Journal of the Korean Association for Shell and Spatial Structures
Vol. 4, No. 4 (통권 14호), pp. 113~128, December, 2004

# Vibrations of Complete Paraboloidal Shells with Variable Thickness from a Three-Dimensional Theory

장 경 호* 심 현 주** 강 재 훈***
Chang, Kyong-Ho Shim, Hyun-Ju Kang, Jae-Hoon

**Abstract**

A three-dimensional (3-D) method of analysis is presented for determining the free vibration frequencies and mode shapes of solid paraboloids and complete (that is, without a top opening) paraboloidal shells of revolution with variable wall thickness. Unlike conventional shell theories, which are mathematically two-dimensional (2-D), the present method is based upon the 3-D dynamic equations of elasticity. The ends of the shell may be free or may be subjected to any degree of constraint. Displacement components $u_r$, $u_\theta$, and $u_z$ in the radial, circumferential, and axial directions, respectively, are taken to be sinusoidal in time, periodic in $\theta$, and algebraic polynomials in the r and z directions. Potential (strain) and kinetic energies of the paraboloidal shells of revolution are formulated, and the Ritz method is used to solve the eigenvalue problem, thus yielding upper bound values of the frequencies by minimizing the frequencies. As the degree of the polynomials is increased, frequencies converge to the exact values. Convergence to four-digit exactitude is demonstrated for the first five frequencies of the complete, shallow and deep paraboloidal shells of revolution with variable thickness. Numerical results are presented for a variety of paraboloidal shells having uniform or variable thickness, and being either shallow or deep. Frequencies for five solid paraboloids of different depth are also given. Comparisons are made between the frequencies from the present 3-D Ritz method and a 2-D thin shell theory.

***Keywords*: *Three-Dimensional Analysis; Vibration; Complete Paraboloidal Shells; Solid Paraboloids; Shells of Revolution; Thick Shell; Variable Thickness; Ritz method***

## 1. INTRODUCTION

Paraboloidal shell structures and components have been widely used in civil, mechanical, and aerospace structures and systems, e.g., horns, nozzles, rocket fairings, solar collectors, communication antennas, optical mirrors, etc. A vast published literature exists for free vibrations of shells. The monograph of Leissa[1] summarized approximately 1000 relevant publications worldwide through the 1960's. Almost all of these dealt with shells of revolution (e.g., circular cylindrical, conical, spherical). Among them were eight references[2-9] considering paraboloidal shells. Some additional investigations of the static and dynamic characteristics of paraboloidal shells have also been uncovered.[10-22] However, these studies were either experimental, or were based upon thin shell theory, which is mathematically two-dimensional (2-D). That is, for thin shells one assumes the Kirchhoff hypothesis that normals to the shell middle surface remain normal to it during deformations (vibratory, in this case), and unstretched in length. This yields an eighth order set of partial differential equations of motion to be solved. For para- boloidal shells they involve variable coefficients, making them quite difficult to solve.

Even so, conventional shell theory is only

* 중앙대학교(서울) 건설환경공학과, 조교수
** 중앙대학교(서울) 건축학부, 박사과정
*** 정회원 · 중앙대학교(서울) 건축학부, 조교수

applicable to thin shells. A higher order shell theory could be used which considers the effects of shear deformation and rotary inertia, and would be useful for the low frequency modes of moderately thick shells. Such a theory would also be 2-D. But for paraboloidal shells the resulting equations would be very complicated. For such moderately thick shells, approximate results can be obtained by the finite element method.

Recently, from a 3-D theory in terms of three displacement components which are tangent or normal to the shell middle surface, Leissa and Kang[23] analyzed the free vibrations of open (with a top opening) paraboloidal shells of revolution with linear thickness variation along the meridional direction using the Ritz method, but they did not give results for complete (without a top opening) paraboloidal shells due to the singularities arising at the peaks of the shells.

In the present work complete paraboloidal shells of revolution with variable thickness and solid paraboloids are analyzed by a 3-D approach. Instead of attempting to solve equations of motion, an energy approach is followed which, as sufficient freedom is given to the three displacement components, yields frequency values as close to the exact ones as desired. To evaluate the energy integrations over the shell volume exactly (not numerically), displacements and strains are expressed in terms of the circular cylindrical coordinates, instead of related 3-D shell coordinates which are normal and tangent to the shell midsurface, which were employed by Leissa and Kang[23]. Results are obtained for five solid paraboloids and fifteen complete, shallow and deep paraboloidal shells of revolution with both uniform and variable thickness, which are completely free. Comparisons are also made between the frequencies from the present 3-D Ritz method and a 2-D thin shell theory.

## 2. METHOD OF ANALYSIS

A representative cross-section of a complete paraboloidal shell of revolution having variable thickness is shown in Fig. 1. The shell thicknesses ($h$) at the top and bottom are $h_t(\overline{AB})$ and $h_b(\overline{CD})$, respectively. The curve $z_m$ passing through the origin of the circular cylindrical coordinate system and a point $(r, z) = (R, H)$ in Fig. 1 is the midsurface of the shell and its equation is expressed as

$$z_m = r^2 / 4a, \tag{1}$$

where $a = R^2 / 4H$ is the focal distance of $z_m$. The straight line at the boundary of the shell passes through the point $(r, z) = (R, H)$ is normal to the curve $z_m$ and its equation is expressed as

$$z_b = -\frac{R}{2H}(r - R - \frac{2H^2}{R}). \tag{2}$$

Each of the two points C $(R_o, H_o)$ and D $(R_i, H_i)$ on the line $z_b$ are at a distance of $h_b/2$ from the mid surface point $(r, z) = (R, H)$, and thus $H_{i,o}$ and $R_{i,o}$ are located at

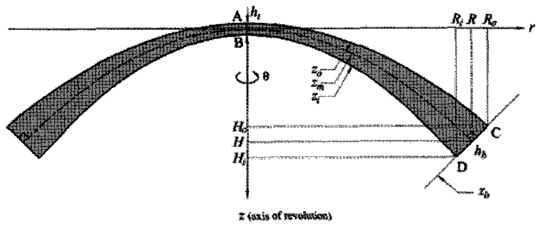


Fig. 1. A cross section of a complete paraboloidal shell of revolution and the circular cylindrical coordinate system (r, z, θ).

$$H_{i,o} = \frac{R}{2k}\left[2k^2 + 1 - \frac{R_{i,o}}{R}\right], \tag{3}$$

$$R_{i,o} = R\left[1 \mp \frac{k\alpha}{\sqrt{1+4k^2}}\right], \tag{4}$$

where

$$k \equiv \frac{H}{R}, \quad \alpha \equiv \frac{h_b}{R}. \tag{5}$$

Also, the curves $z_{i,o}$, which generate inner and outer surfaces of the shell, respectively, pass through the points $B$ (0, $h_t/2$) and $D$ ($R_i, H_i$), and $A$ (0, $-h_t/2$) and $C$ ($R_o, H_o$), respectively, and thus their equations are

$$z_{i,o} = \frac{r^2}{4a_{i,o}} \pm \frac{h_t}{2}, \tag{6}$$

where $a_{i,o}$ are the focal distances of the curves $z_{i,o}$, respectively,

$$a_{i,o} = \frac{kR_{i,o}{}^2}{2[(2k^2 \mp \alpha\beta k + 1)R - R_{i,o}]} \tag{7}$$

with $\beta$ is thickness ratio, defined by

$$\beta \equiv h_t / h_b. \tag{8}$$

Thus the domain $(\Lambda)$ of the shell is described by

$$0 \le r \le R_i, \quad z_o(r) \le z \le z_i(r), \quad 0 \le \theta \le 2\pi, \tag{9a}$$

and

$$R_i \le r \le R_o, \quad z_o(r) \le z \le z_b(r), \quad 0 \le \theta \le 2\pi, \tag{9b}$$

where $\theta$ is the circumferential angle. For a solid paraboloid with $z_m$ as an outer surface and $R$ as a radius of circular bottom, the domain $(\Lambda)$ is given by

$$0 \le r \le R, \quad z_m(r) \le z \le H, \quad 0 \le \theta \le 2\pi. \tag{10}$$

For mathematical convenience, the radial ($r$) and axial ($z$) coordinates are made dimensionless as

$$\psi \equiv r/R, \quad \zeta \equiv z/H. \tag{11}$$

Thus the domain of the shell in terms of the nondimensional coordinates $(\psi, \zeta, \theta)$ is given by

$$0 \le \psi \le \psi_i, \quad \zeta_o(\psi) \le \zeta \le \zeta_i(\psi), \quad 0 \le \theta \le 2\pi, \tag{12a}$$

and

$$\psi_i \le \psi \le \psi_o, \quad \zeta_o(\psi) \le \zeta \le \zeta_b(\psi), \quad 0 \le \theta \le 2\pi, \tag{12b}$$

where

$$\psi_{i,o} \equiv \frac{R_{i,o}}{R} = 1 \pm \frac{k\alpha}{\sqrt{1+4k^2}} \tag{13}$$

$$\zeta_b = 1 - \frac{1}{2k^2}(\psi - 1), \tag{14}$$

$$\zeta_{i,o} = \left(\frac{2k^2 \mp \alpha\beta k + 1 - \gamma_{i,o}}{2k^2\,\gamma_{i,o}{}^2}\right)\psi^2 \pm \frac{\alpha\beta}{2k}, \tag{15}$$

with

$$\gamma_{i,o} \equiv \frac{R_{i,o}}{R} = 1 \mp \frac{\alpha k}{\sqrt{1+4k^2}}. \tag{16}$$

