

# Output Feedback Semi-Global Stabilization for Input-Affine Nonlinear Systems

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**Abstract :** In this paper, the output feedback stabilizing problem is solved using any given state feedback control law. Compared to the linear systems for which any combination of a state feedback control law and a state observer solves the problem, the output feedback stabilizing problem is not so straightforward for nonlinear systems. We briefly explain the intrinsic obstructions for this problem and provide new output feedback scheme which achieves the semi-global stabilization with the high-gain state observer. The overall controller is explicitly given and the stability is analyzed. Our result only assumes the asymptotic stabilizability and the completely uniform observability of the plant. Therefore, the result can be regarded as an extension of the separation principle for linear systems in some aspect.

**Keywords :** semi-global stabilization, output feedback, nonlinear systems

## I. Introduction

Consider a single-input single-output nonlinear system,

$$\dot{\xi} = f_{\xi}(\xi) + g_{\xi}(\xi)u \quad (1a)$$

$$y = h_{\xi}(\xi) \quad (1b)$$

where  $\xi \in R^n$  is the states,  $f_{\xi}$  and  $g_{\xi}$  are smooth vector fields, and  $h_{\xi}$  is a smooth function. It is assumed that  $f_{\xi}(0) = 0$  and  $h_{\xi}(0) = 0$ .

In the linear control theory, stabilizability and detectability of the system guarantee the existence of output feedback controller, i.e., any pole-placement state feedback and any Luenberger observer can be combined to construct an output feedback controller (*separation principle*). However, for nonlinear control, it has been understood that such a desirable property does not hold in general. Especially, a counterexample has been presented in [1] which shows that global stabilizability and global observability are not sufficient for global output feedback stabilization. As a consequence, succeeding research activities have been devoted into two fields. One of them is imposing additional conditions on the system for global output feedback stabilization, for example, differential geometric conditions on the system structure [2], or an existence assumption of a certain Lyapunov function [3]. The other approach is focused on the semi-global output feedback stabilization instead of the global stabilization [4][5]. In particular, Teel and Praly [6], [7] constructed a semi-global output feedback stabilizing controller only under global stabilizability and observability. For the discussion to be clearer, some definitions are provided here.

**Definition 1:** An equilibrium point  $\xi = 0$  of (1) is *globally state (respectively, output) feedback stabilizable* if there exists a feedback control law using the information of the state  $\xi$  (respectively, the output  $y$ ) such that the closed-loop system is globally asymptotically stable, more precisely, the region of attraction is the whole space of  $R^n$ .

**Definition 2:** An equilibrium point  $\xi = 0$  of (1) is *semi-globally state (respectively, output) feedback stabilizable* if, for

each compact set  $\mathcal{K}$  which is a neighborhood of the origin, there exists a feedback control law using the information of the state  $\xi$  (respectively, the output  $y$ ) such that the region of attraction contains  $\mathcal{K}$ .

**Definition 3:** An equilibrium point  $\xi = 0$  of (1) is *locally state (respectively, output) feedback stabilizable* if there exists a feedback control law using the information of the state  $\xi$  (respectively, the output  $y$ ) such that the closed-loop system is locally asymptotically stable, more precisely, there is an open region of attraction containing the origin.

However, the aforementioned results [6], [7] used the 'dynamic extension' technique and the high-gain observer [8] which estimates the derivatives of the output  $y$ . As a result, the order of controller is greater than that of the plant in general, which is unnecessary in the case of linear output feedback stabilization.

To establish nonlinear output feedback stabilization, which is a natural extension of linear one, some crucial properties of linear version should be pointed out.

P1) Only stabilizability and observability are sufficient for output feedback stabilization. No more conditions are needed.

P2) If the system is observable when  $u \equiv 0$ , it is also observable for every known  $u$ .

P3) The order of observer is the same as that of the plant. Thus, the order of output feedback controller is  $n$ .

P4) The procedures to design output feedback controller is completely separated, that is, any state feedback controller and any observer can be combined.

