a characteristic polynomial. In [3], another pole assignment method working with a spectral radius and a pulse transfer function is proposed. The procedure is simple, but it is used only for checking the positions of closed loop poles, not for designing the controller.

In this paper, we deal with the procedure to design robustly a state feedback controller which assigns all the closed loop poles in a specified disk for a linear time invariant system with perturbations of physical parameters. A distinct point of the proposed robust design procedure is that the nominal closed loop matrix A_{cl} is firstly established such that all its eigenvalues are positioned in a specified disk. The formula for determining the closed loop perturbations $\triangle A_{cl}$ is rearranged as a function of the nominal closed loop matrix only. Thus, if the nominal closed loop matrix is chosen, then the perturbed closed loop matrix A_{clp} can be determined from the equation $A_{clp} = A_{cl} + \triangle A_{cl}$. That means, we can obtain the desired A_{clp} by choosing A_{cl} appropriately. Next, there is an algebraic step, in which we introduce a procedure to determine the state feedback control law F from the matrix equation $A_{cl} = A + BF$, where B is a non- square matrix. The procedure is based on a partition of every matrix in the above equation in the horizontal direction. This paper also introduces a "useful control disturbance" which is used to move the closed loop poles into a desired region. Finally, numerical example is shown to illustrate the proposed robust pole assignment in a specified disk.

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II. Problem formulation

We discuss a robust pole assignment problem in a specified disk by a state feedback for linear time invariant systems with uncertainties. Those systems are described as follows:

$$\dot{x} = (A + \triangle A)x + B + \triangle B)u \tag{1a}$$

or

$$x_{k+1} = (A + \triangle A)x_k + (B + \triangle B)u_k \tag{1b}$$

where $x(x_k) \in R^n$ is the state vector, and $u(u_k) \in R^m$ is the control input. Matrices A and B are of constant with appropriate dimension, $\triangle A$ and $\triangle B$ are perturbations of the matrices A and B respectively. Without loss of generality, it is assumed that the pair (A, B) is controllable, rank $B = m$, and the matrices $\triangle A$ and $\triangle B$ are given in the following form:

$$|\triangle A| \leq_e E_a \qquad E_a \geq_e 0 \tag{2a}$$

or

$$\| \triangle A \| \leq a \qquad a \geq 0 \tag{2b}$$

and

$$\triangle B = bE_b \qquad b \geq 0 \tag{2c}$$

where $E_a \in R^{n \times n}$ and $E_b \in R^{n \times m}$ are fixed known matrices representing structured information for perturbations on the entries of the dynamic state matrix A, and the input matrix B respectively, and the symbol $\|\cdot\|$ denotes any norm.

The robust pole assignment problem leads to determine the state feedback control law:

$$u = Fx \tag{3a}$$

or

$$u_k = Fx_k \tag{3b}$$

and a change interval of the parameter b, $(0 \le b \le b_{\max})$ such that all the closed loop poles of the given perturbed system (1) lies in the specified disk $D(-\alpha, r)$ with the center $(-\alpha + j0)$ and the radius $r(\alpha \ge -1, r \ge 0)$.

III. Robust pole assignment

1. Boundness of a closed loop matrix

In this subsection, we introduce the basic theories related a boundness of a closed loop matrix which are the background to our approach for solving the pole assignment problem stated in Section 2.

Theorem 1: If the perturbation matrix of the input matrix has the form of Eq. (2c), then there exists a matrix M satisfying:

$$\triangle B = bMB \tag{4}$$

where

$$M = E_b V \Sigma_m^{-1} U_0^T \tag{5}$$

and $V \in R^{m \cdot m}$, $\Sigma_m \in R^{m \cdot m}$, and $U_0 \in R^{n \cdot m}$ are the decomposition matrices of the matrix B by using a singular value decomposition. ■

Proof: See Appendix A.

When the state feedback control law F is applied to the given perturbed system (1), the closed loop system has the form:

$$\dot{x} = [(A + \triangle A) + (B + \triangle B)F]x. \tag{6}$$

By applying Theorem 3.1, Eq. (6) is rearranged as follows:

$$\dot{x} := A_{clp} x \tag{7a}$$

where

$$A_{clp} := A_{cl} + \triangle A_{cl} \tag{7b}$$

$$A_{cl} := A + BF \tag{7c}$$

$$\triangle A_{cl} := \triangle A + bM(A_{cl} - A). \tag{7d}$$

Remark 1: If the matrix A_{cl} is known, then we can calculate the matrices F and $\triangle A_{cl}$ from Eqs. (7c) and (7d) respectively. ■

Thus, our robust pole assignment procedure included two steps. Firstly, we choose the nominal closed loop matrix A_{cl} such that all its eigenvalues are located in a specified disk, and to find a finite change of the parameter b which do not affect the eigenvalue distribution of A_{cl}, i.e., the eigenvalues of the matrix A_{clp} in Eq. (7b) are still positioned in a specified disk. Secondly, we determine the feedback control law F from Eq. (7c). This state feedback control law will assign all the closed loop poles of the given perturbed system (1) in a specified disk.