In the case of solid paraboloids, it is given by

$$0 \le \psi \le 1, \quad \psi^2 \le \zeta \le 1, \quad 0 \le \theta \le 2\pi. \tag{17}$$

Utilizing tensor analysis, the three equations of motion in the circular cylindrical coordinate system $(r, z, \theta)$ were found to be[24)]

$$\sigma_{rr,r} + \sigma_{rz,z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta} + \sigma_{r\theta,\theta}) = \rho \ddot{u}_r, \quad (18a)$$

$$\sigma_{rz,r} + \sigma_{zz,z} + \frac{1}{r}(\sigma_{rz} + \sigma_{z\theta,\theta}) = \rho \ddot{u}_z, \quad (18b)$$

$$\sigma_{r\theta,r} + \sigma_{z\theta,z} + \frac{1}{r}(2\sigma_{r\theta} + \sigma_{\theta\theta,\theta}) = \rho \ddot{u}_\theta, \quad (18c)$$

where the $\sigma_{ij}$ are the normal $(i = j)$ and shear $(i \neq j)$ stress components; $u_r$, $u_z$, and $u_\theta$ are the displacement components in the $r$, $z$, and $\theta$ directions, respectively; $\rho$ is mass density per unit volume; the commas indicated spatial derivatives; and the dots denote time derivatives.

The well-known relationships between the tensorial stresses $(\sigma_{ij})$ and strains $(\varepsilon_{ij})$ of isotropic, linear elasticity are

$$\sigma_{ij} = \lambda \varepsilon \delta_{ij} + 2G\varepsilon_{ij}, \quad (19)$$

where $\lambda$ and $G$ are the Lamé parameters, expressed in terms of Young's modulus ($E$) and Poisson's ratio $(\nu)$ for an isotropic solid as:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}, \quad (20)$$

$\varepsilon \equiv \varepsilon_{rr} + \varepsilon_{zz} + \varepsilon_{\theta\theta}$ is the trace of the strain tensor, and $\delta_{ij}$ is Kronecker's delta.

The three-dimensional tensorial strains $(\varepsilon_{ij})$ are found to be related to the three displacements $u_r$, $u_z$, and $u_\theta$, by Sokolnikoff [24)]

$$\varepsilon_{rr} = u_{r,r}, \quad \varepsilon_{zz} = u_{z,z}, \quad \varepsilon_{\theta\theta} = \frac{u_r + u_{\theta,\theta}}{r} \quad (21a)$$

$$2\varepsilon_{rz} = u_{r,z} + u_{z,r}, \quad 2\varepsilon_{r\theta} = u_{\theta,r} + \frac{u_{r,\theta} - u_\theta}{r},$$

$$2\varepsilon_{z\theta} = u_{\theta,z} + \frac{u_{z,\theta}}{r}. \quad (21b)$$

Substituting Eqs. (19) and (21) into Eqs. (18), one obtains a set of three second-order partial differential equations in $u_r$, $u_z$, and $u_\theta$ governing free vibrations. However, in the case of paraboloidal shells, exact solutions are intractable because of the variable coefficients that appear in many terms. Alternatively, one may approach the problem from an energy perspective.

Because the strains are related to the displacement components by Eqs. (21), unacceptable strain singularities may be encountered exactly at $r$=0 due to the term $1/r$. Since a very small hole does not affect the frequencies,[25)] such singularities may be avoided by replacing the range for $\psi(\equiv r/R)$, $0 \le \psi \le \psi_i$ in Eq. (12a) and $0 \le \psi \le 1$ in Eq. (17), with $10^{-5} \le \psi \le \psi_i$ and $10^{-5} \le \psi \le 1$, respectively.

During vibratory deformation of the body, its strain (potential) energy ($V$) is the integral over the domain $(\Lambda)$:

$$V = \frac{1}{2}\int_\Lambda (\sigma_{rr}\varepsilon_{rr} + \sigma_{zz}\varepsilon_{zz} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + 2\sigma_{rz}\varepsilon_{rz} + 2\sigma_{r\theta}\varepsilon_{r\theta} + 2\sigma_{z\theta}\varepsilon_{z\theta})\, r\, dr\, dz\, d\theta \quad (22)$$

Substituting Eqs. (19) and (21) into Eq. (22) results in the strain energy in terms of the three displacements:

$$V = \frac{1}{2}\int_\Lambda [\lambda(\varepsilon_{rr} + \varepsilon_{zz} + \varepsilon_{\theta\theta})^2 + 2G\{\varepsilon_{rr}^2 + \varepsilon_{zz}^2 + \varepsilon_{\theta\theta}^2 + 2(\varepsilon_{rz}^2 + \varepsilon_{z\theta}^2 + \varepsilon_{r\theta}^2)\}]\, r\, dr\, dz\, d\theta \quad (23)$$

where the tensorial strains $\varepsilon_{ij}$ are expressed in terms of the three displacements by Eqs. (21).

The kinetic energy ($T$) is simply

$$T = \frac{1}{2}\int_{\Lambda} \rho(\dot{u}_r^2 + \dot{u}_z^2 + \dot{u}_\theta^2)\, r\, dr\, dz\, d\theta \quad (24)$$

For the free, undamped vibration, the time ($t$) response of the three displacements is sinusoidal and, moreover, the circular symmetry of the body of revolution allows the displacements to be expressed by

$$u_r(\psi,\zeta,\theta,t) = U_r(\psi,\zeta)\cos n\theta \sin(\omega t + \alpha), \quad (25a)$$

$$u_z(\psi,\zeta,\theta,t) = U_z(\psi,\zeta)\cos n\theta \sin(\omega t + \alpha), \quad (25b)$$

$$u_\theta(\psi,\zeta,\theta,t) = U_\theta(\psi,\zeta)\sin n\theta \sin(\omega t + \alpha), \quad (25c)$$

where $U_r$, $U_z$, and $U_\theta$ are displacement functions of $\psi$ and $\zeta$, $\omega$ is a natural frequency, and $\alpha$ is an arbitrary phase angle determined by the initial conditions. The circumferential wave number is taken to be an integer ($n$=0, 1, 2, 3, ⋯, $\infty$), to ensure periodicity in $\theta$. That the variables separable form of Eqs. (25) does apply may be verified by substituting the displacements into the 3-D equations of motion.[26] Then Eqs. (25) account for all free vibration modes except for the torsional ones. These modes arise from an alternative set of solutions which are the same as Eqs. (25), except that $\cos n\theta$ and $\sin n\theta$ are interchanged. For $n \geq 1$, this set duplicates the solutions of Eqs. (25), with the symmetry axes of the mode shapes being rotated. But for $n$=0 the alternative set reduces to $u_r = u_z$=0, $u_\theta = U_\theta^*(\psi,\zeta)\sin(\omega t + \alpha)$, which corresponds to the torsional modes. The displacements uncouple by circumferential wave number ($n$), leaving only coupling in $r$ (or $\psi$) and $z$ (or $\zeta$).

The Ritz method uses the maximum potential (strain) energy ($V_{\max}$) and the maximum kinetic energy ($T_{\max}$) functionals in a cycle of vibratory motion. The functionals for the shells are obtained by setting $\sin^2(\omega t + \alpha)$ and $\cos^2(\omega t + \alpha)$ equal to unity in Eqs. (23) and (24) after the displacements (25) are substituted, and by using the non-dimensional coordinates $\psi$ and $\zeta$ as follows:

$$V_{\max} = \frac{HG}{2}\left[\int_0^{\psi_i}\int_{\zeta_o}^{\zeta_i} I_V\, \psi\, d\zeta\, d\psi + \int_{\psi_i}^{\psi_o}\int_{\zeta_o}^{\zeta_b} I_V\, \psi\, d\zeta\, d\psi\right], \quad (26)$$

$$T_{\max} = \frac{\rho\omega^2 HR^2}{2}\left[\int_0^{\psi_i}\int_{\zeta_o}^{\zeta_i} I_T\, \psi\, d\zeta\, d\psi + \int_{\psi_i}^{\psi_o}\int_{\zeta_o}^{\zeta_b} I_T\, \psi\, d\zeta\, d\psi\right], \quad (27)$$

where

$$I_V \equiv \left[\frac{\lambda}{G}(\kappa_1 + \kappa_2 + \kappa_3)^2 + 2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) + \kappa_4^2\right] \Gamma_1 + (\kappa_5^2 + \kappa_6^2)\Gamma_2, \quad (28)$$

$$I_T \equiv (U_r^2 + U_z^2)\Gamma_1 + U_\theta^2 \Gamma_2, \quad (29)$$

and

$$\kappa_1 \equiv \frac{U_r + nU_\theta}{\psi}, \quad \kappa_2 \equiv U_{r,\psi}, \quad \kappa_3 \equiv \frac{U_{z,\zeta}}{k}, \quad (30a)$$

$$\kappa_4 \equiv U_{z,\psi} + \frac{U_{r,\zeta}}{k}, \quad \kappa_5 \equiv \frac{nU_z}{\psi} - \frac{U_{\theta,\zeta}}{k},$$

$$\kappa_6 \equiv \frac{nU_r + U_\theta}{\psi} - U_{\theta,\psi}, \quad (30b)$$

and $\Gamma_1$ and $\Gamma_2$ are constants, defined by

$$\Gamma_1 \equiv \int_0^{2\pi} \cos^2 n\theta\, d\theta = \begin{cases} 2\pi & \text{if } n = 0 \\ \pi & \text{if } n \geq 1 \end{cases},$$

$$\Gamma_2 \equiv \int_0^{2\pi} \sin^2 n\theta\, d\theta = \begin{cases} 0 & \text{if } n = 0 \\ \pi & \text{if } n \geq 1 \end{cases}. \quad (31)$$

