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SOME SEQUENCES OF IMPROVEMENT OVER LINDLEY TYPE ESTIMATOR

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Abstract. In this paper, the problem of estimating a $p-$variate $(p \geq 4)$ normal mean vector is considered in a decision-theoretic setup. Using a simple property of the noncentral chi-square distribution, a sequence of smooth estimators dominating the Lindley type estimator has been produced and each improved estimator is better than previous one.

1. Introduction

Let $\mathbf{X} = (X_1, \cdots, X_p)'$ be a $p-$variate random vector and $\mathbf{X} \sim N_p(\theta,\ \mathbf{I}_p)$,
$\theta \in R^p$. For any estimator $\delta(\mathbf{X})$ of θ, the loss in estimating θ by $\delta(\mathbf{X})$ is

$$L(\delta,\ \theta) = \|\delta - \theta\|^2 = (\delta - \theta)'(\delta - \theta). \tag{1.1}$$

The performance of an estimator δ is evaluated by its risk function defined as

$$R(\delta, \theta) = E[L(\delta, \theta)].$$

The standard estimator (MLE as well as the best location estimator) of θ is

$$\delta^0 = \mathbf{X}$$

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which is admissible for $p \leq 2$. Stein(1956) and James and Stein(1961) showed that δ^0 is inadmissible for $p \geq 3$ and it is dominated by

$$\delta^{JS} = \left(1 - \frac{(p-2)}{\|\mathbf{X}\|^2}\right)\mathbf{X}, \ p \geq 3.$$

Since this pioneering work, many shrinkage estimators which dominate δ^0 have been proposed. Guo and Pal(1992) considered a sequence of improved estimators providing successive improvement over δ^{JS}. Kubokawa (1991) and Guo and Pal(1992) constructed a sequence of estimators (dominating δ^{JS}) where each estimator is better than the previous one and the sequence converges to an admissible estimator. Subsequently a number of authors provided classes of Stein-type estimators dominating $\mathbf{X}$(Efron and Morris(1976), Ghosh, Hwang, and Tsui(1984) where other references are cited). One common feature of the above classes of estimators dominating $\mathbf{X}$ is that they are all spherically symmetric shrinking $\mathbf{X}$ toward some particular point, not necessarily the origin.

The Lindley(1962) type estimator is

$$\delta^{\mathbf{1}} = \overline{\mathbf{X}}\mathbf{1} + \left(1 - \frac{\mathbf{p}-\mathbf{3}}{\|\mathbf{X} - \overline{\mathbf{X}}\mathbf{1}\|^2}\right)(\mathbf{X} - \overline{\mathbf{X}}\mathbf{1}), \ \mathbf{p} \geq \mathbf{4},$$

where $\overline{X} = (X_1 + \cdots + X_p)/p$ and $\mathbf{1} = (1, \cdots, 1)'$. The Lindley type estimator possesses better risk properties than the ordinary James-Stein estimator over a large region of the parameter space, suggesting that from a sampling theoretic viewpoint the shrinkage should be taken toward $\overline{X}\mathbf{1}$ as opposed to the origin.

In this paper, a sequence of improved estimators providing successive improvements over δ^1 is constructed. Each of these improved estimators again, in turn, can be dominated by using a technique of Kubokawa (1991).

In Section 2, such improved estimators are derived. The above results are generalized when $\mathbf{X} \sim N_p(\theta, \mathbf{\Sigma})$, where the covariance matrix $\mathbf{\Sigma}$ is either completely unknown or $\mathbf{\Sigma} = \sigma^2\mathbf{I}_p$ for some unknown scalar $\sigma^2 > 0$ in Section 3.

2. Improved estimators dominating δ^1

Consider a sequence of estimators of the form

$$\delta^n = \overline{X}\mathbf{1}+K_n(\mathbf{X}-\overline{X}\mathbf{1}),\ n = 1, 2, 3, \cdots , \tag{2.1}$$

where $K_n = K_n(\mathbf{X})$ is a suitable function of $\mathbf{X}$.

We choose $K_1 = \left(1 - \dfrac{p-3}{\|\mathbf{X} - \overline{X}\mathbf{1}\|^2}\right)$ to make the first element δ^1 of the sequence $\{\delta^n\}$. Our goal is to construct δ^n $(n \geq 2)$ such that for any integer $n \geq 1$ and $p \geq 4$,

$$R(\delta^{n+1}, \theta) \leq R(\delta^n,\ \theta),\ \forall\ \theta \in R^p.$$

To dominate the estimator δ^n for any $n \geq 1$, define δ^{n+1} as

$$\delta^{n+1} = \delta^n + r_n^*(\mathbf{X} - \overline{X}\mathbf{1}),\ \text{i.e.,}\ K_{n+1} = K_n + r_n^*,$$

where $r_n^* = r_n^*(\mathbf{X})$ is a suitable real valued function.