The following theorem can be used for choosing the matrices A_{cl} and A_{clp}.

Theorem 2: The eigenvalues of the matrices A_{cl} and A_{clp} are located within the specified disk $D(-\alpha, r)$ if the following condition is satisfied:

$$\|A_{cl} + \alpha I\| + \|\triangle A\| + b\|M(A_{cl} - A)\| \le r \tag{8}$$

where, $\|\cdot\|$ indicates any norm and I is an identity matrix with appropriate dimension. ■

Proof: See Appendix A.

Thus, the inequality (8) is a condition for robust pole assignment in the specified disk $D(-\alpha, r)$, and our robust procedure is as the following. Firstly, we solve the robust pole assignment problem with $b = 0$, i.e., the given system is subjected only to perturbations of the dynamic state matrix A. In this case, our robust design procedure is very simple comparing to the other known methods, for example, the method proposed in [5]. We just look for the closed loop matrix A_{cl} such that it satisfies the following condition:

$$\|A_{cl} + \alpha I\| \le r - \|\triangle A\| = r - a. \tag{9}$$

When the perturbations of the input matrix B are added to perturbations of the dynamic state matrix A, the elements of the closed loop matrix A_{clp} change continuously with the parameter b. The eigenvalues of A_{clp} will be changed by the same manner. Therefore, secondly, we increase the parameter b to determine an interval $(0 \le b \le b_{\max})$ such that the condition (8) is still satisfied, i.e., all the closed loop poles of the given perturbed system (1) are still located in the specified disk $D(-\alpha, r)$. From the condition (6) we can obtain the following equation to determine a change interval of the parameter b:

$$b_{\max} = \frac{r - a - \|A_{cl} + \alpha I\|}{\|M(A_{cl} - A)\|}. \tag{10}$$

Remark 2: We can use Gershgorin's, Schur's theorem, and the properties of an orthogonal or unitary matrix to establish the matrices A_{cl} and A_{clp} such that all their eigenvalues are located in the specified disk $D(-\alpha, r)$ ■

2. Robust feedback control law

As we mentioned before, the next step in our robust pole assignment procedure is to determine the state feedback control law F which assigns all the closed loop poles of the perturbed system (1) in the specified disk $D(-\alpha, r)$. If the input matrix B is a square and non-singular form, from Eq. (7c) we can find out:

$$F = B^{-1}(A_{cl} - A). \tag{11}$$

But, when the input matrix B is not a square, i.e. $n \ne m$, Eq. (11) can not be solved by a common procedure. In this case we can partition those matrices in the horizontal direction by the following way:

$$A_{cl} := \left[\frac{A_{cl1}}{A_{cl2}}\right], \quad A := \left[\frac{A_1}{A_2}\right], \quad B := \left[\frac{B_1}{B_2}\right]$$

where $A_{cl1},\ A_1 \in R^{m\times n}$,
$A_{cl2},\ A_2 \in R^{(n-m)\times n}$,
$B_1 \in R^{m\times m},\ B_2 \in R^{(n-m)\times m}$,

and Eq. (11) is equivalent to:

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} F = \begin{bmatrix} A_{cl1} \\ A_{cl2} \end{bmatrix} - \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}. \tag{12}$$

And Eq. (12) can be separated as:

$$B_1F = A_{cl1} - A_1 \tag{13a}$$

$$B_2F = A_{cl2} - A_2. \tag{13b}$$

If the sub-matrix B_1 is non-singular, then the feedback control law F is determined from Eq. (13a), i.e.,

$$F = B_1^{-1}(A_{cl1} - A_1) \tag{14}$$

and the condition for solving Eq. (11) is Eq. (13b).

Therefore, in our procedure, the sub-matrix A_{cl1} can be firstly established to determine the feedback control law F of Eq. (13a). After that, we calculate the sub-matrix A_{cl2} from Eq. (13b). Both A_{cl1} and A_{cl2} must satisfy the condition (9). If the sub-matrix A_{cl2} does not satisfy that condition, then we choose repeatedly the sub-matrix A_{cl1} in order to get a desired sub-matrix A_{cl2}.

