Honam Mathematical J. 26(2004). pp. 17–41

INTUITIONISTIC FUZZY SUBGROUPS AND COSETS

KUL HUR, SU YOUN JANG AND HEE WON KANG

Abstract. In this paper, we obtain the intuitionistic fuzzy subgroups generated by intuitionistic fuzzy sets and some properties preserved by a ring homomorphism. Furthermore, we introduce the concept of intuitionistic fuzzy coset and study some of it's properties.

0. Introduction

In 1965, Zadeh [14] introduced the concept of fuzzy sets. After that time, several researchers[6,10,12,13] have applied the notion of fuzzy sets to algebra.

In 1986, Atanassov[1] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. Recently, Çoker and his colleagues [4,5,7] applied the notion of intuitionistic fuzzy sets to topology. In 1989, Biswas[3] introduced the concept of intuitionistic fuzzy subgroups and investigated some of it's properties. In 2003, Baldev Banerjee and Dhiren Kr. Basnet[2] studied intuitionistic fuzzy subrings and ideals using intuitionistic fuzzy sets. Moreover, Hur and his colleagues[8,9] redefined the concepts of intuitionistic fuzzy subgroupoids, subgroups and rings, and studied some of their properties.

In this paper, we introduce the concept of intuitionistic fuzzy cosets and investigate some of it's properties. Furthermore we obtain the intuitionistic fuzzy subgroups generated by intuitionistic fuzzy sets, and investigate some properties preserves by a ring homomorphism.

Received October 15, 2003. Revised December 4, 2003.
2000 Mathematics Subject Classification: 03F55, 20N25.

1. Preliminaries

We will list some concepts and one result needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I.

Definition 1.1[1]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) in X if $\mu_A + \nu_A \leq 1$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2[1]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs in X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\ ]A = (\mu_A, 1 - \mu_A)$, $< > A = (1 - \nu_A, \nu_A)$.

Definition 1.3[4]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(a) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.

(b) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4[4]. $0_\sim = (0, 1)$ and $1_\sim = (1, 0)$.

Definition 1.5[4]. Let X and Y be nonempty sets and let $f : X \to Y$ be a mapping. Let $A = (\mu_A, \nu_A)$ be an IFS in X and $B = (\mu_B, \nu_B)$ be an IFS in Y. Then

(a) the *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by:

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)),$$

where $f^{-1}(\mu_B) = \mu_B \circ f$.

(b) the *image* of A under f, denoted by $f(A)$, is the IFS in Y defined by:

$$f(A) = (f(\mu_A), f(\nu_A)),$$

where for each $y \in Y$

$$f(\mu_A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset, \end{cases}$$

and

$$f(\nu_A)(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 1 & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

Result 1.A[4, Corollary 2.10]. Let $A, A_i (i \in J)$ be IFSs in X, let $B, B_j (j \in K)$ IFSs in Y and let $f : X \to Y$ be a mapping. Then

(1) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

(2) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$.

(3) $A \subset f^{-1}(f(A))$.

If f is injective, then $A = f^{-1}(f(A))$.

(4) $f(f^{-1}(B)) \subset B$.

If f is surjective, then $f(f^{-1}(B)) = B$.

(5) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$.

(6) $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$.

(7) $f(\bigcup A_i) = \bigcup f(A_i)$.

(8) $f(\bigcap A_i) \subset \bigcap f(A_i)$.

If f is injective, then $f(\bigcap A_i) = \bigcap f(A_i)$.

(9) $f(1_\sim) = 1_\sim$, if f is surjective and $f(0_\sim) = 0_\sim$.

(10) $f^{-1}(1_\sim) = 1_\sim$ and $f^{-1}(0_\sim) = 0_\sim$.

(11) $[f(A)]^c \subset f(A^c)$, if f is surjective.

(12) $f^{-1}(B^c) = [f^{-1}(B)]^c$.

Definition 1.6[4]. Let X be a set and let $\lambda, \mu \in I$ with $0 \leq \lambda + \mu \leq 1$. Then the IFS $C_{(\lambda,\mu)}$ in X is defined by: for each $x \in X$, $C_{(\lambda,\mu)}(x) = (\lambda, \mu)$, i.e., $\mu_{C_{(\lambda,\mu)}}(x) = \lambda$ and $\nu_{C_{(\lambda,\mu)}}(x) = \mu$.

Definition 1.7[8]. Let $(G, \cdot)$ be a groupoid and let $A \in IFS(X)$. Then A is called an *intuitionistic fuzzy subgroupoid* (in short, $IFGP$) of G if for any $x, y \in G$, $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

2. Intuitionistic fuzzy subgroup generated by an intuitionistic fuzzy set

Definition 2.1[8]. Let A be an IFS in a set X and let $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$. Then the set $A^{(\lambda,\mu)} = \{x \in X : \mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \mu\}$ is called a (λ, μ)-*level subset* of A.

The following is the immediate result of Dedinition 2.1:

Proposition 2.2. Let A be an IFS in a set X and let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathrm{Im}(A)$. If $\lambda_1 < \lambda_2$ and $\mu_1 > \mu_2$, then $A^{(\lambda_1,\mu_1)} \supset A^{(\lambda_2,\mu_2)}$.

Definition 2.3[9]. Let G be a group and let $A \in IFS(G)$. Then A is called an *intuitionistic fuzzy subgroup* (in short, IFG) of G if it satisfies the following conditions:

(i) A is an IFGP of G.

(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$ for each $x \in G$.

We will denote the set of all IFGs of G as IFG(G).

Result 2.A[9, Proposition 2.6]. Let A be an IFG of a group G. Then $A(x^{-1}) = A(x)$ and $\mu_A(x) \leq \mu_A(e), \nu_A(x) \geq \nu_A(e)$ for each $x \in G$, where e is the identity element of G.

Result 2.B[9, Proposition 2.18 and 2.19]. Let A be an IFS in a group G. Then A is an IFG of G if and only if $A^{(\lambda,\mu)}$ is a subgroup of G for each $(\lambda, \mu) \in \text{Im}(A)$.

Definition 2.4[8]. Let A be an IFG of a group G and let $(\lambda, \mu) \in \text{Im}(A)$. Then the subgroup $A^{(\lambda,\mu)}$ is called a (λ, μ)-*level subgroup* of A.

Lemma 2.5. Let A be any IFS of a set X. Then
$\mu_A(x) = \bigvee\{\lambda : x \in A^{(\lambda,\mu)}\}$ and $\nu_A(x) = \bigwedge\{\mu : x \in A^{(\lambda,\mu)}\}$,
where $x \in X$ and $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$.

