Honam Mathematical J. **26**(2004), No. 2, pp. 209–218

VARIOUS COMPACT-TYPE PROPERTIES BETWEEN ω-BOUNDEDNESS AND PSEUDOCOMPACTNESS

MYUNG HYUN CHO, JUN-HUI KIM, AND HYO SUN SEO

Abstract. On the analogy of total countable compactness, we study interesting subfamilies in the class of pseudocompact spaces. We show relationships between totally pseudocompact spaces, sequentially pseudocompact spaces, and DFCC spaces. We also prove relationships among densely ξ-pseudocompact, ξ-pseudocompact, and countably pracompact spaces. As a productive result on countably pracompact spaces, we will prove that if X is a countably pracompact space and Y is a countably pracompact k-space, then $X \times Y$ is countably pracompact.

1. Introduction

Throughout this paper, a space always means a T_2 topological space. A space X is called *ω-bounded* if the closure of any countable subset of X is compact. A space X is called *totally countably compact (or strong ω-compact)*[7] if any infinite subset of X contains an infinite subset whose closure is compact. Clearly, every totally countably compact space is countably compact. Total countable compactness is a generalization of both ω-boundedness and sequential compactness.

In this paper, on the analogy of total countable compactness, we study interesting subfamilies in the class of pseudocompact spaces. In section 2, we show relationships between totally pseudocompact spaces [1],

This paper was supported by Wonkwang University Research Grant, 2003.

Received February 28, 2004; Accepted May 12, 2004.

AMS (2000) Subject Classification:Primary 54D30, 54D50, 54B10 .

Keywords and Phrases: DFCC, totally countably compact, totally pseudocompact, sequentially pseudocompact, densely ξ-pseudocompact, ξ-pseudocompact, and countably pracompact.

sequentially pseudocompact spaces, and DFCC spaces. A space X is *sequentially pseudocompact* if for every sequence $\mathcal{A}$ of non-empty open sets in X there exist a subsequence $\mathcal{B}$ and a point $x \in X$ each neighborhood of which meets all but finitely many elements of $\mathcal{B}$. In section 3, we also consider densely ξ-pseudocompact, ξ-pseudocompact [4], and countably pracompact spaces [1]. We prove implications between them and Diagram 1 in section 2 shows the relationships between just mentioned properties. In the rest of section 3, we will be concerned with the product topology. The product of a compact space and a countably compact space is countably compact. It is not true, however, that the product of every two countably compact spaces is countably compact. It is well-known [3] that the product $X \times Y$ of a countably compact space X and a countably compact k-space Y(in particular, a countably compact sequential space) is countably compact.

Let $(X, \mathcal{T})$ be a space and let $\mathcal{C}(X, \mathcal{T})$ denote the family of all compact subsets of $(X, \mathcal{T})$. Let $\mathcal{K}(X, \mathcal{T})$ denote the family of all subsets F of X which have the property that $F \cap K$ is open (or close) in K(w.r.t. the subspace topology on K) for all $K \in \mathcal{C}(X, \mathcal{T})$. Clearly $\mathcal{T} \subset \mathcal{K}(X, \mathcal{T})$ and $\mathcal{K}(X, \mathcal{T})$ is a topology on the set X. A space $(X, \mathcal{T})$ is called a *k-space* (or is said to be *compactly generated*) provided $\mathcal{T} = \mathcal{K}(X, \mathcal{T})$. The *k-leader* of the space $(X, \mathcal{T})$ is defined as the set equipped with the strongest of all topologies $\mathcal{T}'$ on X inducing on each compact subset of the space $(X, \mathcal{T})$ the same topology as $\mathcal{T}$. With this definition the k-leader of an arbitrary space is a k-space.

Theorem 1.1 *[6] A space $(X, \mathcal{T})$ is totally countably compact if and only if the space $(X, \mathcal{K}(X, \mathcal{T}))$(its k-leader) is countably compact. In particular, every countably compact k-space is totally countably compact.*

Theorem 1.1 suggests when the reverse implications hold in theorems of Section 2 and 3.

Pseudocompactness is not finitely multiplicative. However, we have the following theorem which is originally known by Tamano.

Theorem 1.2 *[3] The product $X \times Y$ of a pseudocompact space X and a pseudocompact k-space Y is pseudocompact.*

It follows from the above theorem that the product $X \times Y$ of a pseudocompact space X and a compact space Y is pseudocompact. Also, the product $X \times Y$ of a pseudocompact space X and a pseudocompact sequential space Y is pseudocompact. The product $X \times Y$ of a pseudocompact space X and a sequentially compact Tychonoff space Y is pseudocompact. As a result on countably pracompact spaces, we will prove Theorem 3.4 that if X is a countably pracompact space and Y is a countably pracompact k-space, then $X \times Y$ is countably pracompact.