If we cannot choose A_{cl2} satisfied Eq. (13b), i.e.,

$$A_{cl2} \neq A_2 + B_2F \tag{15}$$

then some closed loop poles locate outside the specified disk $D(-\alpha, r)$. In this case, we can use some known methods, such as the Amin [4] and Solheim [8], to move every pole from outside to inside the disk $D(-\alpha, r)$. Here, we introduce a method to move simultaneously those closed loop poles from outside to inside a specified disk. This method is based on a concept "control useful disturbance" explained in the next theorem.

Theorem 3: Assume that the sub-matrix B_1 is established as to satisfy Eq. (14), there exists a matrix $\triangle A_2$ satisfying:

$$A_{cl2} = A_2 + B_2F + \triangle A_2. \tag{16}$$

And we can rewrite equation (13) as follows

$$A_{cl} = A + BF + \Delta\overline{A}.$$

Here, the matrix F is given by Eq. (14), and the matrix $\triangle\overline{A}$ is named a "control useful disturbance" which is defined as the following:

$$\triangle\overline{A} = \begin{bmatrix} 0 \\ \triangle A_2 \end{bmatrix}.$$

If the matrices A and B are perturbed to $(\triangle A + \triangle\overline{A})$ and $\triangle B$, and $\|\triangle A + \triangle\overline{A}\|$ is known, we can solve the robust pole assignment problem for the given system (1). ■

Proof: See Appendix A.

Assume that after i step the sub-matrix B_{1i} is established to satisfy Eq. (14), there exists the matrix $\triangle A_{2i}$ satisfying:

$$A_{cl2i} = A_{2i} + B_{2i}F_i + \triangle A_{2i} \qquad i \geq 1$$

Now we deal the pole assignment problem with uncertainties. Consider the system (1) with perturbations $(\triangle A + \triangle\overline{A}_i)$ and $\triangle B$. The robust pole assignment condition for this system is as the following:

$$\|A_{cli} + aI\| + b\|M(A_{cli} - A)\| \leq r - \overline{a_i}$$

Remark 3: If rank$(B) < m$, we look for a non-singular sub-matrix $\overline{B_1}$ and determine the feedback control law F from:

$$F = \overline{B_1}^{-1}(A_{cl1} - A_1).$$

Thus,

$$A_{cl1} \neq A_1 + B_1F$$

and in the worst case we have:

$$A_{cl2} \neq A_2 + B_2F.$$

If so, there exist sub-matrices $\triangle A_1$ and $\triangle A_2$ such that:

$$A_{cl1} = A_1 + B_1F + \triangle A_1$$

and

$$A_{cl2} = A_2 + B_2F + \triangle A_2$$

or

$$A_{cl} = A + BF + \triangle\overline{\overline{A}}$$

where $\triangle\overline{\overline{A}}$ is a "control useful disturbance" which has the form:

$$\triangle\overline{\overline{A}} = \begin{bmatrix} \triangle A_1 \\ \triangle A_2 \end{bmatrix}.$$

Now, the robust pole assignment design is leaded to the same problem as just discussed before.

IV. Numerical example

Let us consider a problem to assign robustly the closed loop system's poles in the disk $D(-6, 2)$ for the following perturbed system matrices:

$$A = \begin{bmatrix} -7 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 5 & 0 \\ -2 & 4 \end{bmatrix}.$$

with perturbations:

$$\|\triangle A\|_\infty \leq a = 0.4$$

$$\triangle B = bE_b = b\begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}.$$

Using the condition (9) we can choose the nominal closed loop matrix of the system as the following:

$$A_{cl} = \begin{bmatrix} -7 & 0 & 0 \\ 0.2 & -6 & 0.8 \\ 0.2 & 0.2 & -6 \end{bmatrix}.$$

We determine the matrix M from Eq. (5).

$$M = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0.020 & 0 \\ 0 & 0.030 & 0.025 \end{vmatrix}.$$

If the system is subjected to perturbations of both matrices A and B, then from Eq. (16) we can calculate the maximum value of the parameter b such that the closed loop poles of the perturbed system are still located in the specified disk $D(-6,2)$.

$$b_{\max} = \frac{r-a-\|A_{cl}+aI\|_{INF}}{\|M(A_{cl}-A)\|_{INF}} = 1.4286.$$

It means that the perturbed system has the closed loop poles in the specified disk $D(-6,2)$ if the parameter b is in the interval $0 \le b \le 1.4286$.

In the next step, we determine the state feedback control law F. We can choose the second and the third row vectors of the input matrix B to form the sub-matrix B_1:

$$B_1 = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}.$$

Then

$$B_2 = [0 \quad 0]$$

and

$$A_2 = [-7 \quad 0 \quad 0] \qquad A_{cl2} = [-7 \quad 0 \quad 0]$$

$$A_1 = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \qquad A_{cl1} = \begin{bmatrix} 0.2 & -6 & 0.8 \\ 0.2 & 0.2 & -6 \end{bmatrix}.$$

We determine the state feedback control law F by Eq. (14)

$$F = \begin{bmatrix} -0.1600 & -0.8000 & -0.2400 \\ -0.2800 & -0.3500 & -2.3700 \end{bmatrix}.$$

This feedback control law satisfies Eq. (13b) and assigns all the closed loop poles of the perturbed system (1) in the disk $D(-6,2)$.