Proof. Let $\alpha = \bigvee\{\lambda : x \in A^{(\lambda,\mu)}\}$, let $\beta = \bigwedge\{\mu : x \in A^{(\lambda,\mu)}\}$ and let $\varepsilon > 0$ be arbitrary. Then $\alpha - \varepsilon < \bigvee\{\lambda : x \in A^{(\lambda,\mu)}\}$ and $\beta + \varepsilon > \bigwedge\{\mu : x \in A^{(\lambda,\mu)}\}$. Thus there exist $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$ such that $x \in A^{(\lambda,\mu)}$, $\alpha - \varepsilon < \lambda$ and $\beta + \varepsilon > \mu$. Since $x \in A^{(\lambda,\mu)}$, $\mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \mu$. Thus $\mu_A(x) > \alpha - \varepsilon$ and $\nu_A(x) < \beta + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$. We now show that $\mu_A(x) \leq \alpha$ and $\nu_A(x) \geq \beta$. Suppose $\mu_A(x) = t_1$ and $\nu_A(x) = t_2$. Then $t_1 + t_2 \leq 1$. Thus $x \in A^{(t_1,t_2)}$. So $t_1 \in \{\lambda : x \in A^{(\lambda,\mu)}\}$ and $t_2 \in \{\mu : x \in A^{(\lambda,\mu)}\}$. Thus $t_1 \leq \bigvee\{\lambda : x \in A^{(\lambda,\mu)}\}$ and $t_2 \geq \bigwedge\{\mu : x \in A^{(\lambda,\mu)}\}$, i.e., $\mu_A(x) \leq \alpha$ and $\nu_A(x) \geq \beta$. This completes the proof.

We shall denote by (A) the IFG generated by the IFS A in G. We shall use the same notation $(A^{(\lambda,\mu)})$ for the ordinary subgroup of the group G generated by the level subset $A^{(\lambda,\mu)}$.

Theorem 2.6. Let G be a group and let $A \in$ IFS(G). Let $A^* \in$ IFS(G) be defined as follows: for each $x \in G$,

$$\mu_{A^*}(x) = \bigvee\{\lambda : x \in (A^{(\lambda,\mu)})\} \text{ and } \nu_{A^*}(x) = \bigwedge\{\mu : x \in (A^{(\lambda,\mu)})\},$$

where $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$. Then A^* is an IFG of G such that $A^* = \bigcap\{B \in IFG(G) : A \subset B\}$. In this case, A^* is called the *intuitionistic fuzzy subgroup generated by A in G* and denoted by (A).

Proof. Let $(t_1, t_2) \in \text{Im}(A^*)$ and let $\alpha = t_1 - \frac{1}{n}$ and $\beta = t_2 + \frac{1}{n}$, where n is any sufficiently large positive integer. Let $x \in G$. Suppose $x \in A^{*(t_1,t_2)}$. Then $\mu_{A^*}(x) \geq t_1$ and $\nu_{A^*}(x) \leq t_2$. Thus there exist $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$ such that $\lambda > \alpha, \mu < \beta$ and $x \in A^{(\lambda,\mu)}$. Since $(\alpha, \beta) < (\lambda, \mu)$ and $\alpha + \beta \leq 1$, $A^{(\lambda,\mu)} \subset A^{(\alpha,\beta)}$. So $x \in A^{(\alpha,\beta)}$ i.e., $x \in (A^{(\alpha,\beta)})$.

Now suppose $x \in (A^{(\alpha,\beta)})$. Then $\alpha \in \{\lambda : x \in (A^{(\lambda,\mu)})\}$ and $\beta \in \{\mu : x \in (A^{(\lambda,\mu)})\}$. Thus $\alpha \leq \bigvee\{\lambda : x \in (A^{(\lambda,\mu)})\}$ and $\beta \geq \bigwedge\{\mu : x \in (A^{(\lambda,\mu)})\}$. So $t_1 - \frac{1}{n} \leq \mu_{A^*}(x)$ and $t_2 + \frac{1}{n} \geq \nu_{A^*}(x)$ i.e., $t_1 \leq \mu_{A^*}(x)$ and $t_2 \geq \nu_{A^*}(x)$. Hence $x \in A^{*(t_1,t_2)}$ i.e., $(A^{(\alpha,\beta)}) \subset A^{*(t_1,t_2)}$. Hence $A^{*(t_1,t_2)} = (A^{(\alpha,\beta)})$. Since $(A^{(\alpha,\beta)})$ is a subgroup of G, $A^{*(t_1,t_2)}$ is a subgroup of G. By Result 2.B, A^* is an IFG of G.

Now, we show that $A \subset A^*$. Let $x \in G$. Then, by Lemma 2.5, $\mu_A(x) = \bigvee\{\lambda : x \in A^{(\lambda,\mu)}\}$ and $\nu_A(x) = \bigwedge\{\mu : x \in A^{(\lambda,\mu)}\}$. Thus $\mu_A(x) \leq \bigvee\{\lambda : x \in (A^{(\lambda,\mu)})\}$ and $\nu_A(x) \geq \bigwedge\{\mu : x \in (A^{(\lambda,\mu)})\}$. So $A \subset A^*$.

Finally, let B be any IFG of G such that $A \subset B$. We show that $A^* \subset B$. Let $x \in G$ and $A^*(x) = (t_1, t_2)$. Then $A^{*(t_1,t_2)} = (A^{(\alpha,\beta)})$, where $\alpha = t_1 - \frac{1}{n}$, $\beta = t_2 + \frac{1}{n}$, and n is any sufficiently large positive integer. Thus $x \in (A^{(\alpha,\beta)})$. So $x = a_1 a_2 \cdots a_m$, where a_i or a_i^{-1} belongs to $A^{(\alpha,\beta)} (i = 1, \cdots, m)$.

On the other hand,

$$\begin{aligned}\mu_B(x) &= \mu_B(a_1a_2\cdots a_m)\\ &\geq \mu_B(a_1)\wedge\mu_B(a_2)\wedge\cdots\wedge\mu_B(a_m)\\ &\geq \mu_A(a_1)\wedge\mu_A(a_2)\wedge\cdots\wedge\mu_A(a_m)\\ &\geq \alpha = t_1-\frac{1}{n}\end{aligned}$$

and

$$\begin{aligned}\nu_B(x) &= \nu_B(a_1a_2\cdots a_m)\\ &\leq \nu_B(a_1)\vee\nu_B(a_2)\vee\cdots\vee\nu_B(a_m)\\ &\leq \nu_A(a_1)\vee\nu_A(a_2)\vee\cdots\vee\nu_A(a_m)\\ &\leq \beta = t_1+\frac{1}{n}.\end{aligned}$$

Since n is sufficiently large positive integer, $\mu_B(x)\geq t_1$ and $\nu_B(x)\leq t_2$. So $A^*\subset B$. Hence $A^*=\bigcap\{B\in IFG(G): A\subset B\}$. This completes the proof.

It is possible that card Im(A^*) be less than card Im(A). Moreover, Im(A^*) need not be contained in Im(A) as shown in the following examples.

Example 2.7. Let $G=\{e,a,b,ab\}$ be the Klein four-group, where $a^2=b^2=e$ and $ab=ba$. Define an IFS A of G by:
$A(e)=(0.5,0.5)$, $A(a)=(0.8,0.2)$, $A(b)=(0.7,0.3)$ and $A(ab)=(0.6,0.4)$. Then $A^{(0.8,0.2)}=\{a\}$, $A^{(0.7,0.3)}=\{a,b\}$, $A^{(0.6,0.4)}=\{a,b,ab\}$ and $A^{(0.5,0.5)}=G$. So $(A^{(0.8,0.2)})=\{e,a\}$ and $(A^{(0.7,0.3)})=G$. Moreover, by definition, we have $A^*(e)=A^*(a)=(0.8,0.2)$ and $A^*(b)=A^*(ab)=(0.7,0.3)$.