From now on, three aspects of output feedback are presented. These give some motivation and justification of the treatment in this paper.

## 1. Nonlinear Observers

Global nonlinear observers have been actively studied in the literature. Most of them require some additional conditions as well as global observability, e.g., linearity up to output injection [9], [2], or input boundedness and restriction of certain nonlinear growth [10], [11].

The fact that, in nonlinear systems, the observability can be destroyed by an input  $u$  [12, p.415], is another obstacle for general construction of observer. In order to construct an output feedback controller by designing state feedback controller and

observer separately as in (P), some strong notion of observability independent of a state feedback controller as stated in (P) is required. Gauthier and Bornard [13] showed that a necessary and sufficient condition for the system (1) to be *observable for any input* is that (1) is diffeomorphic to a system of the form

$$\dot{x} = f(x) + g(x)u \tag{2a}$$

$$y = Cx = x_1 \tag{2b}$$

where

$$f(x) = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ \psi(x) \end{bmatrix} \quad g(x) = \begin{bmatrix} g_1(x_1) \\ g_2(x_1, x_2) \\ \vdots \\ g_{n-1}(x_1, \dots, x_{n-1}) \\ g_n(x_1, x_2, \dots, x_n) \end{bmatrix} \tag{2c}$$

which property they called *completely uniform observability*. Although Teel and Praly [6] defined completely uniform observability in a slightly different way using the derivatives of input and output, throughout this paper it is defined as follows.

**Definition 4:** The system (1) is *completely uniformly observable* if (1) is diffeomorphic to (2) on  $R^n$ .

In [11] a high gain observer is also constructed for the system of the form (2), i.e., for a completely uniformly observable system. Since it fits the purpose stated in (P) and has useful properties for output feedback, we will utilize the observer of [11] in this paper. Although the observer [11] requires that the input  $u$  is uniformly bounded and that the vector fields  $f$  and  $g$  are globally Lipschitz (which is in opposition to (P)), we will eliminate them by saturating the input and using the semi-global approach.

## 2. Feedback Control using Estimated States

Another key obstruction for global output feedback is ‘finite escape time’ phenomenon which is well discussed in [1].

Suppose a system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2^3 + u \\ y &= x_1 \end{aligned}$$

which is globally state feedback stabilizable with  $u(x) = -x_1 - x_2 - x_2^3$ , and completely uniformly observable. Suppose also that a global observer is constructed which estimates the true state asymptotically, that is,

$$u(\hat{x}(t)) \rightarrow u(x(t)) \quad \text{as } t \rightarrow \infty. \tag{3}$$

Hence, the system with output feedback is

$$\dot{x}_1 = x_2 \tag{4a}$$

$$\dot{x}_2 = x_2^3 + u(\hat{x}) = \frac{1}{2}x_2^3 + \left[\frac{1}{2}x_2^3 + u(\hat{x})\right]. \tag{4b}$$

Though the global observer guarantees the convergence of (3), it takes some time for the control value  $u(\hat{x}(t))$  to converge to the true control  $u(x(t))$ . During that time interval, some state may escape to infinity. For the example in (4) with  $x_1(0) = 0$ ,  $x_2(0) = 10$  and  $\hat{x}(0) = 0$ , the state  $x_2$  goes to infinity within 0.01 seconds unless the second term of (4b) ( $[\frac{1}{2}x_2^3(t) + u(\hat{x}(t))]$ ) becomes negative during that time. This facts shows

that, for the output feedback stabilization, the convergence rate of the observer should be sufficiently fast.