V. Conclusion

This paper presents a method to solve the robust pole assignment problem with the closed loop poles in a specified disk for a linear time system with perturbations.

Firstly, the formula using to determine the closed loop perturbations is established so that these perturbations are a function of a nominal closed loop matrix. Thus, when we choose this matrix we must pay attention to the closed loop perturbations such that the closed loop poles is always located in a specified disk.

Secondly, the feedback control law F that assigns the closed loop poles of a perturbed system in a specified disk is determined from the equation $A_{cl} = A + BF$. The procedure for solving F is based on partitioning every matrix in this equation in the horizontal direction. This paper also proposes a "control useful disturbance" which is used to simultaneously move the closed loop poles into a desired region. Hopefully, the idea of using a "control useful disturbance" will be extended to solve a pole assignment problem for a system that is uncontrollable.

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Appendix A

Proof of Theorem 1

Using a singular value decomposition approach we can decompose the matrix B as follows:

$$B = U\Sigma V^* \tag{A.1}$$

where

$$U^*U = I \qquad V^*V = I.$$

We can rewrite Eq. (A.1) in the form:

$$B = [U_0 \mid U_1]\begin{bmatrix} \Sigma_m \\ \hline 0 \end{bmatrix} V^*$$

or

$$B = U_0 \Sigma_m V^*. \tag{A.2}$$

Postmultiplying both sides of Eq. (15) with B and substituting Eq. (A.2) into the right side of this equation we can get:

$$MB = E_b V \Sigma_m^{-1} U_0^T U_0 \Sigma_m V^* = E_b.$$

Thus, the theorem is proved. ■

Proof of Theorem 2

From the inequality (5) and Eq. (7d) we can obtain:

$$\| A_{cl} + aI \| + \| \triangle A_{cl} \| \leq r.$$

And this equation is equivalent to:

$$\| A_{cl} + aI \| \leq r \tag{A.3}$$

and

$$\| A_{clp} + aI \| \leq r. \tag{A.4}$$

On the other hand we known that any eigenvalue of a matrix in modulus is smaller than any norm, i.e.,

$$| \lambda_i [A_{cl} + aI] | \leq \| A_{cl} + aI \| \tag{A.5}$$

and

$$| \lambda_i [A_{clp} + aI] | \leq \| A_{clp} + aI \| . \tag{A.6}$$

Comparing inequalities (A.3) and (A.4) with inequalities (A.5) and (A.6), respectively, we can complete the proof of Theorem 3.2. •

Proof of Theorem 3

If there exists Eq. (16), then Eq. (12) becomes:

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} F = \begin{bmatrix} A_{cl1} \\ A_{cl2} \end{bmatrix} - \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \triangle A_2 \end{bmatrix}$$

or

$$BF = A_{cl} - A - \triangle \overline{A}. \tag{A.7}$$

Substituting A_{cl} from Eq. (A.7) into Eq. (7a) we can get:

$$\dot{x} = [A_{cl} + (\triangle A + \triangle \overline{A}) + bM(A_{cl} - A)]x. \tag{A.8}$$

Thus, if $\| \triangle A + \triangle \overline{A} \|$ is given, i.e.,

$$\| \triangle A + \triangle \overline{A} \| \leq \overline{a}$$

then comparing Eq. (A.8) with Eq. (7) we can obtain the robust pole assignment conditions for the given system (1) with perturbations $(\triangle A + \triangle \overline{A})$ and $\triangle B$ as the following:

$$\| A_{cl} + aI \| \leq r - \overline{a} \tag{A.9}$$

and

$$b_{\max} = \frac{r - \overline{a} - \| A_{cl} + aI \|}{\| M(A_{cl} - A) \|}. \tag{A.10}$$

The conditions (A.9) and (A.10) are respectively similar to the conditions (9) and (10)

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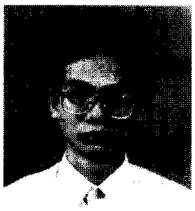

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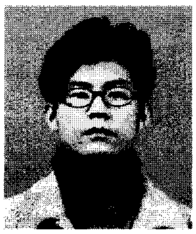

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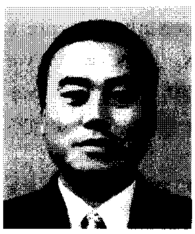

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