Now an attempt is made to obtain a necessary and sufficient condition for a p-group to be cyclic.

Lemma 2.8. Let G be a finite group. Suppose there exists an IFG A of G satisfying the following conditions: for any $x, y \in G$,

(i) $A(x) = A(y) \Rightarrow (x) = (y)$.

(ii) $\mu_A(x) > \mu_A(y)$ and $\nu_A(x) < \nu_A(y) \Rightarrow (x) \subset (y)$.

Then G is cyclic.

Proof. Suppose A is constant on G. Then $A(x) = A(y)$ for any $x, y \in G$. By the condition (i), $(x) = (y)$. So $G = (x)$. Now suppose A is not constant on G. Let Im(A)=$\{(t_0, s_0), (t_1, s_1), \cdots, (t_n, s_n)\}$, where $t_0 > t_1 > \cdots > t_n$ and $s_0 < s_1 < \cdots < s_n$. Then, by Proposition 2.2 and Result 2.B, we obtain the chain of level subgroups of A:

$$A^{(t_0,s_0)} \subset A^{(t_1,s_1)} \subset \cdots \subset A^{(t_n,s_n)} = G.$$

Let $x \in G - A^{(t_{n-1},s_{n-1})}$. We show that $G = (x)$. Let $g \in G - A^{(t_{n-1},s_{n-1})}$. Since $t_0 > t_1 > \cdots > t_n$ and $s_0 < s_1 < \cdots < s_n$, $A(g) = A(x) = A^{(t_{n-1},s_{n-1})}$. By the condition (i), $(g) = (x)$. Thus $G - A^{(t_{n-1},s_{n-1})} \subset (x)$. Now let $g \in A^{(t_{n-1},s_{n-1})}$. Then $\mu_A(g) \geq t_{n-1} > t_n = \mu_A(x)$ and $\nu_A(g) \leq s_{n-1} < s_n = \nu_A(x)$. By the condition (ii), $(g) \subset (x)$. Thus $A^{(t_{n-1},s_{n-1})} \subset (x)$. So $G = (x)$. Hence, in either case, G is cyclic.

Lemma 2.9. Let G be a cyclic group of order p^n, where p is prime. Then there exists an IFG A of G satisfying the following conditions: for any $x, y \in G$,

(i) $A(x) = A(y) \Rightarrow (x) = (y)$.

(ii) $\mu_A(x) > \mu_A(y)$ and $\nu_A(x) < \nu_A(y) \Rightarrow (x) \subset (y)$.

Proof. Consider the following chain of subgroups of G:

$$(e) = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset G_n = G,$$

where G_i is the subgroup of G generated by an element of order p^i, $i = 0, 1, \cdots, n$ and e is the identity of G. We define a complex mapping $A = (\mu_A, \nu_A) : G \to I \times I$ as follows: for each $x \in G$,

$$A(e) = (t_0, s_0)$$

and

$$A(x) = (t_i, s_i) \text{ if } x \in G_i - G_{i-1} \text{ for any } i = 1, 2, \cdots, n,$$

where $t_i, s_i \in I$ such that $t_i + s_i \leq 1$, $t_0 > t_1 > \cdots > t_n$ and $s_0 < s_1 < \cdots < s_n$. Then we can easily check that A is an IFG of G satisfying the conditions (i) and (ii).

From Lemma 2.8 and Lemma 2.9 we obtain the following:

Theorem 2.10. Let G be a group of order p^n. Then G is cyclic if and only if there exists an IFG A of G such that for any $x, y \in G$,

(i) $A(x) = A(y) \Rightarrow (x) = (y)$.

(ii) $\mu_A(x) > \mu_A(y)$ and $\nu_A(x) < \nu_A(y) \Rightarrow (x) \subset (y)$.

3. Intuitionistic fuzzy ideals and homomorphisms

Definition 3.1[9]. Let $(R, +, \cdot)$ be a ring and let $0_\sim \neq A \in IFS(R)$. Then A is called an *intuitionistic fuzzy subring* (in short, $IFSR$) in R if it satisfies the following conditions:

(i) A is an IFG with respect to the operation "+" (in the sense of Definition 2.3),

(ii) A is an IFGP with respect to the operation " $\cdot$ " (in the sense of Definition 1.7)

It is clear that subrings of R are IFSRs of R. We will denote the set of all IFSRs of R as IFSR(R).

Definition 3.2[9]. Let R be a ring and let $0_\sim \neq A \in$ IFS(R). Then A is called an *intuitionistic fuzzy ideal* (in short,IFI) of R if it satisfies the following conditions:

(i) A is an IFSR of R.

(ii) $\mu_A(xy) \geq \mu_A(x)$, $\nu_A(xy) \leq \nu_A(x)$ and $\mu_A(xy) \geq \mu_A(y)$, $\nu_A(xy) \leq \nu_A(y)$ for any $x, y \in R$.

We will denote the set of all IFIs of R as IFI(R). It is well-known that if $A, B \in IFI(R)$, then $A \cap B \in IFI(R)$ (See Theorem 4.4 in [2]).

Result 3.A[9, Proposition 4.5]. Let R be a ring and let $0_\sim \neq A \in$ IFS(R). Then A is an IFI of R if and only if for any $x, y \in R$,

(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$,

(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

It is clear that if A is an IFI of R, then $A(-x) = A(x) \leq A(0)$ for each $x \in R$, where 0 is the identity in R with respect to " + ".

Proposition 3.3. Let A be an IFS in a ring R. Then $A \in IFI(R)$ if and only if $A^{(\lambda,\mu)}$ is an ideal of R for each $(\lambda, \mu) \in \mathrm{Im}(A)$.

Proof. ($\Rightarrow$): Suppose A is an IFI of R. For each $(\lambda, \mu) \in \mathrm{Im}(A)$, let $x, y \in A^{(\lambda,\mu)}$. Then $\mu_A(x) \geq \lambda, \nu_A(x) \leq \mu$ and $\mu_A(y) \geq \lambda, \nu_A(y) \leq \mu$. By Result 3.A (i), $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$. Thus $\mu_A(x-y) \geq \lambda$ and $\nu_A(x-y) \leq \mu$. So $x - y \in A^{(\lambda,\mu)}$. Let $x \in R$ and $y \in A^{(\lambda,\mu)}$. Then $\mu_A(y) \geq \lambda$ and $\nu_A(y) \leq \mu$. Since A is an IFI of R, by Result 3.A (ii), $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$. Thus $\mu_A(xy) \geq \lambda$ and $\nu_A(xy) \leq \mu$. So $xy \in A^{(\lambda,\mu)}$. Similarly, we have $yx \in A^{(\lambda,\mu)}$. Hence $A^{(\lambda,\mu)}$ is an ideal of R.

($\Leftarrow$): Suppose the necessary condition holds. For any $x, y \in R$, let $A(x) = (t_1, s_1)$ and $A(y) = (t_2, s_2)$. Then clearly, $x \in A^{(t_1,s_1)}$ and

$y \in A^{(t_2,s_2)}$. Since $A^{(t_1,s_1)}$ is an ideal of R, $x - y \in A^{(t_1,s_1)}$. Then

$$\mu_A(x-y) \geq t_1 \geq t_1 \wedge t_2 = \mu_A(x) \wedge \mu_A(y)$$

and

$$\nu_A(x-y) \leq s_1 \leq s_1 \vee s_2 = \nu_A(x) \vee \nu_A(y).$$

Thus A satisfies the condition (i) of Result 3.A.