From Eqs. (20) it is seen that the non-dimensional constant $\lambda/G$ in Eq. (28) involves only $\nu$ as follows

$$\frac{\lambda}{G}=\frac{2\nu}{1-2\nu}. \qquad (32)$$

For solid paraboloids, the maximum energy functionals are given simply by

$$V_{\max}=\frac{HG}{2}\int_0^1\int_{\psi^2}^1 I_V\,\psi\,d\zeta\,d\psi, \qquad (33)$$

$$T_{\max}=\frac{\rho\omega^2 HR^2}{2}\int_0^1\int_{\psi^2}^1 I_T\,\psi\,d\zeta\,d\psi. \qquad (34)$$

The displacement functions $U_r$, $U_z$ and $U_\theta$ in Eqs. (25) are further assumed as algebraic polynomials,

$$U_r(\psi,\zeta)=\eta_r\sum_{i=0}^{I}\sum_{j=0}^{J}A_{ij}\,\psi^i\zeta^j \qquad (35a)$$

$$U_z(\psi,\zeta)=\eta_z\sum_{k=0}^{K}\sum_{l=0}^{L}B_{kl}\,\psi^k\zeta^l \qquad (35b)$$

$$U_\theta(\psi,\zeta)=\eta_\theta\sum_{m=0}^{M}\sum_{n=0}^{N}C_{mn}\,\psi^m\zeta^n \qquad (35c)$$

and similarly for $U_\theta^*$, where *i, j, k, l, m,* and *n* are integers; *I, J, K, L, M,* and *N* are the highest degrees taken in the polynomial terms; $A_{ij}$, $B_{kl}$ and $C_{mn}$ are arbitrary coefficients to be determined, and the $\eta$ are functions depending upon the geometric boundary conditions to be enforced. For example:

1. completely free: $\eta_r=\eta_z=\eta_\theta=1$,
2. the bottom edge fixed: $\eta_r=\eta_z=\eta_\theta=\zeta_b$,

The functions of $\eta$ shown above, impose only the necessary geometric constraints related to displacement boundary conditions. Together with the algebraic polynomials in Eqs. (35), they form function sets which are mathematically complete (Kantorovich and Krylov[27], pp. 266-268). Thus, the function sets are capable of representing any 3-D motion of the shell with increasing accuracy as the indices *I, J,* ⋯, *N* are increased. In the limit, as sufficient terms are taken, all internal kinematic constraints vanish, and the functions (35) will approach the exact solution as closely as desired.

The eigenvalue problem is formulated by minimizing the free vibration frequencies with respect to the arbitrary coefficients $A_{ij}$, $B_{kl}$ and $C_{mn}$, thereby minimizing the effects of the internal constraints present, when the function sets are finite. This corresponds to the equations (Ritz[28]):

$$\frac{\partial}{\partial A_{ij}}(V_{\max}-T_{\max})=0,\quad (i=0,1,2,\ldots,I;\ j=0,1,2,\ldots,J) \qquad (36a)$$

$$\frac{\partial}{\partial B_{kl}}(V_{\max}-T_{\max})=0,\quad (k=0,1,2,\ldots,K;\ l=0,1,2,\ldots,L) \qquad (36b)$$

$$\frac{\partial}{\partial C_{mn}}(V_{\max}-T_{\max})=0,\quad (m=0,1,2,\ldots,M;\ n=0,1,2,\ldots,N) \qquad (36c)$$

Equations (36) yield a set of (*I*+1)(*J*+1) +(*K*+1)(*L*+1)+(*M*+1)(*N*+1) linear, homogeneous, algebraic equations in the unknowns $A_{ij}$, $B_{kl}$, and $C_{mn}$. The equations can be written in the form

$$(\mathbf{K}-\Omega\,\mathbf{M})\,\mathbf{x}=\mathbf{0}, \qquad (37)$$

where **K** and **M** are stiffness and mass matrices resulting from the maximum strain energy ($V_{\max}$) and the maximum kinetic energy

($T_{max}$), respectively, and $\Omega$ is an eigenvalue of the vibrating system, expressed as the square of non-dimensional frequency, $\Omega \equiv \rho\omega^2 R^2 / G$, and the vector $\mathbf{x}$ takes the form

$$\mathbf{x} = (A_{00}, A_{01}, ..., A_{IJ}; B_{00}, B_{01}, ..., B_{KL}; C_{00}, C_{01}, ..., C_{MN})^T \quad (38)$$

For a nontrivial solution, the determinant of the coefficient matrix is set equal to zero, which yields the frequencies (eigenvalues); that is to say $|\mathbf{K} - \Omega\,\mathbf{M}| = 0$. These frequencies are upper bounds on the exact values. The mode shape (eigenfunction) corresponding to each frequency is obtained, in the usual manner, by substituting each $\omega$ back into the set of algebraic equations, and solving for the ratios of coefficients.

## 3. CONVERGENCE STUDIES

To guarantee the accuracy of frequencies obtained by the procedure described above, it is necessary to conduct some convergence studies to determine the number of terms required in the power series of Eqs. (35). A convergence study is based upon the fact that, if the displacements are expressed as power series, all the frequencies obtained by the Ritz method should converge to their exact values in an upper bound manner. If the results do not converge properly, or converge too slowly, it would be likely that the assumed displacement functions chosen are poor ones, or be missing some functions from a minimal complete set of polynomials.

Table 1 is such a study for a completely free, shallow ($H/R$=1/3), complete paraboloidal shell of revolution having variable thickness ($h_t/h_b$ =1/3) with $h_b/R$=1/10. The table lists the first

Table 1. Convergence of frequencies in $\omega R\sqrt{\rho/G}$ of a free, complete paraboloidal shell of revolution having variable thickness ($h_t/h_b$ =1/3) for the five lowest bending modes (n=2) with $H/R$ =1/3 and $h_b/R$ =1/10 for $\nu$ =0.3.

| TZ | TR | DET | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 18 | 0.1828 | 1.881 | 2.285 | 4.142 | 7.945 |
| 2 | 4 | 24 | 0.1746 | 1.701 | 1.929 | 4.087 | 5.117 |
| 2 | 5 | 30 | 0.1718 | 1.439 | 1.901 | 4.048 | 4.095 |
| 2 | 6 | 36 | 0.1711 | 1.379 | 1.899 | 2.936 | 4.084 |
| 2 | 7 | 42 | 0.1706 | 1.375 | 1.899 | 2.593 | 4.083 |
| 2 | 8 | 48 | 0.1704 | 1.372 | 1.898 | 2.560 | 4.083 |
| 2 | 9 | 54 | 0.1703 | 1.371 | 1.898 | 2.530 | 4.083 |
| 3 | 3 | 27 | 0.1693 | 1.732 | 1.938 | 4.098 | 7.232 |
| 3 | 4 | 36 | 0.1613 | 1.432 | 1.901 | 3.731 | 4.090 |
| 3 | 5 | 45 | 0.1608 | 1.295 | 1.897 | 2.874 | 4.083 |
| 3 | 6 | 54 | 0.1603 | 1.279 | **1.896** | 2.305 | 4.083 |
| 3 | 7 | 63 | 0.1600 | 1.274 | 1.896 | 2.210 | 3.861 |
| 3 | 8 | 72 | 0.1598 | 1.273 | 1.896 | 2.183 | 3.642 |
| 3 | 9 | 81 | 0.1597 | 1.272 | 1.896 | 2.171 | 3.549 |
| 4 | 3 | 36 | 0.1636 | 1.588 | 1.912 | 4.086 | 4.774 |
| 4 | 4 | 48 | 0.1604 | 1.346 | 1.898 | 3.248 | 4.085 |
| 4 | 5 | 60 | 0.1600 | 1.280 | 1.897 | 2.521 | 4.083 |
| 4 | 6 | 72 | 0.1598 | 1.274 | 1.896 | 2.217 | 4.082 |
| 4 | 7 | 84 | 0.1596 | 1.272 | 1.896 | 2.186 | 3.643 |
| 4 | 8 | 96 | 0.1596 | **1.271** | 1.896 | 2.170 | 3.563 |
| 4 | 9 | 108 | 0.1596 | 1.271 | 1.896 | 2.166 | 3.502 |
| 5 | 3 | 45 | 0.1612 | 1.466 | 1.903 | 3.696 | 4.087 |
| 5 | 4 | 60 | 0.1601 | 1.307 | 1.897 | 2.904 | 4.083 |
| 5 | 5 | 75 | 0.1597 | 1.276 | 1.896 | 2.328 | 4.083 |
| 5 | 6 | 90 | 0.1596 | 1.272 | 1.896 | 2.193 | 3.835 |
| 5 | 7 | 105 | 0.1596 | 1.271 | 1.896 | 2.176 | 3.581 |
| 5 | 8 | 120 | 0.1595 | 1.271 | 1.896 | 2.167 | 3.526 |
| 5 | 9 | 135 | 0.1595 | 1.271 | 1.896 | **2.164** | 3.488 |
| 6 | 3 | 54 | 0.1610 | 1.413 | 1.901 | 3.457 | 4.085 |
| 6 | 4 | 72 | 0.1597 | 1.290 | 1.897 | 2.636 | 4.083 |
| 6 | 5 | 90 | 0.1596 | 1.273 | 1.896 | 2.242 | 4.082 |
| 6 | 6 | 108 | 0.1596 | 1.272 | 1.896 | 2.183 | 3.664 |
| 6 | 7 | 126 | 0.1595 | 1.271 | 1.896 | 2.171 | 3.551 |
| 6 | 8 | 144 | 0.1595 | 1.271 | 1.896 | 2.165 | 3.503 |
| 6 | 9 | 162 | 0.1595 | 1.271 | 1.896 | 2.164 | 3.481 |
| 6 | 10 | 180 | 0.1595 | 1.271 | 1.896 | 2.164 | **3.478** |
| 7 | 3 | 63 | 0.1608 | 1.401 | 1.901 | 3.140 | 4.084 |
| 7 | 4 | 84 | 0.1596 | 1.281 | 1.897 | 2.468 | 4.083 |
| 7 | 5 | 105 | **0.1595** | 1.272 | 1.896 | 2.203 | 3.995 |
| 7 | 6 | 126 | 0.1595 | 1.271 | 1.896 | 2.178 | 3.580 |
| 7 | 7 | 147 | 0.1595 | 1.271 | 1.896 | 2.167 | 3.529 |
| 7 | 8 | 168 | 0.1595 | 1.271 | 1.896 | 2.164 | 3.489 |
| 7 | 9 | 189 | 0.1595 | 1.271 | 1.896 | 2.164 | 3.479 |
| 7 | 10 | 210 | 0.1595 | 1.271 | 1.896 | 2.164 | 3.478 |