However, at this point, there are two obstacles. The first one is the fact that no matter how fast the convergence rate of observer is, there always exists an initial condition of  $x_2$  whose trajectory blows up in finite time. Indeed, for a system  $\dot{z} = \frac{1}{2}z^3$ , the solution  $z(t)$  from  $z(0) = z_0 > 0$  blows up at  $t = \frac{1}{\frac{3}{2}z_0^2}$ , which can be made arbitrarily small by increasing the initial  $z_0$  [1]. The semi-global approach is now appealing since it restricts possible initial conditions, which is practically reasonable. The second obstacle is the so-called ‘peaking phenomenon’ [14] which is generally inevitable when the convergence of observer is forced to be sufficiently fast. For fast convergence rate, most observers use high-gain, or place their poles far left. This ensures fast convergence but may generate initial peaking, i.e. large mismatched value between  $u(x(t))$  and  $u(\hat{x}(t))$  for the short initial period. This mismatching again may reduce the escape time of the system, thus, the observer needs to converge faster. A remedy for this vicious cycle is saturating the value of control  $u(\hat{x})$ , which is based on the idea of [4].

## 3. State Feedback Stabilization for Completely Uniformly Observable Systems

As pointed out in the above discussion, the approach in this paper is based on the completely uniform observability. To construct an output feedback controller, globally stabilizing state feedback is indispensable for the system (2). Unfortunately, in spite of global stabilizability of (2) there has been no general procedure for such controller in the literature. Nevertheless, several well-known methods can be used for the state feedback. Feedback linearizable systems, or partially linearizable systems with ISS-zero dynamics, is globally stabilizable [2], thus, can be used in the proposed approach if it is completely uniformly observable. A system with global relative degree of order  $n$  also satisfies Assumption 1. If a control Lyapunov function is known, Artstein and Sontag’s control can be used if the small control property holds [9]. In fact, since the semi-global stabilization rather than global one is dealt with, only semi-global state feedback results are required in the above discussion, as pointed out in [6].

In summary, by using the idea of input saturation [5], a nonlinear observer is constructed and finite escape time is avoided in this paper. The main difference between our result and that of [6] is the order of controller. For example, even for a simple linear system

$$\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= x_1 \\ y &= x_1, \end{aligned}$$

the strategy of their paper gives a controller of order 4. This is mainly due to the ‘dynamic extension’ used in their paper. For this example, the dynamic extension technique causes order 2 in addition to the order 2 of the observer. On the other hand, this paper still yields a controller of order 2 which is the same order as the plant. This is achieved by incorporating the observer proposed in [11] instead of the dynamic extension.

## II. Main Results

In this section, an output feedback scheme is presented and analyzed. A sufficient condition for the proposed scheme is just the following.

**Assumption 1:** The system (1) is globally state feedback stabilizable<sup>1</sup> and completely uniformly observable.

### 1. Design Steps

Consider the system (2).

#### Step 1. (State Feedback Stabilization)

Choose a compact set  $\mathcal{K} \in R^n$  in which the initial state  $x(0)$  can be located. Design a  $C^1$  state feedback control  $\alpha(x)$  with a continuously differentiable Lyapunov function  $V(x)$ , such that

1.  $\mathcal{K} \subset \Omega_0 := \{x : V(x) \leq c\}$  for a positive constant  $c$ .
2.  $\Omega_1 := \{x : V(x) \leq c + \delta\}$  is connected and compact, in which  $\delta$  is a positive constant.
3.  $\dot{V} = L_f V + L_g V \alpha = -W(x)$  where  $W(x)$  is positive definite function ( $W(0) = 0$ ) on  $\Omega_1$ .

The existence of such  $\alpha$  and  $V(x)$  is guaranteed by the global state feedback stabilizability. Practically these can be found by the existing methods discussed at section I-3. Also, define  $\Omega_{\frac{1}{2}} = \{x : V(x) \leq c + \frac{\delta}{2}\}$  for future use. Hence,  $\mathcal{K} \subset \Omega_0 \subset \Omega_{\frac{1}{2}} \subset \Omega_1$ .

**Remark 1:** This step need not be performed for the form of (2), i.e. in  $x$ -coordinates. By the global state feedback stabilizability of the plant (1), the globally stabilizing  $\alpha_\xi(\xi)$  and  $V_\xi(\xi)$  for (1) can be found. Then, they can be transformed to  $\alpha(x)$  and  $V(x)$ , since (1) is diffeomorphic to (2) by the completely uniform observability.