Now for each $x \in R$, let $A(x) = (\lambda, \mu)$. Then clearly $x \in A^{(\lambda,\mu)}$. Let $y \in R$. Since $A^{(\lambda,\mu)}$ is an ideal of R, $xy \in A^{(\lambda,\mu)}$ and $yx \in A^{(\lambda,\mu)}$. Then

$$\mu_A(xy) \geq \lambda = \mu_A(x), \nu_A(xy) \leq \mu = \nu_A(x)$$

and

$$\mu_A(yx) \geq \lambda = \mu_A(y), \nu_A(yx) \leq \mu = \nu_A(y).$$

Thus A satisfies the condition (ii) of Definition 3.2. Hence A is an IFI of R.

Example 3.4. Let R denote the ring of real numbers under the usual operations of addition and multiplication. We define a complex mapping $A = (\mu_A, \nu_A) : R \to I \times I$ as follows : for each $x \in R$,

$$A(x) = \begin{cases} (t, s) & \text{if } x \text{ is rational,} \\ (t', s') & \text{if } x \text{ is irrational,} \end{cases}$$

where $(t, s), (t', s') \in I \times I$ such that $t+s \leq 1, t'+s' \leq 1$ and $t > t', s < s'$. Then we can see that $A \in IFSR(R)$ but $A \notin IFI(R)$.

Definition 3.5[2]. Let X and Y be sets, let $f : X \longrightarrow Y$ be a mapping and let $A \in IFS(X)$. Then A is said to be *f-invariant* if $f(x) = f(y)$ implies $A(x) = A(y)$, i.e., $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$.

Result 3.B[2,Proposition 6.6]. Let X and Y be sets, let $f : X \longrightarrow Y$ a mapping and let $A \in IFS(X)$. If A is f-invariant, then $f^{-1}(f(A)) = A$.

Definition 3.6[8]. Let $(X, \cdot)$ be a groupoid and let $A, B \in IFS(X)$. Then the intuitionistic fuzzy product of A and B, $A \circ B$, is defined as follows: for each $x \in X$,

$$\mu_{A\circ B}(x) = \begin{cases} \bigvee_{(y,z)\in X\times X}[\mu_A(y) \wedge \mu_B(z)] & \text{if } x = yz, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nu_{A\circ B}(x) = \begin{cases} \bigwedge_{(y,z)\in X\times X}[\nu_A(y) \vee \nu_B(z)] & \text{if } x = yz, \\ 1 & \text{otherwise.} \end{cases}$$

Similarly, we have the following definition:

Definition 3.7. Let A and B be any two IFIs of a ring R. Then the *intuitionistic fuzzy sum* of A and B, $A + B$, is defined as follows: for any $x \in R$,

$$\mu_{A+B}(x) = \begin{cases} \bigvee_{(y,z)\in R\times R}[\mu_A(y) \wedge \mu_B(z)] & \text{if } x = y + z, \\ 0 & \text{if } x \neq y + z, \end{cases}$$

and

$$\nu_{A+B}(x) = \begin{cases} \bigwedge_{(y,z)\in R\times R}[\nu_A(y) \vee \nu_B(z)] & \text{if } x = y + z, \\ 1 & \text{if } x \neq y + z. \end{cases}$$

Let $f : R \to R'$ be a ring epimorphism. If A is an IFR [resp. IFI] of R, then so is $f(A)$; and B is an IFR [resp. IFI] of R', then so is $f^{-1}(B)$ (See Theorem 6.2 and Theorem 6.3 in [2]).

This section reflects the effect of a homomorphism on the sum, product and intersection of any two IFIs of a ring.

Theorem 3.8. Let $f : R \to R'$ be a ring epimorphism. If A and B are IFIs of R, then

(1) $f(A + B) = f(A) + f(B)$.

(2) $f(A \circ B) = f(A) \circ f(B)$.

(3) $f(A \cap B) \subset f(A) \cap f(B)$, with equality if at least one of A or B is f-invariant.

Proof. (1) Let $y \in R'$ and let $\epsilon > 0$ be arbitrary. Let $(\alpha, \alpha') = f(A+B)(y)$ and let $(\beta, \beta') = (f(A) + f(B))(y)$. Then:

$$\alpha = \mu_{f(A+B)}(y) = \bigvee_{z \in f^{-1}(y)} \mu_{A+B}(z),$$
$$\alpha' = \nu_{f(A+B)}(y) = \bigwedge_{z \in f^{-1}(y)} \nu_{A+B}(z)$$

and

$$\beta = \mu_{f(A)+f(B)}(y) = \bigvee_{y=z+z'} [\mu_{f(A)}(z) \wedge \mu_{f(B)}(z')],$$
$$\beta' = \nu_{f(A)+f(B)}(y) = \bigwedge_{y=z+z'} [\nu_{f(A)}(z) \vee \nu_{f(B)}(z')].$$

Thus $\alpha - \epsilon < \bigvee_{z \in f^{-1}(y)} \mu_{A+B}(z)$ and $\alpha' + \epsilon > \bigwedge_{z \in f^{-1}(y)} \nu_{A+B}(z)$. So there exist $z_0, z_0' \in R$ with $f(z_0) = y$ and $f(z_0') = y$ such that $\alpha - \epsilon < \mu_{A+B}(z_0)$ and $\alpha' + \epsilon > \nu_{A+B}(z_0')$. By the definition of sum,

$\alpha - \epsilon < \bigvee_{z_0 = a+b} [\mu_A(a) \wedge \mu_B(b)]$ and $\alpha' + \epsilon > \bigwedge_{z_0' = a'+b'} [\nu_A(a') \vee \nu_B(b')]$.
Then there exist $a_0, b_0 \in R$ with $z_0 = a_0 + b_0$ such that $\alpha - \epsilon < \mu_A(a_0) \wedge \mu_B(b_0)$ and there exist $a_0', b_0' \in R$ with $z_0' = a_0' + b_0'$ such that $\alpha' + \epsilon > \nu_A(a_0') \vee \nu_B(b_0')$.
On the other hand,

$$\begin{aligned} \beta &\geq \mu_{f(A)}(f(a_0)) \wedge \mu_{f(B)}(f(b_0)) \\ &= f(\mu_A)(f(a_0)) \wedge f(\mu_B)(f(b_0)) \\ &= f^{-1}(f(\mu_A))(a_0) \wedge f^{-1}(f(\mu_B))(b_0) \\ &\geq \mu_A(a_0) \wedge \mu_B(b_0). \end{aligned}$$

Similarly, we have

$$\beta' \leq \nu_A(a_0') \vee \nu_B(b_0').$$

So $\beta > \alpha - \epsilon$ and $\beta' < \alpha' + \epsilon$. Since ϵ is arbitrary, $\beta \geq \alpha$ and $\beta' \leq \alpha'$. Hence $(f(A) + f(B))(y) \geq f(A+B)(y)$ for each $y \in R'$. (*)