five nondimensional frequencies in $\omega R\sqrt{\rho/G}$ for $\nu$=0.3, for modes having two circumferential waves ($n$=2) in their mode shapes.

To make the study of convergence less complicated, equal numbers of polynomial terms were taken in both the $r$ (or $\Psi$) coordinate (i.e., $I$=$K$=$M$) and $z$ (or $\zeta$) coordinate (i.e., $J$=$L$=$N$), although some computational optimization could be obtained for some configurations and some mode shapes by using unequal numbers of polynomial terms.

The symbols ***TZ*** and ***TR*** in the table indicate the total numbers of polynomial terms used in the $z$ (or $\zeta$) and $r$ (or $\Psi$) directions, respectively. Note that the frequency determinant order ***DET*** is related to ***TZ*** and ***TR*** as follows:

$$\boldsymbol{DET}=\begin{cases} \boldsymbol{TZ}\times\boldsymbol{TR} & \text{for torsional modes } (n=0) \\ 2\times\boldsymbol{TZ}\times\boldsymbol{TR} & \text{for axisymmetric modes } (n=0) \\ 3\times\boldsymbol{TZ}\times\boldsymbol{TR} & \text{for general modes } (n\geq 1) \end{cases} \quad (39)$$

Table 1 shows the monotonic convergence of all five frequencies as ***TZ*** (= $J$+1, $L$+1, and $N$+1 in Eqs. (35)) are increased, as well as ***TR*** (= $I$+1, $K$+1, and $M$+1 in Eqs. (35)). One sees, for example, that the fundamental (i.e., lowest) nondimensional frequency in $\omega R\sqrt{\rho/G}$ converges to four digits (0.1595) when 3×(7×5)=105 terms are used, which results in ***DET***=105. Moreover, this accuracy requires using at least seven terms through the axial direction (***TZ***=7) and five ones through the radial direction (***TR***=5).

Table 2 is a similar convergence study for a completely free, deep ($H/R$=3), complete paraboloidal shell of revolution having variable thickness ($h_t/h_b$=1/3) with $h_b/R$=1/10. The table lists the first five nondimensional fre-

Table 2. Convergence of frequencies in $\omega R\sqrt{\rho/G}$ of a free, complete paraboloidal shell of revolution having variable thickness ($h_t/h_b$ =1/3) for the five lowest torsional modes (n=0T) with $H/R$=3 and $h_b/R$=1/10 for $\nu$=0.3.

| *TR* | *TZ* | *DET* | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 6 | 1.449 | 3.178 | 16.39 | 37.29 | 50.92 |
| 2 | 4 | 8 | 1.425 | 2.516 | 5.064 | 19.08 | 36.25 |
| 2 | 5 | 10 | **1.423** | 2.455 | 3.597 | 7.346 | 21.50 |
| 2 | 6 | 12 | 1.423 | 2.435 | 3.504 | 4.728 | 10.03 |
| 2 | 7 | 14 | 1.423 | **2.434** | 3.430 | 4.612 | 5.944 |
| 2 | 8 | 16 | 1.423 | 2.434 | 3.427 | 4.423 | 5.808 |
| 2 | 9 | 18 | 1.423 | 2.434 | 3.425 | 4.416 | 5.422 |
| 3 | 3 | 9 | 1.432 | 2.832 | 4.862 | 22.30 | 36.82 |
| 3 | 4 | 12 | 1.424 | 2.467 | 4.126 | 6.851 | 21.94 |
| 3 | 5 | 15 | 1.423 | 2.442 | 3.511 | 5.366 | 9.260 |
| 3 | 6 | 18 | 1.423 | 2.434 | 3.453 | 4.593 | 6.630 |
| 3 | 7 | 21 | 1.423 | 2.434 | 3.426 | 4.476 | 5.741 |
| 3 | 8 | 24 | 1.423 | 2.434 | 3.425 | 4.416 | 5.520 |
| 3 | 9 | 27 | 1.423 | 2.434 | **3.424** | 4.409 | 5.412 |
| 4 | 3 | 12 | 1.425 | 2.601 | 4.357 | 7.600 | 23.98 |
| 4 | 4 | 16 | 1.424 | 2.443 | 3.714 | 5.892 | 9.419 |
| 4 | 5 | 20 | 1.423 | 2.437 | 3.462 | 4.764 | 7.424 |
| 4 | 6 | 24 | 1.423 | 2.434 | 3.433 | 4.509 | 5.865 |
| 4 | 7 | 28 | 1.423 | 2.434 | 3.425 | 4.430 | 5.597 |
| 4 | 8 | 32 | 1.423 | 2.434 | 3.424 | 4.410 | 5.435 |
| 4 | 9 | 36 | 1.423 | 2.434 | 3.424 | 4.408 | 5.398 |
| 5 | 3 | 15 | 1.424 | 2.497 | 4.082 | 6.156 | 11.02 |
| 5 | 4 | 20 | 1.423 | 2.439 | 3.517 | 5.177 | 7.896 |
| 5 | 5 | 25 | 1.423 | 2.434 | 3.445 | 4.542 | 6.264 |
| 5 | 6 | 30 | 1.423 | 2.434 | 3.426 | 4.463 | 5.604 |
| 5 | 7 | 35 | 1.423 | 2.434 | 3.424 | 4.412 | 5.497 |
| 5 | 8 | 40 | 1.423 | 2.434 | 3.424 | 4.409 | 5.401 |
| 5 | 9 | 45 | 1.423 | 2.434 | 3.424 | **4.407** | 5.391 |
| 6 | 3 | 18 | 1.424 | 2.491 | 3.736 | 5.933 | 8.669 |
| 6 | 4 | 24 | 1.423 | 2.437 | 3.455 | 4.783 | 6.858 |
| 6 | 5 | 30 | 1.423 | 2.434 | 3.433 | 4.482 | 5.751 |
| 6 | 6 | 36 | 1.423 | 2.434 | 3.424 | 4.428 | 5.535 |
| 6 | 7 | 42 | 1.423 | 2.434 | 3.424 | 4.409 | 5.426 |
| 6 | 8 | 48 | 1.423 | 2.434 | 3.424 | 4.408 | 5.393 |
| 6 | 9 | 54 | 1.423 | 2.434 | 3.424 | 4.407 | 5.388 |
| 6 | 10 | 60 | 1.423 | 2.434 | 3.424 | 4.407 | **5.387** |
| 7 | 3 | 21 | 1.424 | 2.471 | 3.693 | 5.217 | 8.322 |
| 7 | 4 | 28 | 1.423 | 2.435 | 3.449 | 4.560 | 6.341 |
| 7 | 5 | 35 | 1.423 | 2.434 | 3.427 | 4.451 | 5.578 |
| 7 | 6 | 42 | 1.423 | 2.434 | 3.424 | 4.412 | 5.472 |
| 7 | 7 | 49 | 1.423 | 2.434 | 3.424 | 4.408 | 5.398 |
| 7 | 8 | 56 | 1.423 | 2.434 | 3.424 | 4.407 | 5.390 |
| 7 | 9 | 63 | 1.423 | 2.434 | 3.424 | 4.407 | 5.387 |
| 7 | 10 | 70 | 1.423 | 2.434 | 3.424 | 4.407 | 5.387 |

quencies in $\omega R\sqrt{\rho/G}$ for the torsional modes ($n$=$0^T$). One sees that the first torsional frequency (1.423) requires using only (***TR***, ***TZ***)=(2, 5), which results in ***DET***=10, for four significant figure exactitude.