2. Totally pseudocompact and sequentially pseudocompact spaces

A space X is called *totally pseudocompact* [1] if for any infinite family $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ of non-empty open sets in X there exists a compact subspace K of X that intersects infinitely many elements of $\mathcal{A}$, i.e.,

$$|\{\alpha \in \Lambda : K \cap A_\alpha \neq \emptyset\}| \geq \aleph_0.$$

Theorem 2.1 *Every totally countably compact space is totally pseudocompact.*

Proof. Let X be a totally countably compact space and $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ be an infinite family of non-empty open sets in X. For each $\alpha \in \Lambda$, choose $a_\alpha \in A_\alpha$ and let $A = \{a_\alpha : \alpha \in \Lambda\}$. Since X is totally countably compact, A contains an infinite subset A_0 such that $\overline{A_0}$ is compact. Clearly $\overline{A_0}$ meets infinitely many elements of $\mathcal{A}$. Hence X is totally pseudocompact. □

Recall that a space X has the *discrete finite chain condition* (henceforth abbreviated DFCC) provided every discrete family of nonempty open sets is finite. It is well-known [3] that every pseudocompact normal space is countably compact. In fact, the property wD, which is weaker than normality, is enough to imply the countable compactness of a DFCC space.

Theorem 2.2 *Every totally pseudocompact space is DFCC.*

Proof. Let X be a totally pseudocompact space and let $\mathcal{D} = \{D_\alpha \subset X : \alpha \in \Lambda\}$ be an infinite discrete family of non-empty open sets in X. Then there exists a compact subspace A of X such that $A \cap D_\alpha \neq \emptyset$ for infinitely many $\alpha \in \Lambda$. Choose $d_\alpha \in A \cap D_\alpha$ for infinitely many $\alpha \in \Lambda$. Then $\{d_\alpha : \alpha \in \Lambda\}$ is a relatively discrete infinite subset of A. This contradicts compactness of A. Therefore $\mathcal{D}$ must be finite and so X is DFCC. □

In the class of Tychonoff spaces, DFCC is equivalent to pseudocompactness. The analogy with sequential compactness allows us to identify another interesting subfamily in the class of pseudocompact spaces.

Definition 2.1 *A space X is sequentially pseudocompact if for every sequence $\mathcal{A}$ of non-empty open sets in X there exist a subsequence $\mathcal{B}$ and a point $x \in X$ each neighborhood of which meets all but finitely many elements of $\mathcal{B}$.*

Theorem 2.3 *Every sequentially compact space is sequentially pseudocompact.*

Proof. Assume that X is sequentially compact. Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a sequence of non-empty open sets in X. Choose $a_n \in A_n$ for each $n \in \omega$ and let $A = \{a_n : n \in \omega\}$. Then A is an infinite sequence of points in X. Since X is sequentially compact, A has a convergent subsequence $\{a_{n_k} : k \in \omega\}$, say $a_{n_k} \to x$ for some $x \in X$. Then $\mathcal{B} = \{A_{n_k} : k \in \omega\}$, where $a_{n_k} \in A_{n_k}$ for each $k \in \omega$, is a subsequence of $\mathcal{A}$. Clearly each neighborhood of x meets all but finitely many elements of $\mathcal{B}$ being $a_{n_k} \in A_{n_k}$ and $a_{n_k} \to x$. Therefore X is sequentially pseudocompact. □

Theorem 2.4 *Every sequentially pseudocompact space is totally pseudocompact.*

Proof. Let X be sequentially pseudocompact and let $\mathcal{A}$ be an infinite family of non-empty open sets in X. Without loss of generality, assume $\mathcal{A}$ is countably infinite so that $\mathcal{A}$ is a sequence of non-empty open sets in X. By sequential pseudocompactness, there exist a subsequence $\mathcal{B}$ and a point $x \in X$ each neighborhood of which meets all but finitely many elements of $\mathcal{B}$. From this, we may construct a compact subspace that intersect infinite many elements of $\mathcal{A}$. □

Proposition 2.1 *A first countable pseudocompact space is sequentially pseudocompact.*

3. Around ξ-compact spaces

Denote by ω^* the family of all free ultrafilters on ω. Thus if $\xi \in \omega^*$, then ξ is a maximal family with f.i.p. of subsets of ω, and $\cap\xi = \emptyset$. For a fixed $\xi \in \omega^*$, a point x of a space X is called a *ξ-limit* of a sequence $f : \omega \to X$ of points of X, denoted by $x = \xi\text{-}lim f$, if for every neighborhood U of x the set $\{n \in \omega : f(n) \in U\}$ belongs to ξ.