Now we will show that $\beta \leq \alpha$ and $\beta' \geq \alpha'$. Clearly,

$$\beta - \epsilon < \bigvee_{y=z+z'} [\mu_{f(A)}(z) \wedge \mu_{f(B)}(z')]$$

and

$$\beta' + \epsilon > \bigwedge_{y=z+z'} [\nu_{f(A)}(z) \vee \nu_{(B)}(z')].$$

Then there exist $z_1, z_1' \in R'$ with $y = z_1 + z_1'$ such that

$$\beta - \epsilon < \mu_{f(A)}(z_1) = \bigvee_{x \in f^{-1}(z_1)} \mu_A(x),$$

$$\beta - \epsilon < \mu_{f(B)}(z_1') = \bigvee_{x \in f^{-1}(z_1')} \mu_B(x)$$

and there exist $z_0, z_0' \in R'$ with $y = z_0 + z_0'$ such that

$$\beta + \epsilon > \nu_{f(A)}(z_0) = \bigwedge_{x \in f^{-1}(z_0)} \nu_A(x),$$

$$\beta + \epsilon > \nu_{f(B)}(z_0') = \bigwedge_{x \in f^{-1}(z_0')} \nu_B(x).$$

Thus there exist $x_1, x_1' \in R$ with $f(x_1) = z_1$ and $f(x_1') = z_1'$ such that

$$\beta - \epsilon < \mu_A(x_1),\ \beta - \epsilon < \mu_B(x_1')$$

and there exist $x_0, x_0' \in R$ with $f(x_0) = z_0$ and $f(x_0') = z_0'$ such that

$$\beta + \epsilon > \nu_A(x_0),\ \beta + \epsilon > \nu_B(x_0').$$

So

$$\begin{aligned}\beta - \epsilon &< \mu_A(x_1) \wedge \mu_B(x_1') \leq \mu_{A+B}(x_1 + x_1') \\ &\leq \bigvee_{x \in f^{-1}(y)} \mu_{A+B}(x) = \mu_{f(A+B)}(y)\end{aligned}$$

and

$$\begin{aligned}\beta' + \epsilon &> \nu_A(x_0) \vee \nu_B(x_0') \geq \nu_{A+B}(x_0 + x_0') \\ &\geq \bigwedge_{x \in f^{-1}(y)} \nu_{A+B}(x) = \nu_{f(A+B)}(y).\end{aligned}$$

Hence $\beta - \epsilon < \alpha$ and $\beta' + \epsilon > \alpha'$. Since $\epsilon > 0$ is arbitrary, $\beta \leq \alpha$ and $\beta' \geq \alpha'$.
So $(f(A) + f(B))(y) \leq f(A + B)(y)$ for each $y \in R'$ $(**)$
Therefore, by $(*)$ and $(**)$, $f(A) + f(B) = f(A + B)$.

(2) Let $y \in R'$ and let $\epsilon > 0$ be arbitrary. Let $(\alpha, \alpha') = f(A \circ B)(y)$ and let $(\beta, \beta') = (f(A) \circ f(B))(y)$. Then

$$(*)' \quad \begin{aligned} \alpha = \mu_{f(A \circ B)}(y) = \bigvee_{z \in f^{-1}(y)} \mu_{A \circ B}(z), \\ \alpha' = \nu_{f(A \circ B)}(y) = \bigwedge_{z \in f^{-1}(y)} \nu_{A \circ B}(z) \end{aligned}$$

and

$$(**)' \quad \begin{aligned} \beta = \mu_{f(A) \circ f(B)}(y) = \bigvee_{y = y_1 y_2} [\mu_{f(A)}(y_1) \wedge \mu_{f(B)}(y_2)], \\ \beta' = \nu_{f(A) \circ f(B)}(y) = \bigwedge_{y = y_1 y_2} [\nu_{f(A)}(y_1) \vee \nu_{f(B)}(y_2)]. \end{aligned}$$

In $(*)'$ $\alpha - \epsilon < \bigvee_{z \in f^{-1}(y)} \mu_{A \circ B}(z)$ and $\alpha' + \epsilon > \bigwedge_{z \in f^{-1}(y)} \nu_{A \circ B}(z)$. Thus there exist $x, x' \in f^{-1}(y)$ such that $\alpha - \epsilon < \mu_{A \circ B}(x)$ and $\alpha' + \epsilon > \nu_{A \circ B}(x')$. Since $\mu_{A \circ B}(x) = \bigvee_{x = x_1 x_2} [\mu_A(x_1) \wedge \mu_B(x_2)]$ and $\nu_{A \circ B}(x') = \bigwedge_{x' = x'_1 x'_2} [\nu_A(x'_1) \vee \nu_B(x'_2)]$, there exist $x_1, x_2, x'_1, x'_2 \in R$ with $x = x_1 x_2$ and $x' = x'_1 x'_2$ such that $\alpha - \epsilon < \mu_A(x_1) \wedge \mu_B(x_2)$ and $\alpha' + \epsilon > \nu_A(x'_1) \vee \nu_B(x'_2)$. By Result 1.A(3), since $A \subset f^{-1}(f(A))$, $\mu_A \leq \mu_{f^{-1}(f(A))}$ and $\nu_A \geq \nu_{f^{-1}(f(A))}$. On the other hand, $\mu_{f^{-1}(f(A))} = f^{-1}(\mu_{f(A)}) = f^{-1}(f(\mu_A))$ and $\nu_{f^{-1}(f(A))} = f^{-1}(\nu_{f(A)}) = f^{-1}(f(\nu_A))$. Thus

$$\begin{aligned} \alpha - \epsilon &< f^{-1}(\mu_{f(A)})(x_1) \wedge f^{-1}(\mu_{f(B)})(x_2) \\ &= \mu_{f(A)}(f(x_1)) \wedge \mu_{f(B)}(f(x_2)) \\ &\leq \bigvee_{y = y_1 y_2} [\mu_{f(A)}(y_1) \wedge \mu_{f(B)}(y_2)] \\ &= \mu_{f(A) \circ f(B)}(y) = \beta \end{aligned}$$

and

$$\begin{aligned}\alpha' + \epsilon &> f^{-1}(\nu_{f(A)})(x'_1) \vee f^{-1}(\nu_{f(B)})(x'_2)\\ &= \nu_{f(A)}(f(x'_1)) \vee \nu_{f(B)}(f(x'_2))\\ &\geq \bigwedge_{y=y_1y_2} [\nu_{f(A)}(y'_1) \vee \nu_{f(B)}(y'_2)]\\ &= \nu_{f(A)\circ f(B)}(y) = \beta'.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\alpha \leq \beta$ and $\alpha' \geq \beta'$.
In $(**)'$,

$$\begin{aligned}\beta - \epsilon &< \bigwedge_{y=y_1y_2} [\mu_{f(A)}(y_1) \wedge \mu_{f(B)}(y_2)\\ &= \bigwedge_{y=y_1y_2} [(\bigvee_{z_1 \in f^{-1}(y_1)} \mu_A(z_1)) \wedge (\bigvee_{z_2 \in f^{-1}(y_2)} \mu_B(z_2))]\end{aligned}$$

and

$$\begin{aligned}\beta' + \epsilon &> \bigwedge_{y=y_1y_2} [\nu_{f(A)}(y_1) \vee \nu_{f(B)}(y_2)]\\ &= \bigwedge_{y=y_1y_2} [(\bigwedge_{z_1 \in f^{-1}(y_1)} \nu_A(z_1)) \vee (\bigwedge_{z_2 \in f^{-1}(y_2)} \nu_B(z_2))].\end{aligned}$$