It is interesting to note in both Tables 1 and 2 that the modes for $n$=2 require much larger size of ***DET*** compared with the torsional modes ($n$=$0^T$), at least for the first two frequencies. This is primarily because only the circumferential displacement components ($u_\theta$) are involved in the torsional modes, whereas all three components enter into the modes having $n \geq 1$, as seen in Eqs. (39).

Frequencies in underlined, bold-faced type in Tables 1 and 2 are the exact values (to four significant figures) achieved with the smallest determinant sizes. Comparing the numbers of terms (***TZ***, ***TR***) yielding the underlined 10 frequencies, one finds that, as the shell becomes deeper, i.e., as $H/R$ becomes larger, more polynomial terms in the axial direction (***TZ***) are typically required for accurate frequencies converged to four significant figures with an exception of the first frequency (0.1595) for $n$=2 in Table 1 for a shallow shell ($H/R$=1/3), which resulted from (***TZ***, ***TR***)=(7, 5).

## 4. NUMERICAL RESULTS AND DISCUSSION

Tables 3~7 present the nondimensional frequencies in $\omega R\sqrt{\rho/G}$ of free, complete paraboloidal shells of revolution with $H/R$=1/3, 1/2, 1, 2, and 3, respectively. Each table is for three shell configurations of ($h_b/R$, $h_t/h_b$) =(1/30, 1), (1/10, 1/3), and (1/10, 1). That is, in

Table 3. Frequencies in $\omega R\sqrt{\rho/G}$ of free, complete paraboloidal shells of revolution with $H/R$ =1/3 for $\nu$=0.3.

| n | s | $h_b/R$=1/30, $h_t/h_b$=1 | $h_b/R$=1/10, $h_t/h_b$=1/3 | $h_b/R$=1/10, $h_t/h_b$=1 | n | s | $h_b/R$=1/30, $h_t/h_b$=1 | $h_b/R$=1/10, $h_t/h_b$=1/3 | $h_b/R$=1/10, $h_t/h_b$=1 |
|---|---|---|---|---|---|---|---|---|---|
| $0^T$ | 1 | 4.754 | 5.096 | 4.751 | 3 | 1 | **0.1967(2)** | **0.4255(2)** | **0.5634(2)** |
| | 2 | 7.839 | 8.068 | 7.834 | | 2 | 1.176 | 1.653 | 2.282 |
| | 3 | 10.84 | 11.01 | 10.83 | | 3 | 1.829 | 2.790 | 3.519 |
| | 4 | 13.81 | 13.94 | 13.81 | | 4 | 2.798 | 3.073 | 4.236 |
| | 5 | 16.78 | 16.90 | 16.80 | | 5 | 3.519 | 4.306 | 5.748 |
| $0^A$ | 1 | **0.9423(6)** | **1.030(4)** | **0.9984(4)** | 4 | 1 | **0.3467(3)** | **0.7858(3)** | **0.9800(3)** |
| | 2 | 1.121 | 1.352 | 1.793 | | 2 | 1.372 | 2.179 | 2.998 |
| | 3 | 1.568 | 2.150 | 3.427 | | 3 | 2.200 | 3.525 | 4.577 |
| | 4 | 2.372 | 3.125 | 3.541 | | 4 | 3.316 | 4.134 | 5.204 |
| | 5 | 3.490 | 3.466 | 5.592 | | 5 | 4.578 | 5.208 | 7.320 |
| 1 | 1 | **0.9854** | **1.094(5)** | **1.243(5)** | 5 | 1 | **0.5305(4)** | 1.233 | 1.481 |
| | 2 | 1.286 | 1.672 | 2.502 | | 2 | 1.634 | 2.816 | 3.797 |
| | 3 | 1.917 | 2.656 | 2.684 | | 3 | 2.623 | 4.345 | 5.576 |
| | 4 | 2.686 | 2.759 | 4.434 | | 4 | 3.876 | 5.145 | 6.217 |
| | 5 | 2.883 | 4.288 | 5.649 | | 5 | 5.425 | 6.170 | 8.850 |
| 2 | 1 | **0.08287(1)** | **0.1595(1)** | **0.2417(1)** | 6 | 1 | **0.7474(5)** | 1.759 | 2.057 |
| | 2 | 1.050 | 1.271 | 1.681 | | 2 | 1.954 | 3.542 | 4.658 |
| | 3 | 1.521 | 1.896 | 2.306 | | 3 | 3.093 | 5.233 | 6.547 |
| | 4 | 2.306 | 2.164 | 3.328 | | 4 | 4.475 | 6.128 | 7.262 |
| | 5 | 2.330 | 3.478 | 4.180 | | 5 | 6.169 | 7.192 | 10.07 |

Table 4. Frequencies in $\omega R\sqrt{\rho/G}$ of free, complete paraboloidal shells of revolution with $H/R$ =1/2 for $\nu$ =0.3.

| n | s | $h_b/R$=1/30 $h_t/h_b$=1 | $h_b/R$=1/10 $h_t/h_b$=1/3 | $h_b/R$=1/10 $h_t/h_b$=1 | n | s | $h_b/R$=1/30 $h_t/h_b$=1 | $h_b/R$=1/10 $h_t/h_b$=1/3 | $h_b/R$=1/10 $h_t/h_b$=1 |
|---|---|---|---|---|---|---|---|---|---|
| $0^T$ | 1 | 4.375 | 4.698 | 4.370 | 3 | 1 | **0.1864(2)** | **0.4287(2)** | **0.5375(2)** |
| | 2 | 7.269 | 7.502 | 7.261 | | 2 | 1.303 | 1.702 | 2.171 |
| | 3 | 10.08 | 10.25 | 10.06 | | 3 | 1.851 | 2.736 | 3.451 |
| | 4 | 12.85 | 12.99 | 12.84 | | 4 | 2.634 | 3.077 | 3.857 |
| | 5 | 15.62 | 15.73 | 15.60 | | 5 | 3.450 | 4.070 | 5.669 |
| $0^A$ | 1 | **1.204(6)** | **1.282(5)** | **1.250(4)** | 4 | 1 | **0.3325(3)** | **0.7920(3)** | **0.9443(3)** |
| | 2 | 1.422 | 1.664 | 1.844 | | 2 | 1.424 | 2.161 | 2.795 |
| | 3 | 1.739 | 2.243 | 3.148 | | 3 | 2.129 | 3.395 | 4.482 |
| | 4 | 2.318 | 3.278 | 3.614 | | 4 | 3.058 | 4.123 | 4.734 |
| | 5 | 3.205 | 3.325 | 5.009 | | 5 | 4.231 | 4.897 | 6.976 |
| 1 | 1 | 1.248 | 1.356 | **1.411(5)** | 5 | 1 | **0.5127(4)** | **1.241(4)** | 1.436 |
| | 2 | 1.521 | 1.859 | 2.383 | | 2 | 1.620 | 2.751 | 3.525 |
| | 3 | 1.972 | 2.624 | 2.636 | | 3 | 2.474 | 4.157 | 5.461 |
| | 4 | 2.638 | 2.723 | 4.005 | | 4 | 3.536 | 5.122 | 5.673 |
| | 5 | 2.715 | 4.024 | 5.298 | | 5 | 4.831 | 5.802 | 8.051 |
| 2 | 1 | **0.07685(1)** | **0.1597(1)** | **0.2257(1)** | 6 | 1 | **0.7255(5)** | 1.769 | 2.002 |
| | 2 | 1.253 | 1.396 | 1.692 | | 2 | 1.885 | 3.442 | 4.331 |
| | 3 | 1.648 | 1.932 | 2.281 | | 3 | 2.876 | 5.000 | 6.418 |
| | 4 | 2.268 | 2.212 | 3.063 | | 4 | 4.061 | 6.095 | 6.656 |
| | 5 | 2.279 | 3.337 | 4.122 | | 5 | 5.473 | 6.769 | 9.152 |

Table 5. Frequencies in $\omega R\sqrt{\rho/G}$ of free, complete paraboloidal shells of revolution with $H/R$ =1 for $\nu$ =0.3.