#### Step 2. (Preparation for Semi-global Output Feedback)

Construct the control  $u$  such that

$$u = U \cdot \text{sat} \left( \frac{\alpha(\hat{x})}{U} \right), \quad U \geq \max_{x \in \Omega_1} |\alpha(x)|. \quad (5)$$

Next, modify  $f(x)$  and  $g(x)$  outside the region  $\Omega_1$  to be globally Lipschitz when they are not, and denote them by  $\hat{f}(x)$  and  $\hat{g}(x)$ , respectively. More precisely, find<sup>2</sup> globally Lipschitz  $\hat{\psi}(x)$  and  $\hat{g}_i(x_1, \dots, x_i)$ ,  $1 \leq i \leq n$  such that  $\hat{\psi}(x) = \psi(x)$  and  $\hat{g}_i(x) = g_i(x)$  when  $x \in \Omega_1$ . Then,

$$\dot{x} = \hat{f}(x) + \hat{g}(x)u, \quad y = Cx \quad (6)$$

describes the dynamics of the plant in the region of interest  $\Omega_1$ .

#### Step 3. (Observer Construction)

Suppose the observer's initial state  $\hat{x}(0)$  is located in  $\Omega_0$ . Construct the observer as

$$\dot{\hat{x}} = \hat{f}(\hat{x}) + \hat{g}(\hat{x})u - S_\theta^{-1} C'(C\hat{x} - y) \quad (7)$$

where  $S_\theta$  satisfies<sup>3</sup>

$$0 = -\theta S_\theta - A'S_\theta - S_\theta A + C'C, \quad (8)$$

where  $A$  is such that  $(A)_{i,j} = \delta_{i,j-1}$  and  $C = [1, \dots, 0]$ .

Now, there exists a  $\theta^*$  such that for any  $\theta > \theta^*$ , the states of the closed-loop system is asymptotically stable on  $\Omega_0$ .

<sup>1</sup>In fact, semi-global state feedback stabilizability is necessary rather than the global state feedback stabilizability.

<sup>2</sup>This procedure is termed as 'Lipschitz extension' in [15], [16]. Practical methods finding the suitable extension can be found there.

<sup>3</sup>When  $\theta > 0$ , the existence of such solution  $S_\theta$ , which is positive definite and symmetric, is followed by the observability of  $A$  and  $C$ .

## 2. Analysis

In this subsection, analyses of stability and convergence are presented for the output feedback scheme of last subsection. Firstly, the following lemma recalls the result of [11].

**Lemma 1:** Consider the plant (6) and the observer (7). With the saturated input  $u$  in (5), there exists a constant  $\theta_0^* \geq 1$  such that for any  $\theta > \theta_0^*$ , the observer (7) guarantees

$$|\hat{x}(t) - x(t)| \leq K(\theta) \exp\left(-\frac{\theta}{3}t\right) |\hat{x}(0) - x(0)|. \quad (9)$$

Moreover, for a fixed  $\tau > 0$ ,

$$K(\theta) \exp\left(-\frac{\theta}{3}\tau\right) \rightarrow 0 \quad \text{as} \quad \theta \rightarrow \infty. \quad (10)$$

**Proof:** The first part of the claim follows from the work of [11, Theorem 3], since the system is completely uniformly observable (by the form (2)),  $\hat{f}$  and  $\hat{g}$  are globally Lipschitz, and the input  $u(\hat{x})$  is uniformly bounded, i.e.,

$$|u(\hat{x}(t))| \leq U$$

for all  $t$ , by (5). The inequality (10) is also an easy consequence of their work, but is unclear in [11].