Thus there exist $y_1, y_2 \in R'$ with $y = y_1y_2$ such that

$$\begin{aligned}\beta - \epsilon &< (\bigvee_{z_1 \in f^{-1}(y_1)} \mu_A(z_1)) \wedge (\bigvee_{z_2 \in f^{-1}(y_2)} \mu_B(z_2))\\ &= \bigvee_{z_1 \in f^{-1}(y_1)} \bigvee_{z_2 \in f^{-1}(y_2)} [\mu_A(z_1) \wedge \mu_B(z_2)]\end{aligned}$$

and

$$\begin{aligned}\beta' + \epsilon &> (\bigwedge_{z_1 \in f^{-1}(y_1)} \nu_A(z_1)) \vee (\bigwedge_{z_2 \in f^{-1}(y_2)} \nu_B(z_2))\\ &= \bigwedge_{z_1 \in f^{-1}(y_1)} \bigwedge_{z_2 \in f^{-1}(y_2)} [\nu_A(z_1) \vee \nu_B(z_2)].\end{aligned}$$

So there exist $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ such that

$$\beta - \epsilon < \mu_A(x_1) \wedge \mu_B(x_2) \text{ and } \beta' + \epsilon > \nu_A(x_1) \vee \nu_B(x_2).$$

Let $x = x_1x_2$. Since f is a ring homomorphism, $y = y_1y_2 = f(x_1x_2) = f(x)$. Thus

$$\begin{aligned}\mu_A(x_1) \wedge \mu_B(x_2) &\le \bigvee_{x=x_1x_2} [\mu_A(x_1) \wedge \mu_B(x_2)] \\ &= \mu_{A\circ B}(x) \le \bigvee_{x\in f^{-1}(y)} \mu_{A\circ B}(x) \\ &= \mu_{f(A\circ B)}(y) = \alpha\end{aligned}$$

and

$$\begin{aligned}\nu_A(x_1) \vee \nu_B(x_2) &\ge \bigwedge_{x=x_1x_2} [\nu_A(x_1) \vee \nu_B(x_2)] \\ &= \nu_{A\circ B}(x) \ge \bigwedge_{x\in f^{-1}(y)} \nu_{A\circ B}(x) \\ &= \nu_{f(A\circ B)}(y) = \alpha'\end{aligned}$$

So $\beta - \epsilon < \alpha$ and $\beta' + \epsilon > \alpha'$. Since $\epsilon > 0$ is arbitrary, $\beta \le \alpha$ and $\beta' \ge \alpha'$. Hence $(\alpha, \beta) = (\alpha', \beta')$. This completes the proof.

(3) Clearly, $A \cap B \subset A$ and $A \cap B \subset B$. By Result 1.A (1) ,$f(A \cap B) \subset f(A)$ and $f(A \cap B) \subset f(B)$. So $f(A \cap B) \subset f(A) \cap f(B)$. Suppose B is f-invariant. By Result 3.B, $f^{-1}(f(B)) = B$. Let $y \in R'$ and let $\epsilon > 0$ be arbitrary. Let $(\alpha, \beta) = [f(A) \cap f(B)](y)$ and let $(\alpha', \beta') = [f(A \cap B)](y)$. Then

$$\alpha = \mu_{f(A)\cap f(B)}(y) = (\bigvee_{x\in f^{-1}(y)} \mu_A(x)) \wedge \mu_{f(B)}(y)$$

and

$$\beta = \nu_{f(A)\cap f(B)}(y) = (\bigwedge_{x\in f^{-1}(y)} \nu_A(x)) \vee \nu_{f(B)}(y).$$

Thus $\alpha - \epsilon < (\bigvee_{x\in f^{-1}(y)} \mu_A(x)) \wedge \mu_{f(B)}(y)$ and $\beta + \epsilon > (\bigwedge_{x\in f^{-1}(y)} \nu_A(x)) \vee \nu_{f(B)}(y)$. So there exists an $x \in f^{-1}(y)$ such that $\alpha - \epsilon < \mu_A(x)$, $\alpha - \epsilon < \mu_{f(B)}(y)$ and $\beta + \epsilon > \nu_A(x)$, $\beta + \epsilon > \nu_{f(B)}(y)$. Since B is f-invariant, by Result 3.B, $f^{-1}(f(B)) = B$. Then

$$\mu_{f(B)}(y) = \mu_{f(B)}(f(x)) = f^{-1}(\mu_{f(B)})(x) = \mu_{f^{-1}(f(B))}(x) = \mu_B(x)$$

and

$$\nu_{f(B)}(y) = \nu_{f(B)}(f(x)) = f^{-1}(\nu_{f(B)})(x) = \nu_{f^{-1}(f(B))}(x) = \nu_B(x).$$

Thus

$$\alpha - \epsilon < \mu_A(x), \alpha - \epsilon < \mu_B(x)$$

and

$$\beta + \epsilon > \nu_A(x), \beta + \epsilon > \nu_B(x).$$

So

$$\alpha - \epsilon < \mu_A(x) \wedge \mu_B(x) = \mu_{A\cap B}(x)$$

and

$$\beta + \epsilon > \nu_A(x) \vee \nu_B(x) = \nu_{A\cap B}(x).$$

Hence

$$\alpha - \epsilon < \bigvee_{x\in f^{-1}(y)} \mu_{A\cap B}(x) = \mu_{f(A\cap B)}(y) = \alpha'$$

and

$$\beta + \epsilon > \bigwedge_{x\in f^{-1}(y)} \nu_{A\cap B}(x) = \nu_{f(A\cap B)}(y) = \beta'.$$

Since $\epsilon > 0$ is arbitrary, $\alpha \leq \alpha'$ and $\beta \geq \beta'$. Thus $f(A) \cap f(B) \subset f(A \cap B)$. Therefore $f(A) \cap f(B) = f(A \cap B)$.

4. Intuitionistic fuzzy cosets

Definition 4.1. Let A be any IFI of a ring R and let $x \in R$. Then A_x $\in$IFS(R) is called the *intuitionistic fuzzy coset* determined by x and A if $A_x(r) = A(r - x)$ for each $r \in R$.

Proposition 4.2. Let A be any IFI of a ring R and let R/A the set of all intuitionistic fuzzy cosets of A in R. Then R/A is a ring under the following operations:

$A_x + A_y = A_{x+y}$ and $A_xA_y = A_{xy}$ for any $x, y \in R$.