| n | s | $h_b/R$=1/30 $h_t/h_b$=1 | $h_b/R$=1/10 $h_t/h_b$=1/3 | $h_b/R$=1/10 $h_t/h_b$=1 | n | s | $h_b/R$=1/30 $h_t/h_b$=1 | $h_b/R$=1/10 $h_t/h_b$=1/3 | $h_b/R$=1/10 $h_t/h_b$=1 |
|---|---|---|---|---|---|---|---|---|---|
| $0^T$ | 1 | 3.234 | 3.456 | 3.228 | 3 | 1 | **0.1611(2)** | **0.4204(2)** | **0.4725(2)** |
| | 2 | 5.504 | 5.708 | 5.493 | | 2 | 1.171 | 1.435 | 1.662 |
| | 3 | 7.707 | 7.880 | 7.691 | | 3 | 1.686 | 2.325 | 2.788 |
| | 4 | 9.881 | 10.03 | 9.860 | | 4 | 2.172 | 3.117 | 3.301 |
| | 5 | 12.04 | 12.16 | 12.01 | | 5 | 2.752 | 3.264 | 4.099 |
| $0^A$ | 1 | 1.494 | 1.540 | 1.528 | 4 | 1 | **0.2953(3)** | **0.7819(3)** | **0.8542(3)** |
| | 2 | 1.731 | 1.946 | 1.945 | | 2 | 1.168 | 1.790 | 2.103 |
| | 3 | 2.056 | 2.510 | 2.620 | | 3 | 1.750 | 2.768 | 3.393 |
| | 4 | 2.402 | 3.077 | 3.565 | | 4 | 2.329 | 3.824 | 4.263 |
| | 5 | 2.796 | 3.630 | 3.626 | | 5 | 3.009 | 4.110 | 4.854 |
| 1 | 1 | 1.514 | 1.598 | 1.590 | 5 | 1 | **0.4636(4)** | **1.229(4)** | **1.321(4)** |
| | 2 | 1.777 | 2.065 | 2.120 | | 2 | 1.269 | 2.311 | 2.697 |
| | 3 | 2.116 | 2.504 | 2.485 | | 3 | 1.910 | 3.375 | 4.119 |
| | 4 | 2.460 | 2.674 | 2.991 | | 4 | 2.577 | 4.535 | 5.203 |
| | 5 | 2.531 | 3.380 | 4.031 | | 5 | 3.349 | 5.062 | 5.699 |
| 2 | 1 | **0.06285(1)** | **0.1542(1)** | **0.1864(1)** | 6 | 1 | **0.6647(5)** | 1.754 | 1.863 |
| | 2 | 1.265 | **1.249(5)** | **1.434(5)** | | 2 | 1.459 | 2.948 | 3.393 |
| | 3 | 1.703 | 2.058 | 2.268 | | 3 | 2.158 | 4.099 | 4.931 |
| | 4 | 2.106 | 2.148 | 2.368 | | 4 | 2.903 | 5.347 | 6.134 |
| | 5 | 2.272 | 2.856 | 3.467 | | 5 | 3.757 | 6.004 | 6.612 |

Table 6. Frequencies in $\omega R\sqrt{\rho/G}$ of free, complete paraboloidal shells of revolution with $H/R$ =2 for $\nu$ =0.3.

| n | s | $h_b/R=1/30$ | $h_b/R=1/10$ | $h_b/R=1/10$ | n | s | $h_b/R=1/30$ | $h_b/R=1/10$ | $h_b/R=1/10$ |
|---|---|---|---|---|---|---|---|---|---|
| | | $h_t/h_b=1$ | $h_t/h_b=1/3$ | $h_t/h_b=1$ | | | $h_t/h_b=1$ | $h_t/h_b=1/3$ | $h_t/h_b=1$ |
| $0^T$ | 1 | 1.948 | 2.048 | 1.945 | 3 | 1 | **0.1452(2)** | **0.4069(2)** | **0.4258(2)** |
| | 2 | 3.363 | 3.478 | 3.357 | | 2 | **0.6135(5)** | **0.8765(5)** | **0.9728(5)** |
| | 3 | 4.757 | 4.872 | 4.748 | | 3 | 1.115 | 1.465 | 1.627 |
| | 4 | 6.142 | 6.253 | 6.130 | | 4 | 1.484 | 2.023 | 2.266 |
| | 5 | 7.522 | 7.626 | 7.507 | | 5 | 1.795 | 2.609 | 2.970 |
| $0^A$ | 1 | 1.581 | 1.603 | 1.593 | 4 | 1 | **0.2700(3)** | **0.7593(4)** | **0.7845(3)** |
| | 2 | 1.721 | 1.833 | 1.813 | | 2 | 0.6438 | 1.253 | 1.374 |
| | 3 | 1.924 | 2.194 | 2.175 | | 3 | 1.064 | 1.803 | 2.032 |
| | 4 | 2.140 | 2.581 | 2.579 | | 4 | 1.438 | 2.387 | 2.733 |
| | 5 | 2.395 | 2.906 | 2.838 | | 5 | 1.790 | 3.022 | 3.513 |
| 1 | 1 | 1.486 | 1.518 | 1.499 | 5 | 1 | **0.4279(4)** | 1.197 | 1.228 |
| | 2 | 1.593 | 1.661 | 1.630 | | 2 | 0.7758 | 1.767 | 1.909 |
| | 3 | 1.761 | 1.987 | 1.948 | | 3 | 1.155 | 2.340 | 2.611 |
| | 4 | 1.977 | 2.335 | 2.319 | | 4 | 1.528 | 2.956 | 3.368 |
| | 5 | 2.206 | 2.558 | 2.579 | | 5 | 1.908 | 3.634 | 4.209 |
| 2 | 1 | **0.05420(1)** | **0.1486(1)** | **0.1606(1)** | 6 | 1 | 0.6184 | 1.711 | 1.748 |
| | 2 | 0.7165 | **0.7258(3)** | **0.8033(4)** | | 2 | 0.9769 | 2.375 | 2.536 |
| | 3 | 1.336 | 1.445 | 1.516 | | 3 | 1.347 | 2.996 | 3.297 |
| | 4 | 1.643 | 1.934 | 2.049 | | 4 | 1.728 | 3.655 | 4.111 |
| | 5 | 1.908 | 2.203 | 2.250 | | 5 | 2.136 | 4.376 | 5.024 |

Table 7. Frequencies in $\omega R\sqrt{\rho/G}$ of free, complete paraboloidal shells of revolution with $H/R$ =3 for $\nu$ =0.3.

| n | s | $h_b/R=1/30$ | $h_b/R=1/10$ | $h_b/R=1/10$ | n | s | $h_b/R=1/30$ | $h_b/R=1/10$ | $h_b/R=1/10$ |
|---|---|---|---|---|---|---|---|---|---|
| | | $h_t/h_b=1$ | $h_t/h_b=1/3$ | $h_t/h_b=1$ | | | $h_t/h_b=1$ | $h_t/h_b=1/3$ | $h_t/h_b=1$ |
| $0^T$ | 1 | 1.369 | 1.423 | 1.368 | 3 | 1 | **0.1403(2)** | **0.4004(2)** | **0.4099(2)** |
| | 2 | 2.364 | 2.434 | 2.362 | | 2 | **0.3734(4)** | **0.6705(4)** | **0.7228(4)** |
| | 3 | 3.349 | 3.424 | 3.344 | | 3 | 0.7171 | 1.034 | 1.132 |
| | 4 | 4.330 | 4.407 | 4.324 | | 4 | 1.043 | 1.416 | 1.551 |
| | 5 | 5.311 | 5.387 | 5.305 | | 5 | 1.324 | 1.807 | 1.985 |
| $0^A$ | 1 | 1.533 | 1.572 | 1.533 | 4 | 1 | **0.2616(3)** | **0.7468(5)** | **0.7595(5)** |
| | 2 | 1.644 | 1.699 | 1.676 | | 2 | 0.4595 | 1.065 | 1.126 |
| | 3 | 1.786 | 1.931 | 1.905 | | 3 | 0.7194 | 1.405 | 1.525 |
| | 4 | 1.952 | 2.186 | 2.168 | | 4 | 0.9887 | 1.769 | 1.950 |
| | 5 | 2.131 | 2.411 | 2.414 | | 5 | 1.252 | 2.165 | 2.416 |
| 1 | 1 | 1.111 | 1.178 | 1.113 | 5 | 1 | 0.4154 | 1.178 | 1.194 |
| | 2 | 1.372 | 1.383 | 1.376 | | 2 | 0.6166 | 1.567 | 1.637 |
| | 3 | 1.575 | 1.638 | 1.614 | | 3 | 0.8453 | 1.932 | 2.068 |
| | 4 | 1.723 | 1.909 | 1.884 | | 4 | 1.086 | 2.318 | 2.525 |
| | 5 | 1.908 | 2.205 | 2.192 | | 5 | 1.336 | 2.737 | 3.055 |
| 2 | 1 | **0.05193(1)** | **0.1468(1)** | **0.1533(1)** | 6 | 1 | 0.6017 | 1.686 | 1.705 |
| | 2 | **0.4066(5)** | **0.4538(3)** | **0.4958(3)** | | 2 | 0.8225 | 2.152 | 2.232 |
| | 3 | 0.9152 | 0.9770 | 1.028 | | 3 | 1.047 | 2.560 | 2.709 |
| | 4 | 1.295 | 1.407 | 1.465 | | 4 | 1.282 | 2.980 | 3.212 |
| | 5 | 1.548 | 1.779 | 1.855 | | 5 | 1.546 | 3.450 | 3.832 |

each of the five tables, frequencies are given for thin and thick shells of uniform thickness, and also one having variable thickness. Poisson's ratio (ν) was taken to be 0.3. Forty frequencies are given for each configuration, which arise from eight circumferential wave numbers ($n$=$0^T$, $0^A$, 1, 2, 3, 4, 5, 6) and the first five modes ($s$ =1, 2, 3, 4, 5) for each value of $n$, where the superscripts $T$ and $A$ indicate torsional and axisymmetric modes, respectively. The numbers in parentheses identify the first five frequencies for each configuration. The zero frequencies of rigid body modes are omitted from the tables.