A sequence $f : \omega \to X$ is called *ξ-convergent* (in X) if it has a ξ-limit in X. The ξ-limit of a sequence in a Hausdorff space is unique since ultrafilters cannot contain a pair of disjoint sets. A point of X is a limit point of a sequence $f : \omega \to X$ iff it is its ξ-limit for some $\xi \in \omega^*$. So in a countably compact space, every sequence f has a ξ-limit for some $\xi \in \omega^*$, but different sequences may require different ξ's. This suggests that a space X is *ξ-compact* if every sequence $f : \omega \to X$ has an ξ-limit point in X. Every ω-bounded space is ξ-compact.

What kind of spaces has the property that it is ξ-compact for all $\xi \in \omega^*$? The answer is true in the class of T_3 spaces. For this see Theorem 4.9 in [7].

Every ξ-compact space is countably compact, but the converse is not true (see 4.8 Example in [7]).

Definition3.1 *A space X is densely ξ- compact if it has a dense subspace Y each infinite subset of which has a ξ-limit point.*

Proposition 3.1 *Every ξ-compact space is densely ξ-compact.*

Proof. It is trivial. □

Definition3.2 *A space X is ξ-pseudocompact if for any sequence $\{V_n : n \in \omega\}$ of nonempty open sets in X there exists a point $p \in X$ such that for every neighborhood U of p, $\{n \in \omega : V_n \cap U \neq \emptyset\} \in \xi$.*

Theorem 3.1 *Every densely ξ-compact space is ξ-pseudocompact.*

proof. Let X be densely ξ-compact. Then there exists a dense $Y \subset X$ such that every infinite subsets of Y has a ξ-limit point. Let $\{V_n : n \in \omega\}$ be a sequence of non-empty open sets in X. Then $Y \cap V_n \neq \emptyset$ for all $n \in \omega$ (since Y is dense in X). Choose $y_n \in Y \cap V_n$ for each $n \in \omega$. Then

$\{y_n : n \in \omega\}$ is an infinite subset of Y and so it has a ξ-limit point p (say). Let U be an open neighborhood of p. Then $\{n \in \omega : y_n \in U\} \in \xi$ and therefore $\{n \in \omega : U \cap V_n \neq \phi\} \in \xi$ (since ξ is an ultrafilter). Hence X is ξ-pseudocompact. □

Recall that a subspace Y of a space X is *relatively countably compact* in X if every infinite subset of Y has a limit point in X. A space X is *countably pracompact* if it has a dense subspace Y such that Y is relatively countably compact in X. Clearly, every countably compact space is countably pracompact, and it is easy to see that every countably pracompact space is DFCC.

Theorem 3.2 *Every densely ξ-compact space is countably pracompact.*

Proof. Let X be a densely ξ-compact space. Then there exists a dense subspace $Y \subset X$ such that every infinite subset of Y has a ξ-limit point $p \in X$. But since p is a limit point of the sequence in Y, Y is relatively countably compact in X. Therefore X is countably pracompact.□

Theorem 3.3 *Every ξ-pseudocompact space is DFCC.*

Proof. Assume that X is ξ-pseudocompact. Let $\{U_n : n \in \omega\}$ be a discrete infinite sequence of non-empty open subsets of X. Then there exists a point $p \in X$ such that for every neighborhood U of p, $\{n \in \omega : U_n \cap U \neq \emptyset\} \in \xi$. However, since $\{U_n : n \in \omega\}$ is discrete, p has a neighborhood V such that $\{n \in \omega : U_n \cap V \neq \emptyset\}$ is at most one. This is a contradiction. (how? $\emptyset \in \xi$?) □

Theorem 3.4 *If X is a countably pracompact space and Y is a countably pracompact k-space, then $X \times Y$ is countably pracompact.*