Proof. For any $a, b, c, d \in R$, suppose $A_a = A_b$ and $A_c = A_d$. Then

$$(1) \qquad A(r-a) = A(r-b) \text{ for each } r \in R$$

and

$$(2) \qquad A(r-c) = A(r-d) \text{ for each } r \in R.$$

Let $r = a + c - d$ in (1), $r = c$ in (2) and $r = a$ in (1). Then

$$(3) \qquad A(a+c-d-a) = A(a+c-d-b) = A(c-d),$$

$$(4) \qquad A(c-c) = A(c-d) = A(0)$$

and

$$(5) \qquad A(a-a) = A(a-b) = A(0).$$

On the other hand,

$$\begin{aligned}
\mu_{A_a+A_c}(r) &= \mu_{A_{a+c}}(r) = \mu_A(r-a-c) \\
&= \mu_A[(r-b-d)-(a+c-b-d)] \\
&\geq \mu_A(r-b-d) \wedge \mu_A(a+c-b-d) \\
&= \mu_A(r-b-d) \wedge \mu_A(0) \text{ (By (3) and (4))} \\
&= \mu_A(r-b-d) \\
&= \mu_{A_{b+d}}(r) = \mu_{A_b+A_d}(r)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{A_a+A_c}(r) &= \nu_{A_{a+c}}(r) = \nu_A(r-a-c) \\
&= \nu_A[(r-b-d)-(a+c-b-d)] \\
&\leq \nu_A(r-b-d) \vee \nu_A(a+c-b-d) \\
&= \nu_A(r-b-d) \vee \nu_A(0) \text{ (By (3) and (4))} \\
&= \nu_A(r-b-d) \\
&= \nu_{A_{b+d}}(r) = \nu_{A_b+A_d}(r).
\end{aligned}$$

Thus $A_b + A_d \subset A_a + A_c$. By the similar arguments, we have

$$A_a + A_c \subset A_b + A_d.$$

So $A_a + A_c = A_b + A_d$. Hence addition is well-defined. Also,

$$\begin{aligned}
\mu_{A_aA_c}(r) &= \mu_{A_{ac}}(r) = \mu_A(r - ac) \\
&= \mu_A[(r - bd) - (ac - bd)] \\
&\geq \mu_A(r - bd) \wedge \mu_A(ac - bd) \\
&= \mu_A(r - bd) \wedge \mu_A((a - b)c - b(d - c)) \\
&\geq \mu_A(r - bd) \wedge \mu_A(a - b)\mu_A(d - c) \\
&= \mu_A(r - bd) \wedge \mu_A(0)\mu_A(0) \text{ (By (4) and (5))} \\
&= \mu_A(r - bd) \\
&= \mu_{A_{bd}}(r) = \mu_{A_bA_d}(r)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{A_aA_c}(r) &= \nu_{A_{ac}}(r) = \nu_A(r - ac) \\
&= \nu_A[(r - bd) - (ac - bd)] \\
&\leq \nu_A(r - bd) \vee \nu_A(ac - bd) \\
&= \nu_A(r - bd) \vee \nu_A((a - b)c - b(d - c)) \\
&\leq \nu_A(r - bd) \vee \nu_A(a - b)\nu_A(d - c) \\
&= \nu_A(r - bd) \vee \nu_A(0)\nu_A(0) \text{ (By (4) and (5))} \\
&= \nu_A(r - bd) \\
&= \nu_{A_{bd}}(r) = \nu_{A_bA_d}(r).
\end{aligned}$$

Thus $A_bA_d \subset A_aA_c$. By the similar arguments, we have $A_aA_c \subset A_bA_d$. So $A_bA_d = A_aA_c$. Hence multiplication is well-defined. Clearly, $A_0(= A)$ acts as the additive identity, A_e as the multiplicative identity (where e is the multiplicative identity of R) and A_{-x} as additive inverse of A_x. It is now a purely routine matter to verify the other properties. This completes the proof.

Lemma 4.3. Let A be an IFSR or an IFI of a ring R. If there exist $x, y \in R$ such that $\mu_A(x) < \mu_A(y)$ and $\nu_A(x) > \nu_A(y)$, then $A(x-y) = A(x) = A(y-x)$.

Proof. Since A is an IFG of R with respect to " + ", by Result 2.A, $A(x-y) = A(y-x)$. Thus it is sufficient to show that $A(x-y) = A(x)$. Since $\mu_A(x) < \mu_A(y)$, $\nu_A(x) > \nu_A(y)$ and A is an IFSR or an IFI of R, $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y) = \mu_A(x)$ and $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y) = \nu_A(x)$. On the other hand, $\mu_A(x) = \mu_A(x-y+y) \geq \mu_A(x-y) \wedge \mu_A(y)$ and $\nu_A(x) = \nu_A(x-y+y) \leq \nu_A(x-y) \vee \nu_A(y)$. Thus $\mu_A(x) \geq \mu_A(x-y)$ and $\nu_A(x) \leq \nu_A(x-y)$. So $A(x-y) = A(x)$. This completes the proof.

Lemma 4.4. If A is any IFI of a ring R, then $A(x) = A(0)$ if and only if $A_x = A_0$, where $x \in R$.

Proof. ($\Rightarrow$): Suppose $A(x) = A(0)$. Since A is an IFG of R with respect to " + ", $A(r) \leq A(0) = A(r)$, i.e., $\mu_A(r) \leq \mu_A(0) = \mu_A(r)$ and $\nu_A(r) \geq \nu_A(0) = \nu_A(r)$ for each $r \in R$.

Case(i): Suppose $A(r) < A(x)$. Then, by Lemma 4.3 $A(r-x) = A(r)$. Thus $A_x(r) = A_0(r)$ for each $r \in R$.

Case(ii): Suppose $A(r) = A(x)$. Then $x, r \in A^{(\lambda,\mu)}$, where $(\lambda, \mu) = A(0)$. Since A is an IFG of R, $A^{(\lambda,\mu)}$ is a subgroup of R. Thus $x - r \in A^{(\lambda,\mu)}$. Thus $\mu_A(x-r) \geq \lambda = \mu_A(0)$ and $\nu_A(x-r) \leq \mu = \nu_A(0)$. Since $\mu_A(x-r) \leq \mu_A(0)$ and $\nu_A(x-r) \geq \nu_A(0)$, $\mu_A(x-r) = \mu_A(0)$ and $\nu_A(x-r) = \nu_A(0)$. Thus $A(x-r) = A(0) = A(x) = A(r)$, i.e., $A_x(r) = A_0(r)$ for each $r \in R$. In either case, $A_x(r) = A_0(r)$ for each $r \in R$. Hence $A_x = A_0$ for each $x \in R$.

($\Leftarrow$): It is straightforward.

Theorem 4.5. Let A be any IFI of a ring R and let $A(0) = (\lambda, \mu)$. Then $R/A^{(\lambda,\mu)} \cong R/A$.

Proof. Define a mapping $f : R \to R/A$ by $f(x) = A_x$ for each $x \in R$. Then it is easy to check that f is a ring epimorphism. By Lemma 4.4,

$$\begin{aligned} \text{Ker}f &= \{x \in R : f(x) = A_0\} = \{x \in R : A_x = A_0\} \\ &= \{x \in R : A(x) = A(0)\} = A^{(\lambda,\mu)}. \end{aligned}$$

Hence $R/A^{(\lambda,\mu)} \cong R/A$.

Theorem 4.6. Let $f : R \longrightarrow R'$ be a ring epimorphism and let A be an IFI of R such that $A^{(\lambda,\mu)} \subset \text{Ker}f$, where $(\lambda, \mu) = A(0)$. Then there exists a unique epimorphism $\bar{f} : R/A \to R'$ such that $f = \bar{f} \circ g$, where $g(x) = A_x$ for each $x \in R$.