It is interesting to note in Tables 3~7 that, irrespective of shell configurations, the fundamental (lowest) and the second frequencies are for modes having two ($n$=2) and three ($n$=3) circumferential half-waves, respectively, and the torsional frequencies ($n$=$0^T$) are all for higher modes. It is also seen that as the shells become shallower (smaller $H/R$) the axisymmetric modes ($n$=$0^A$) are more important. That is, they are among the lowest frequencies of the body.

Table 8 gives the nondimensional frequencies in $\omega R\sqrt{\rho/G}$ of completely free, solid paraboloids with a radius of circular bottom $R$ and a height $H$ for $H/R$=1/3, 1/2, 1, 2, and 3. Poisson's ratio (ν) was again taken to be 0.3. It is interesting to note that $H/R$=1/3 (and perhaps $H/R$=1/2) may be regarded as a shallow shell, with thickness varying such that the bottom surface is a plane.

One sees in Table 8 that for solid paraboloids having small values (1/3, 1/2, 1) of $H/R$ the fundamental frequencies are for modes having ($n$=2), while for large $H/R$=2, 3 they are for $n$=1. The torsional modes ($n$=$0^T$) become more important with increasing $H/R$. In general, one notes in Table 8 that the paraboloids of larger mass (larger $H$ for a fixed $R$) have higher frequencies for the higher circumferential wave numbers ($n$), but lower ones for $n$=1. Thus, the stiffnesses of the larger paraboloids increase more than their masses for most vibration modes, but not for $n$=1. Compared with paraboloidal shells in Tables 3~7, the axisymmetric modes ($n$=$0^A$) of solid paraboloids are more significant.

Table 8. Frequencies in $\omega R_o\sqrt{\rho/G}$ of free, solid paraboloids for ν =0.3.

| n | s | $H/R$ | | | | |
|---|---|---|---|---|---|---|
| | | 1/3 | 1/2 | 1 | 2 | 3 |
| | 1 | 5.279 | 5.072 | 3.858 | **2.163(3)** | **1.473(2)** |
| | 2 | 8.616 | 8.192 | 6.000 | 3.679 | 2.520 |
| $0^T$ | 3 | 11.79 | 8.987 | 6.805 | 5.137 | 3.546 |
| | 4 | 12.03 | 11.16 | 8.517 | 5.766 | 4.563 |
| | 5 | 14.88 | 12.19 | 9.506 | 6.628 | 5.535 |
| | 1 | **1.465(3)** | **1.956(2)** | **2.639(2)** | **2.667(4)** | **1.953(3)** |
| | 2 | 3.719 | 4.383 | 4.023 | 3.019 | 2.957 |
| $0^A$ | 3 | 4.462 | 4.540 | 5.011 | 4.006 | 3.273 |
| | 4 | 6.296 | 7.199 | 5.974 | 4.872 | 4.073 |
| | 5 | 9.027 | 8.807 | 7.478 | 5.579 | 4.627 |
| | 1 | **2.430(5)** | **3.072(5)** | **2.975(4)** | **2.020(1)** | **1.193(1)** |
| | 2 | 3.125 | 3.109 | 3.650 | **2.752(5)** | **2.061(4)** |
| 1 | 3 | 4.905 | 5.768 | 4.236 | 3.170 | 2.697 |
| | 4 | 6.662 | 6.320 | 5.319 | 3.768 | 3.017 |
| | 5 | 7.209 | 6.916 | 5.919 | 4.512 | 3.398 |
| | 1 | **0.8717(1)** | **1.203(1)** | **1.768(1)** | **2.053(2)** | **2.107(5)** |
| | 2 | 3.163 | 3.159 | **3.103(5)** | 2.902 | 2.767 |
| 2 | 3 | 3.351 | 4.139 | 4.667 | 4.180 | 3.644 |
| | 4 | 4.814 | 4.788 | 4.801 | 4.358 | 3.957 |
| | 5 | 6.030 | 6.978 | 5.991 | 4.754 | 4.460 |
| | 1 | **1.462(2)** | **1.965(3)** | **2.728(3)** | 3.091 | 3.173 |
| | 2 | 4.242 | 4.687 | 4.538 | 4.257 | 4.101 |
| 3 | 3 | 4.719 | 5.164 | 5.906 | 5.522 | 5.014 |
| | 4 | 6.517 | 6.474 | 6.300 | 5.628 | 5.218 |
| | 5 | 7.116 | 8.139 | 7.370 | 6.310 | 5.858 |
| | 1 | **2.023(4)** | **2.674(4)** | 3.607 | 4.044 | 4.151 |
| | 2 | 5.115 | 5.956 | 5.754 | 5.433 | 5.264 |
| 4 | 3 | 6.014 | 6.166 | 6.993 | 6.653 | 6.206 |
| | 4 | 8.173 | 8.148 | 7.862 | 6.889 | 6.400 |
| | 5 | 8.215 | 9.269 | 8.621 | 7.703 | 7.119 |
| | 1 | 2.570 | 3.361 | 4.453 | 4.966 | 5.097 |
| | 2 | 5.975 | 7.088 | 6.883 | 6.536 | 6.357 |
| 5 | 3 | 7.203 | 7.185 | 8.056 | 7.741 | 7.313 |
| | 4 | 9.212 | 9.763 | 9.299 | 8.084 | 7.557 |
| | 5 | 9.858 | 10.38 | 9.856 | 8.915 | 8.292 |

## 5. COMPARISON WITH 2-D SHELL THEORY

Lin and Lee[4] analyzed free vibrations of complete paraboloidal shells of revolution applying a 2-D inextensional shell theory, which depends on the assumption that the length of line elements remain invariant under deformation of the shell, based upon Love's[29] equation. They obtained the natural frequencies ($\omega$) for free boundaries as follows:

$$\omega^2 = \frac{n^2(n^2-1)^2 G}{96 a^4 (1-\nu)\rho} \times \left[ \frac{\int_0^{\phi_b} h^3 \tan^{2n-3}\phi \sec^3\phi (\cos^2\phi + \sec^2\phi + 2 - 4\nu)\, d\phi}{\int_0^{\phi_b} h \tan^{2n+1}\phi \sec^3\phi [2n + (n^2+1)\sec^2\phi]\, d\phi} \right], \tag{40}$$

where $h$ is the uniform shell thickness, $a$ (= $R^2/4H$) is the focal distance of the midsurface of a paraboloidal shell, $\phi$ is a meridian coordinate, which is the angle between the normal to the midsurface and the axis of revolution ($z$-axis), and $\phi_b$ (= $\tan^{-1}(2H/R)$) is $\phi$ at the bottom face of the shell.

Comparisons of the present 3-D Ritz method (3DR) with the 2-D shell theory (2DS) by Lin and Lee[4] are made in Tables 9 and 10 for the first ($s$=1) nondimensional frequencies in $\omega R\sqrt{\rho/G}$ for each $n$ (=2, 3, 4, 5, 6) of completely free, complete paraboloidal shells of revolution with uniform thickness ($h_t/h_b$ =1) for $H/R$=1/2 (in Table 9) and 1 (in Table 10), and $\nu$=0.3. The percent difference in frequencies obtained by the two analyses is given by

$$\text{Difference (\%)} = \frac{2\text{DS} - 3\text{DR}}{3\text{DR}} \times 100. \tag{41}$$

Table 9. Comparisons of the first ($s$=1) nondimensional frequencies in $\omega R\sqrt{\rho/G}$ for each $n$ (=2, 3, 4, 5, 6) from the 3 D Ritz (3DR) solutions and 2 D shell (2DS) ones of free, complete paraboloidal shells of revolution with uniform thickness ($h_t/h_b$ =1) for $H/R$ =1/2 and $\nu$ =0.3.

| $h_b/R$ | Methods | n=2 | n=3 | n=4 | n=5 | n=6 |
|---|---|---|---|---|---|---|
| 1/90 | 3DR | .02583 | .06327 | 0.1140 | 0.1774 | 0.2526 |
| | 2DS | .02623 | .06462 | 0.1173 | 0.1840 | 0.2648 |
| | (Difference %) | (1.6%) | (2.1%) | (2.9%) | (3.7%) | (4.8%) |
| 1/30 | 3DR | .07685 | 0.1864 | 0.3325 | 0.5127 | 0.7255 |
| | 2DS | .07871 | 0.1939 | 0.3520 | 0.5521 | 0.7944 |
| | (Difference %) | (2.4%) | (4.0%) | (5.9%) | (7.7%) | (9.5%) |
| 1/10 | 3DR | 0.2257 | 0.5375 | 0.9443 | 1.436 | 2.002 |
| | 2DS | 0.2361 | 0.5816 | 1.055 | 1.656 | 2.383 |
| | (Difference %) | (4.6%) | (8.2%) | (11.7%) | (15.3%) | (19.0%) |

Table 10. Comparisons of the first ($s$=1) nondimensional frequencies in for each $n$ (=2, 3, 4, 5, 6) from the 3 D Ritz (3DR) solutions and 2 D shell (2DS) ones of free, complete paraboloidal shells of revolution with uniform thickness ($h_t/h_b$ =1) for $H/R$ =1 and $\nu$ =0.3.