Let  $S_1$  denote the solution of (8) with  $\theta = 1$ , which is symmetric and positive definite matrix. Then, from [11],

$$(S_\theta)_{i,j} = (S_1)_{i,j} \frac{1}{\theta^{i+j-1}}.$$

Define  $e := \hat{x} - x$ ,  $\zeta_i := e_i/\theta^i$  and  $\|x\|_S := (x'Sx)^{\frac{1}{2}}$ . From these definitions, the following useful relations can be obtained. For  $\theta \geq 1$ ,

$$\|e\|_{S_\theta} = \sqrt{\theta} \|\zeta\|_{S_1} \quad (11)$$

$$\sqrt{\lambda_m(S_1)} \|\zeta\| \leq \|\zeta\|_{S_1} \leq \sqrt{\lambda_M(S_1)} \|\zeta\| \quad (12)$$

$$\frac{1}{\theta^n} \|e\| \leq \|\zeta\| = \left\| \begin{pmatrix} e_1 \frac{1}{\theta^1} \\ e_2 \frac{1}{\theta^2} \\ \vdots \\ e_n \frac{1}{\theta^n} \end{pmatrix} \right\| \leq \frac{1}{\theta} \|e\| \quad (13)$$

where  $\lambda_M(S_1)$  and  $\lambda_m(S_1)$  denote the maximum and minimum eigenvalues of  $S_1$ , respectively.

Now evaluating the derivative of  $\|e\|_{S_\theta}$ , (See [11] for details.)

$$\begin{aligned} \frac{d}{dt} \|e\|_{S_\theta} &\leq -\frac{\theta}{2} \|e\|_{S_\theta} + N \|e\|_{S_\theta} \\ &= -\frac{\theta}{3} \|e\|_{S_\theta} + \left(N - \frac{\theta}{6}\right) \|e\|_{S_\theta} \end{aligned}$$

where  $N$  is a positive constant independent of  $\theta$ . Hence, for  $\theta > \theta_0^* = \max\{1, 6N\}$ ,

$$\|e(t)\|_{S_\theta} \leq \exp\left(-\frac{\theta}{3}t\right) \|e(0)\|_{S_\theta}.$$

Using the relations (11) – (13),

$$\|e(t)\| \leq \theta^{n-1} \sqrt{\frac{\lambda_M(S_1)}{\lambda_m(S_1)}} \exp\left(-\frac{\theta}{3}t\right) \|e(0)\|$$

that is,  $K(\theta) = \theta^{n-1} \sqrt{\frac{\lambda_M(S_1)}{\lambda_m(S_1)}}$  in (9), and hence (10) follows. This completes the proof.  $\blacksquare$

Next, define the deviation of the control as

$$\Delta u(x(t), \hat{x}(t)) := U \cdot \text{sat} \left( \frac{\alpha(\hat{x}(t))}{U} \right) - \alpha(x(t)).$$

Then we can claim the following lemma.

**Lemma 2:** Consider the observer (7) with  $\hat{x}(0) \in \Omega_0$ . For any given  $\tau > 0$  and  $\epsilon > 0$ , there exists  $\theta_2^* > 0$  such that, for any  $\theta \geq \theta_2^*$ ,

$$|\Delta u(t)| \leq \epsilon \exp\left(-\frac{\theta}{3}(t - \tau)\right), \quad t \geq \tau \quad (14)$$

if  $x(t) \in \Omega_{\frac{1}{2}} = \{x : V(x) \leq c + \frac{\delta}{2}\}$  for all  $t$ , with  $x(0) \in \Omega_0$ .

**Proof:** Define

$$d_0 := \sup_{\substack{x(0) \in \Omega_0 \\ \hat{x}(0) \in \Omega_0}} |\hat{x}(0) - x(0)|. \quad (15)$$

And define

$$D := \inf |\hat{x} - x| \quad \forall x \in \{x : V(x) \leq c + \frac{\delta}{2}\} \\ \forall \hat{x} \in \{\hat{x} : V(\hat{x}) > c + \delta\}.$$

Clearly,  $D > 0$ . (If not, i.e.  $D = 0$ , then there are an  $x$  such that  $V(x) \leq c + \frac{\delta}{2}$  and a sequence  $\{\hat{x}_i\}$  such that  $\hat{x}_i \rightarrow x$  and  $V(\hat{x}_i) > c + \delta$ . Since  $V$  is continuous,  $V(\hat{x}_i) \rightarrow V(x)$ , which is a contradiction because  $V(\hat{x}_i) - V(x) \geq \frac{\delta}{2}$ .)