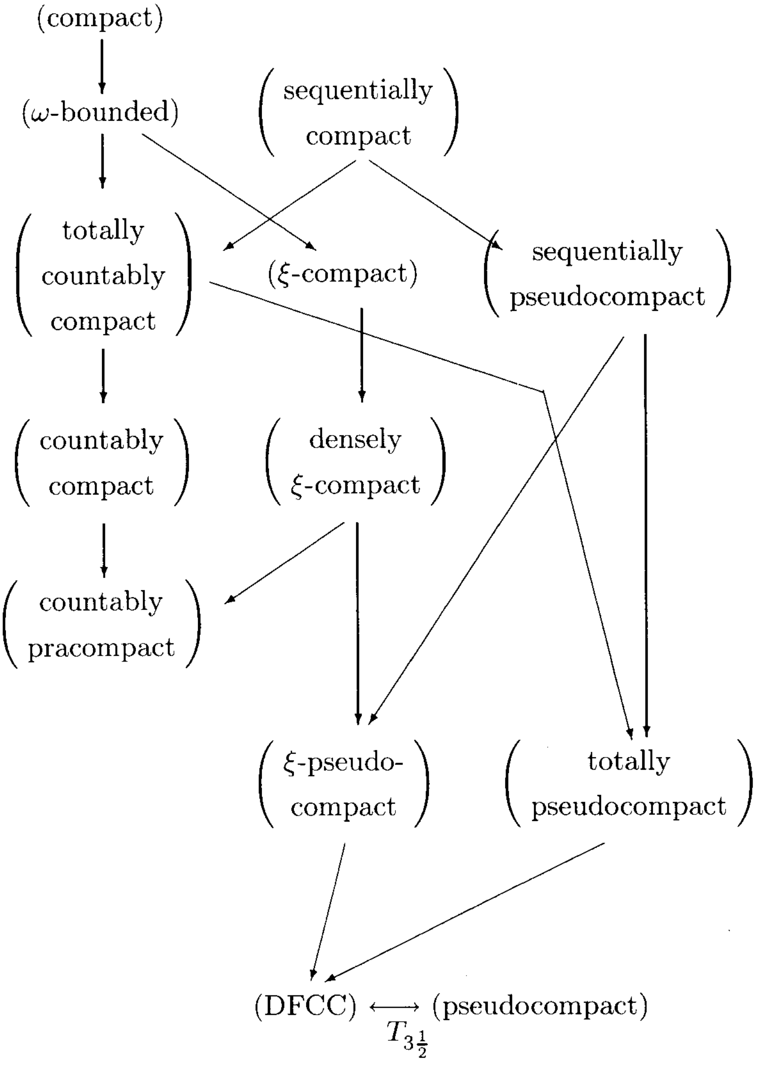

Diagram 1

Proof. Since X is countably pracompact, there is a dense subspace A of X which is relatively countably compact in X. Then $A \times Y$ is dense in $X \times Y$. We claim that $A \times Y$ relatively countably compact in $X \times Y$. It suffices to show from $(i) \Leftrightarrow (iii)$ in 3.10.3 Theorem [3] that every locally finite family of one-point subsets of $A \times Y$ is finite. Let $\mathcal{U} = \{\{a_\alpha\} \times \{y_\alpha\} : \alpha \in \Lambda\}$ be a locally finite family of one-point subsets of $X \times Y$ with $|\Lambda| \geq \omega$. Then by 3.10.12 Lemma [3], there exists an infinite set $\Lambda_0 \subseteq \Lambda$ such that the family $\{\{a_\alpha\} : \alpha \in \Lambda_0\}$ is locally finite or the family $\{\{y_\alpha\} : \alpha \in \Lambda_0\}$ is locally finite. This contradicts the fact $(i) \Leftrightarrow (iii)$ in 3.10.3 Theorem [3] again. Hence $\mathcal{U}$ must be finite. Therefore $A \times Y$ is relatively countably compact. $\square$

References

[1] A. V. Arhangel'skii, *Compactness*, Contemporary Problems in Mathematics. Fundamental Directions. General Topology - 2 Moscow, WINITI Publ. (1989) 5-128 (in Russian); *English translation:* Encyclopedia of Mathematical Sciences. General Topology II, Springer (1996)

[2] A. J. Berner, Spaces with dense conditionally compact subsets, *Proc. Amer. Math. Soc.* , **81** (1981), 137-142

[3] R. Engelking, General Topology, Revised and completed edition, *Heldermann Verlag, Berlin*, (1989)

[4] J. Ginsburg and A. Saks, *Some applications of ultrafilters in topology*, Pacific J. Math. *57:2* (1975)

[5] M. Matveev, A survey on star covering properties, *Topology Atlas*, Preprint No. 330, (1998)

[6] N. Noble, Countably compact and pseudocompact products, *Czsech. Math. J.*, **19** (1969), 390-397

[7] J. E. Vaughan, *Countably compact and sequentially compact spaces*, Handbook of Set-theoretic Topology, K. Kunen and J. E. Vaughan Eds., Elsevier Sci. Pub. (1984) 569-602

Myung Hyun Cho
Division of Mathematics and Informational Statistics
Wonkwang University
Iksan 570-749, Korea mhcho@wonkwang.ac.kr

Jun-Hui Kim
Division of Mathematics and Informational Statistics
Wonkwang University

Iksan 570-749, Korea
junhikim@wonkwang.ac.kr

Hyo Sun Seo
Division of Mathematics and Informational Statistics
Wonkwang University
Iksan 570-749, Korea
solo@wonkwang.ac.kr