| $h_b/R$ | Methods | n=2 | n=3 | n=4 | n=5 | n=6 |
|---|---|---|---|---|---|---|
| 1/90 | 3DR | .02163 | .05548 | 0.1021 | 1.598 | 2.195 |
| | 2DS | .02210 | .05624 | 0.1027 | 1.611 | 2.316 |
| | (Difference %) | (2.2%) | (1.4%) | (0.6%) | (0.8%) | (5.5%) |
| 1/30 | 3DR | .06285 | 0.1611 | 0.2953 | 0.4636 | 0.6647 |
| | 2DS | .06628 | 0.1687 | 0.3080 | 0.4834 | 0.6946 |
| | (Difference %) | (5.5%) | (4.7%) | (4.3%) | (4.3%) | (4.5%) |
| 1/10 | 3DR | 0.1864 | 0.4725 | 0.8542 | 1.321 | 1.863 |
| | 2DS | 0.1989 | 0.5061 | 0.9240 | 1.450 | 2.084 |
| | (Difference %) | (6.7%) | (7.1%) | (8.2%) | (9.8%) | (11.9%) |

It is observed that the 3-D Ritz method yields lower frequencies than the 2-D thin shell results in all the frequencies irrespective of thickness parameter ($h_b/R$), curvature ($H/R$), and circumferential wave number ($n$), as expected. An accurate 3-D analysis should typically yield lower frequencies than those 2-D thin shell theory, mainly because shear deformation and rotary inertia effects are accounted for in a 3-D analysis, but not in 2-D, thin shell theory. Particularly interesting is the fact that the inextensional theory by Lin and Lee[4] includes bending stiffness, but neglects membrane-type stretching effects, and therefore the 2DS frequencies in Table 9 and 10 are close to the accurate 3DR results for the thin shell ($h_b/R$ =1/90). This confirms that the middle surface of the shell can deform inextensionally for these modes ($n$=2, 3, ⋯) when the shell boundary is free. It is noticed in Tables 9 and 10 that the frequency differences become larger as shell thickness ($h_b/R$) increases. It is interesting to note that for the fundamental modes occurring at $n$=2 the differences for $H/R$=1/2 (shallower shell) are smaller than ones for $H/R$=1 (deeper shell), and vice versa for the higher modes ($n$=2, 3, 4, 5) with two exceptions for ($n$, $h_b/R$)=(6, 1/90) and (3, 1/30).

## 6. CONCLUDING REMARKS

Extensive and accurate frequency data determined by the 3-D Ritz analysis have been presented for complete (without a top opening) paraboloidal shells of revolution with variable thickness, and also solid paraboloids. The analysis uses the 3-D equations of the theory of elasticity in their general forms for isotropic materials. They are only limited to small strains. No other constraints are placed upon the displacements. This is in stark contrast with the classical 2-D thin shell theories, which make very limiting assumptions about the displacement variation through the shell thickness.

The method is straightforward, but it is capable of determining frequencies and mode shapes as close to the exact ones as desired. Therefore, the data in Tables 3~8 may be regarded as benchmark results against which 3-D results obtained by other methods, such as finite elements and finite differences, and 2-D shell theories may be compared to determine the accuracy of the latter. Moreover, the frequency determinants required by the present method are at least an order of magnitude smaller than those needed by 3-D finite element analyses of comparable accuracy. This was demonstrated extensively in a paper by McGee and Leissa.[30) The Ritz method guarantees upper bound convergence of the frequencies in terms of functions sets that are mathematically complete, such as algebraic polynomials. Some finite element methods can also accomplish this, but others cannot.

The method presented could also be extended to circumferentially open ($0 \le \theta \le \theta_0$) paraboloidal shells, instead of circumferentially closed ($0 \le \theta \le 2\pi$) paraboloidal shells of revolution considered in the present work. However, the periodicity in $\theta$ would not be present. It would be necessary then to replace the double sums of algebraic polynomials in Eqs. (35) by triple sums, with polynomials in $\theta$ being included.

## REFERENCES

1. Leissa, A.W., *Vibration of shells*, NASA SP 288; U.S. Government Printing Office, 1973 (reprinted by The Acoustical Society of America, 1993).
2. Reissner, E., "On transverse vibrations of thin, shallow elastic shell," Quart. Appl. Math., Vol. 13(2), pp. 169-176, 1955.
3. Johnson, M.W., Reissner E. "On inextensional deformations of shallow elastic shells," J. Math. Phys., Vol. 34(4),

pp. 335-346, 1956.
4. Lin, Y.K. and Lee, F.A., "Vibrations of thin paraboloidal shells of revolution," J. Appl. Mech., Vol. 27(4), pp. 743-746, 1960.
5. Hoppmann, W.H., Cohen, M.I. and Kunukkasseril, V.X., "Elastic vibrations of paraboloidal shells of revolution," J. Acoust. Soc. Ameri., Vol. 36(2), pp. 349-353, 1964.
6. Kalnins, A., "Free vibration of rotationally symmetric shells," J. Acoust. Soc. Amer., Vol. 36(7), pp. 1355-1365, 1964.
7. Graig, R.R., "Analysis of reinforced concrete thin shells - A preliminary study," Tech. Note N-750 (AD 624 197), U.S. Naval Civil Engineering Lab., 1965.
8. Bacon, M.D. and Bert, C.W., "Unsymmetric free vibrations of orthotropic sandwich shells of revolution," AIAA J., Vol. 5(3), pp. 413-417, 1967.
9. Wang, J.T.S. and Lin, C.W., "On the differential equations of the axisymmetric vibration of paraboloidal shells of revolution," NASA CR-932, 1967.
10. Glockner, P.G. and Tawardros, K.Z., "Experiments on free vibration of shells of revolution," Experimental Mechanics, Vol. 13, pp. 411-421, 1973.
11. Beles, A.A. and Soare, M.V., *Elliptic and hyperbolic paraboloidal shells used in constructions*, London-Editura Academiei, Bucuresti: S.P. Christie & Partners, 1976.
12. Utku, S., Shoemaker, W.L. and Salama, M., "Nonlinear equations of dynamics for spinning paraboloidal antennas," Computers and Structures, Vol. 16, pp. 361-370, 1983.
13. Shoemaker, W.L., "The nonlinear dynamics of spinning paraboloidal antennas," Doctoral Dissertation, Duke University, Durham, North Carolina, 1983.
14. Shoemaker, W.L. and Utku, S., "On the vibrations of spinning paraboloids," Journal of Sound and Vibration, Vol. 111, pp. 279-296, 1986.
15. Elliott, G.H., "The evaluation of the modal density of paraboloidal and similar shells," Journal of Sound and Vibration, Vol. 126, pp. 477-483, 1988.
16. Mazurkiewicz, Z.E. and Nagorski, R.T., *Shells of revolution*, Warsaw: PWN-Polish Scientific Publishers, 1991.
17. Kayran, A., Vinson, J.R. and Ardic, E.S., "A method for the calculation of natural frequencies of orthotropic axisymmetrically loaded shells of revolution," Journal of Vibration and Acoustics, Vol. 116, pp. 16-25, 1994.
18. Tzou, H.S. and Ding, J.H., "Distributed modal signals of nonlinear paraboloidal shells with distributed neurons," VIB21545, 2001 Design Technical Conference, Pittsburgh, PA, 2001.
19. Tzou, H.S., Ding, J.H. and Hagiwara, I., "Micro-control actions of segmented actuator patches laminated on deep paraboloidal shells," JSME International Journal Series C, Vol. 45(1), pp. 8-15, 2002.
20. Tzou, H.S. and Ding, J.H., "Distributed modal signals of nonlinear paraboloidal shells with distributed neurons," Journal of Vibration and Acoustics (Paper No. JVA 00 -161), 2002.
21. Ding, J.H., "Micro-piezothermoelastic behavior and distributed sensing/control of nonlinear structronic beam and paraboloidal shell systems," Doctoral Dissertation, University of Kentuchy, Lexington, KY, 2002.
22. Ding, J.H. and Tzou, H.S., "Micro-electromechanics of sensor patches on free paraboloidal shell structronic systems," Mechanical Systems and Signal Processing,

Vol. 18(2), pp. 367-380, 2004.

23. Leissa, A.W. and Kang, J.-H., "Three-dimensional vibration analysis of paraboloidal shells," JSME International Journal Series C, Vol. 45, pp. 2-7, 2002.
24. Sokolnikoff, I.S., *Mathematical theory of elasticity*, Second Edition, New York: McGraw-Hill Book Co., 1956.
25. Kang, J.-H. and Leissa, A.W., "Three-dimensional vibration analysis of solid and hollow hemispheres having varying thickness with and without axial conical holes," Journal of Vibration and Control, Vol. 10(2), pp. 199-214, 2004.
26. Kang, J.-H. and Leissa, A.W., "Three-dimensional field equations of motion, and energy functionals for thick shells of revolution with arbitrary curvature and variable thickness," Journal of Applied Mechanics, Vol. 68, pp. 953-954, 2001.
27. Kantorovich, L.V. and Krylov, V.I., *Approximate methods in higher analysis*, Groningen: Noordhoff, 1958.
28. Ritz, W., "Über eine neue Methode zur Lösung gewisser Variationsprobleme der mathematischen Physik," Journal für die Reine und Angewandte Mathematik, Vol. 135, pp. 1-61, 1909.
29. Love. A.E.H., *The mathematical theory of elasticity*, Dover Publications, New York, N.Y., 1944.
30. McGee, O.G. and Leissa, A.W., "Three-dimensional free vibrations of thick skewed cantilever plates," Journal of Sound and Vibration, Vol. 144, pp. 305-322; Errata Vol. 149, pp. 539-542, 1991.