Now, it follows from Lemma 1 that there exists a  $\theta_1^* (\geq \theta_0^*)$  such that for any  $\theta > \theta_1^*$ ,

$$|\hat{x}(\tau) - x(\tau)| \leq K(\theta) \exp\left(-\frac{\theta}{3}\tau\right) |\hat{x}(0) - x(0)| \\ \leq K(\theta) \exp\left(-\frac{\theta}{3}\tau\right) d_0 \\ < D$$

and thus,  $|\hat{x}(t) - x(t)| < D$  for all  $t \geq \tau$ . This means that  $\hat{x}(t) \in \Omega_1$  for  $t \geq \tau$  (because  $V(x) \leq c + \frac{\delta}{2}$ ,  $V(x^*) > c + \delta \Rightarrow |x^* - x| \geq D$ ). Hence, the control is unsaturated after the time  $\tau$  ( $u = \alpha(\hat{x}(t))$ ,  $t \geq \tau$ ).

By the continuous differentiability,  $\alpha$  is Lipschitz on  $\Omega_1$ . Define  $L$  as a Lipschitz constant of  $\alpha$  on  $\Omega_1$ . Thus, by taking  $\theta_2^* (\geq \theta_1^*)$  such that  $LK(\theta) \exp(-\frac{\theta}{3}\tau)d_0 \leq \epsilon$  for any  $\theta > \theta_2^*$ ,

$$|\Delta u(t)| = |\alpha(\hat{x}(t)) - \alpha(x(t))| \\ \leq L|\hat{x}(t) - x(t)| \\ \leq LK(\theta) \exp\left(-\frac{\theta}{3}t\right) |\hat{x}(0) - x(0)| \\ = LK(\theta) \exp\left(-\frac{\theta}{3}\tau\right) \\ \times |\hat{x}(0) - x(0)| \exp\left(-\frac{\theta}{3}(t - \tau)\right) \\ \leq \epsilon \exp\left(-\frac{\theta}{3}(t - \tau)\right)$$

for  $\theta > \theta_2^*$  and  $t \geq \tau$ . This completes the proof. ■

Finally, the following theorem shows that the semi-global asymptotic stability of the overall system.

**Theorem 3:** Consider the overall system (2), (5) and (7). Under Assumption 1, there exists a  $\theta^* > 0$  such that, for all  $\theta > \theta^*$  and for any initial states  $x(0) \in \Omega_0$ ,  $\hat{x}(0) \in \Omega_0$ , the solution  $(x(t), \hat{x}(t))$  of the closed-loop system (2), (5) and (7) are

uniformly bounded and converge to the origin. Moreover, with the  $\theta$  chosen, the origin of the closed-loop system is stable.

**Proof:** The closed-loop system with  $u$  as in (5) can be written as

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\Delta u \\ \dot{\hat{x}} = \hat{f}(\hat{x}) + \hat{g}(\hat{x})u - S_\theta^{-1}C'(C\hat{x} - Cx)$$

which leads to

$$\dot{V}(x) = L_f V(x) + L_g V(x)\alpha(x) + L_g V(x)\Delta u(t).$$

Define a constant  $\Delta u_{max}$  as

$$\Delta u_{max} := \max_{\substack{x \in \Omega_1 \\ -U \leq v \leq U}} |v - \alpha(x)|,$$

whose existence is guaranteed by the compactness of  $\Omega_1$  and the continuity of  $\alpha(x)$  on  $\Omega_1$ . It can be easily seen that  $|\Delta u(t)| \leq \Delta u_{max}$  while  $x(t)$  remains in  $\Omega_1$ . Since  $L_f V$ ,  $L_g V$  and  $\alpha$  are continuous, there is a constant  $h > 0$  such that

$$|L_f V(x) + L_g V(x)\alpha(x) + L_g V(x)v| \leq h$$

for all  $x \in \Omega_1$  and  $|v| \leq \Delta u_{max}$ .