Proof. Define a mapping $\bar{f} : R_A \to R'$ by $\bar{f}(A_x) = f(x)$ for each $x \in R$. Suppose $A_x = A_y$. Then $A_{x-y} = A_0 = A_x = A_y$. By Lemma 4.4, $A(x-y) = A(0)$. Then $x - y \in A^{(\lambda,\mu)}$. Since $A^{(\lambda,\mu)} \subset \text{Ker}f$, $x - y \in \text{Ker}f$. Thus $f(x) = f(y)$, i.e., $\bar{f}(A_x) = \bar{f}(A_y)$. So $\bar{f}$ is well-defined. Furthermore, since f is surjective, $\bar{f}$ is also surjective. Moreover, it is easy to see that $\bar{f}$ is a homomorphism.

Consider the following diagram:

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {}_g\searrow & & \nearrow_{\bar{f}} \\ & R/A & \end{array}$$

Let $x \in R$. Then $f(x) = \bar{f}(A_x) = \bar{f}(g(x)) = (\bar{f} \circ g)(x)$. Thus the above diagram commutes, i.e., $f = \bar{f} \circ g$.

Suppose there exists an epimorphism $h : R/A \to R'$ such that $f = h \circ g$. Let $x \in R$. Then $\bar{f}(A_x) = f(x) = (h \circ g)(x) = h(g(x)) = h(A_x)$. Thus $\bar{f} = h$. So $\bar{f}$ is unique. This completes the proof.

Corollary 4.6. The induced homomorphism $\bar{f}$ is an isomorphism if and only if A is f-invariant.

Proof. ($\Rightarrow$): Suppose $\bar{f}$ is an isomorphism, i.e., $\bar{f}$ is injective. For any $x, y \in R$, let $f(x) = f(y)$. Then $\bar{f}(A_x) = \bar{f}(A_y)$. Since $\bar{f}$ is injective, $A_x = A_y$. Thus $A_{x-y} = A_0$. By Lemma 4.4, $A(x - y) = A(0)$. By Proposition 2.8 in [9] $A(x) = A(y)$. So A is f-invariant.

($\Leftarrow$): Suppose A is f-invariant and $\bar{f}(A_x) = \bar{f}(A_y)$. Then $f(x) = f(0)$. Since A is f-invariant, $A(x) = A(0)$. By Lemma 4.4, $A_x = A_0$. So $\bar{f}$ is injective. This completes the proof.

Theorem 4.7. Let $f : R \to R'$ be a ring epimorphism and let A be any f-invariant IFI of R. Then $R/A \cong R'/f(A)$.

Proof. Since A is f-invariant, Ker$f \subset A^{(\lambda,\mu)}$, where $(\lambda, \mu) = A(0)$. Consider $f(A)(0') = (f(\mu_A)(0'), f(\nu_A)(0'))$, where $0'$ denotes the additive identity in R'. Then

$$f(\mu_A)(0') = \bigvee_{x \in f^{-1}(0')} \mu_A(x) \text{ and } f(\nu_A)(0') = \bigwedge_{x \in f^{-1}(0')} \nu_A(x).$$

Since $f(0) = 0'$ and $A(x) \leq A(0)$, i.e., $\mu_A(x) \leq \mu_A(0)$, $\nu_A(x) \geq \nu_A(0)$ for each $x \in R$, $f(\mu_A)(0') = \mu_A(0)$ and $f(\nu_A)(0') = \nu_A(0)$, i.e., $f(A)(0') = A(0) = (\lambda, \mu)$. Now,

$$\begin{aligned}
f(x) \in [f(A)]^{(\lambda,\mu)} &\Leftrightarrow \mu_{f(A)}(f(x)) \geq \lambda \text{ and } \nu_{f(A)}(f(x)) \leq \mu \\
&\Leftrightarrow f(\mu_A)(f(x)) \geq \lambda \text{ and } f(\nu_A)((f(x)) \leq \mu \\
&\Leftrightarrow f^{-1}(f(\mu_A))(x) \geq \lambda \text{ and } f^{-1}(f(\nu_A))(x) \leq \mu \\
&\Leftrightarrow \mu_A(x) \geq \lambda \text{ and } \nu_A(x) \leq \mu \text{ (by Result3.B)} \\
&\Leftrightarrow x \in A^{(\lambda,\mu)} \\
&\Leftrightarrow f(x) \in f(A^{(\lambda,\mu)}) \text{ (Since Ker} f \subset A^{(\lambda,\mu)}).
\end{aligned}$$

So $[f(A)]^{(\lambda,\mu)} = f(A^{(\lambda,\mu)})$. By Theorem 4.5, $R/A \cong R/A^{(\lambda,\mu)}$ and $R'/f(A) \cong R'/[f(A)]^{(\lambda,\mu)}$. Hence $R/A \cong R'/f(A)$. This completes the proof.

References

[1] K.Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1986),87-96.

[2] Baldev Banerjee and Dhiren Kr. Basnet, Intuitionistic fuzzy subrings and ideals, J. Fuzzy Math. 11(1)(2003),139-155

[3] R.Biswas, Intuitionistic fuzzy subgroups, Mathematical Forum x(1989),37-46

[4] D. Çoker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88(1997),81-89.

[5] D. Çoker and A.Haydar Es, On fuzzy compactness in intuitionistic fuzzy topological spaces, J. Fuzzy Math. 3(1995),899-909.

[6] P.S.Das, Fuzzy groups and level subgroups, J. Math. Anal. Appl. 84(1981), 264-269.

[7] H.Gürçay, D. Çoker and A.Haydar Es, On fuzzy continuity in intuitionistic fuzzy topological spaces, J. Fuzzy Math. 5(1997), 365-378.

[8] K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy subgroupoids, International Journal of Fuzzy Logic and Intelligent Systems 3(1)(2003), 72-77.

[9] K.Hur, H.W.Kang and H.K.Song, Intuitionistic fuzzy subgroups and subrings, Honam Math. J. 25(1) (2003), 19-41.

[10] R.Kumar, Fuzzy subgroups, fuzzy ideals and fuzzy cosets: Some properties, Fuzzy Sets and Systems 48(1992),267-274.

[11] S.J.Lee and E.P.Lee, The category of intuitionistic fuzzy topological spaces Bull. Korean Math. Soc. 37(1)(2000),63-76.

[12] Wang-jin Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems 8(1982), 133-139.

[13] A.Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35(1971) ,512-517.

[14] L.A.Zadeh, Fuzzy sets, Inform. and Centrol 8 (1965), 338-353.

Kul Hur

Division of Mathematics and Informational Statistics,

and Institute of Basic Natural Science,

Wonkwang University,

Iksan, Chonbuk, Korea 570-749.

E-mail: kulhur@wonkwang.ac.kr

Su Youn Jang
Division of Mathematics and Informational Statistics,
and Institute of Basic Natural Science
Wonkwang University Iksan,
Chonbuk, Korea 570-749.
E-mail: suyoun123@yahoo.co.kr

Hee Won Kang
Dept. of Mathematics Education,
Woosuk University,
Hujong-Ri Samrae-Eup, Wanju-kun Chonbuk, Korea 565-701.
E-mail: khwon@woosuk.ac.kr