Now let  $\tau = \frac{\delta}{2h}$ . It then follows that, for every initial condition  $x(0) \in \Omega_0$ ,

$$V(x(t)) \leq c + \frac{\delta}{2}, \quad \text{for } 0 \leq t \leq \tau,$$

since  $|\Delta u(t)| \leq \Delta u_{max}$  during that time interval. Next, from the fact that  $L_f V(x) + L_g V(x)\alpha(x)$  is strictly negative for  $c \leq V(x) \leq c + \frac{\delta}{2}$ , it follows that there is an  $\epsilon > 0$  such that  $L_f V(x) + L_g V(x)\alpha(x) + L_g V(x)v < 0$  whenever  $c \leq V(x) \leq c + \frac{\delta}{2}$  and  $|v| \leq \epsilon$ . By applying Lemma 2 with the  $\tau$ ,  $\epsilon$  and  $\hat{x}(0) \in \Omega_0$ , it can be shown that there exists a  $\theta^*$  such that, for any  $\theta > \theta^*$ ,  $|\Delta u(t)| \leq \epsilon$  for  $t \geq \tau$ . Therefore, for any  $x(0), \hat{x}(0) \in \Omega_0$ ,

$$V(x(t)) \leq c + \frac{\delta}{2}, \quad \text{for all } t. \quad (16)$$

since  $\dot{V} < 0$  for all  $t \geq \tau$  and all  $x$  such that  $c \leq V(x) \leq c + \frac{\delta}{2}$ . This fact, with the equation (9), shows the uniform boundedness of the state  $x(t)$  and  $\hat{x}(t)$ .

Now it is shown that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . This argument is similar to the one proposed by [14, Theorem 4.1]. Pick an  $\epsilon_1 > 0$ . Then there exists an  $\epsilon'_1 > 0$  such that  $V(x) \leq \epsilon'_1$  implies  $|x| \leq \epsilon_1$  by the positive definiteness of  $V(x)$  on  $\Omega_1$ . By the continuity again, there are positive constants  $h_1$  and  $\delta_1$  such that

$$L_f V(x) + L_g V(x)\alpha(x) + L_g V(x)v \leq -h_1$$

for all  $x \in \{x | \epsilon'_1 \leq V(x) \leq c + \frac{\delta}{2}\}$  and all  $v \in \{v | |v| \leq \delta_1\}$ . Then let  $T \geq \tau$  be such that  $\epsilon \exp(-\frac{\theta}{3}(T - \tau)) \leq \delta_1$ , and let  $T'$  be such that  $h_1(T' - T) > c + \frac{\delta}{2}$ . By (16),  $V(x(t)) \leq c + \frac{\delta}{2}$  for  $0 \leq t \leq T$ . By Lemma 2,  $\Delta u(t) \leq \delta_1$  for  $T \leq t < \infty$ . Thus,  $\dot{V}(x(t)) \leq -h_1$  as long as  $V(x(t)) \geq \epsilon'_1$ . It then follows that there is a  $\hat{t}$  such that  $T \leq \hat{t} \leq T + T'$  and  $V(x(\hat{t})) \leq \epsilon'_1$ . Finally, it is clear that, if  $V(x(\hat{t})) \leq \epsilon'_1$  for some  $\hat{t}$  such that  $\hat{t} \geq T$ , it follows that  $V(x(t)) \leq \epsilon'_1$  for all larger  $t$ . This shows the convergence of  $x(t)$  to the origin. Again, by (9),  $\hat{x}(t)$  also converges to the origin.

From now on, the stability of the overall system is shown for a  $\theta$  selected as above. In fact, it is shown that for any given  $\epsilon_2$ , there exists  $\rho$  such that

$$\begin{aligned} |x(0)| \leq \rho \text{ and } |\hat{x}(0)| \leq \rho \\ \Rightarrow |x(t)| \leq \epsilon_2 \text{ and } |\hat{x}(t)| \leq \epsilon_2. \end{aligned}$$

Since the region of interest is  $\Omega_0$ , without loss of generality,  $\epsilon_2$  is assumed to be small so that  $|x| \leq \epsilon_2 \Rightarrow x \in \Omega_0$ . For given  $\epsilon_2$ , choose  $\rho_1$  such that  $V(x) \leq \rho_1 \Rightarrow |x| \leq \frac{\epsilon_2}{2}$ , and  $\epsilon'_2$  such that  $|x| \leq \epsilon'_2 \Rightarrow V(x) \leq \frac{\rho_1}{2}$ . By the continuity again, there exists  $\delta_2 > 0$  such that

$$L_f V(x) + L_g V(x)\alpha(x) + L_g V(x)v < 0, \quad (17)$$

for all  $x \in \{x \mid \frac{\rho_1}{2} \leq V(x) \leq \rho_1\}$  and all  $v \in \{v \mid |v| \leq \delta_2\}$ . Now, choose  $\rho$  as

$$\rho = \min \left\{ \epsilon'_2, \frac{\epsilon_2}{4K(\theta)}, \frac{\delta_2}{2LK(\theta)} \right\}. \quad (18)$$

Then, for  $|x(0)| \leq \rho$  and  $|\hat{x}(0)| \leq \rho$ ,  $x(0) \in \Omega_0$  and  $\hat{x}(0) \in \Omega_0$ . Suppose that there exists a time  $T$  such that  $\hat{x}(T) \in \partial\Omega_1$ , in which  $\partial\Omega_1$  is the boundary of  $\Omega_1$ , and  $\hat{x}(t) \in \Omega_1$  for  $0 \leq t < T$ . Clearly,  $T > 0$  since  $\hat{x}(t)$  is continuous with respect to  $t$ , and  $T$  may be  $\infty$ . However, the state  $x(t)$  is contained in  $\Omega_1$  for all  $t$  by the previous argument. During the time interval  $(0 \leq t < T)$ ,

$$\begin{aligned} |\Delta u(t)| &\leq L|\hat{x}(t) - x(t)| \\ &\leq LK(\theta)|\hat{x}(0) - x(0)| \\ &\leq LK(\theta)(2\rho) \\ &\leq \delta_2, \end{aligned}$$

which, with (17), implies that  $x(t)$  is captured in the region  $\{x : V(x) \leq \rho_1\}$ . Thus,

$$|x(t)| \leq \frac{\epsilon_2}{2}. \quad (19)$$

On the other hand,

$$\begin{aligned} |\hat{x}(t)| &\leq |x(t)| + K(\theta) \exp\left(-\frac{\theta}{3}t\right)|\hat{x}(0) - x(0)| \\ &\leq \frac{\epsilon_2}{2} + K(\theta)(2\rho) \\ &\leq \epsilon_2 \end{aligned} \quad (20)$$

by (19) and (18). However, since  $\hat{x}(t) \in \Omega_0$  for  $0 \leq t < T$  by (20), the temporary assumption  $\hat{x}(T) \in \partial\Omega_1$  is impossible. Thus,  $T$  should be  $\infty$ , that is, (19) and (20) hold for all  $t \geq 0$ . This completes the proof. ■

### III. Conclusion

In this paper, an output feedback scheme is proposed for semi-global stabilization. The required conditions are only global state feedback stabilizability and completely uniform observability. The scheme satisfies the properties (P), (P) and (P) of section I which are inherited from the linear output feedback stabilization.

On the other hands, (P) is not exactly satisfied, since there should be a procedure to select appropriate  $\theta$  generally depending on the chosen state feedback law  $\alpha(x)$ . This is because that

the observer should be sufficiently faster than the plant dynamics, which is unnecessary for linear output feedback. However, remembering that, for good performance, convergence of observer should be faster than that of the plant even for linear systems, it can be thought as a reasonable drawback.

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