Let $\mathbf{r}_n = r_n^* \cdot (\mathbf{X} - \overline{X}\mathbf{1})$. Define the risk difference (RD) between $R(\delta^{n+1}, \theta)$ and $R(\delta^n, \theta)$ as

$$\begin{aligned} RD(n+1,\ n) &= R(\delta^{n+1},\ \theta) - R(\delta^n\ , \theta) \\ &= E\Big[\ \| \ \delta^n + \mathbf{r}_n - \theta\ \|^2 - \|\ \delta^n - \theta\ \|^2\ \Big] \\ &= E\left[\sum_{i=1}^{p}\Big\{(\delta_i^n + r_{ni} - \theta_i)^2 - (\delta_i^n - \theta_i)^2\Big\}\right] \\ &= E\left[\sum_{i=1}^{p} r_{ni}^2 + 2\sum_{i=1}^{p} r_{ni}(\delta_i^n - \theta_i)\right], \end{aligned} \tag{2.2}$$

where δ_i^n, θ_i and r_{ni} denote the i^{th} elements of δ^n, θ, and $\mathbf{r}_n$ respectively.

The second term of (2.2) can be simplified as

$$\begin{aligned} E\left[\sum_{i=1}^{p} r_{ni}(\delta_i^n - \theta_i)\right] &= \sum_{i=1}^{p} E[r_{ni}\{\overline{X} + K_n(X_i - \overline{X}) - \theta_i\}] \\ &= \sum_{i=1}^{p} E[(K_n - 1)r_{ni}(X_i - \overline{X}) + r_{ni}(X_i - \theta_i)] \\ &= \sum_{i=1}^{p} E[(K_n - 1)r_{ni}(X_i - \overline{X}) + \frac{\partial}{\partial X_i} r_{ni}]. (2.3) \end{aligned}$$

The expression (2.3) is obtained by using Stein's normal identity (Casella and Berger (1990) p187) assuming that r_{ni}'s $(i = 1, 2, \cdots, p)$ satisfy all the regularity conditions of the identity. Combining (2.2) and (2.3), we get

$$RD(n+1,\ n) = E\left[\sum_{i=1}^{p}\{r_{ni}^2 + 2(K_n - 1)r_{ni}(X_i - \overline{X}) + 2\frac{\partial}{\partial X_i} r_{ni}\}\right]. \quad (2.4)$$

We now look for suitable $\mathbf{r}_n = r_n^* \cdot (\mathbf{X} - \overline{X}\mathbf{1})$ such that $RD(n+1,\ n) \leq 0,\ \forall\ n \geq 1$.

Before we derive the general result, let us look at some special cases.

2.1 The case of $\mathbf{n} = \mathbf{1}$

To dominate δ^1(Lindley), take $r_1^* = c_1\|\mathbf{X} - \overline{X}\mathbf{1}\|^{-(2+\alpha_1)}$, where $\alpha_1 > 0$ and c_1 is a suitable constant. Then,

$$\begin{aligned} &2\sum_{i=1}^{p} \frac{\partial}{\partial X_i} r_{1i} \\ &= 2\sum_{i=1}^{p} c_1 \left[\left(1 - \frac{1}{p}\right) \| \mathbf{X} - \overline{X}\mathbf{1} \|^{2+\alpha_1} - \frac{2+\alpha_1}{2}(X_i - \overline{X}) \| \mathbf{X} - \overline{X}\mathbf{1} \|^{\alpha_1} \right. \\ &\times \left\{2(X_1 - \overline{X})\left(-\frac{1}{p}\right) + \cdots + 2(X_i - \overline{X})\left(1 - \frac{1}{p}\right) + \cdots \right. \\ &\left.\left. + 2(X_p - \overline{X})\left(-\frac{1}{p}\right)\right\}\right] / \| \mathbf{X} - \overline{X}\mathbf{1} \|^{2(2+\alpha_1)} \end{aligned}$$

$$= 2\sum_{i=1}^{p} c_1 \left[\left(1-\frac{1}{p}\right) \| \mathbf{X} - \overline{X}\mathbf{1} \|^{2+\alpha_1} - (2+\alpha_1) \| \mathbf{X} - \overline{X}\mathbf{1} \|^{\alpha_1} (X_i - \overline{X})^2\right]$$
$$\times \frac{1}{\| \mathbf{X} - \overline{X}\mathbf{1} \|^{2(2+\alpha_1)}} = \frac{2c_1\{p-(3+\alpha_1)\}}{\|\mathbf{X} - \overline{X}\mathbf{1}\|^{2+\alpha_1}},$$
$$\sum_{i=1}^{p} r_{1i}^2 = \frac{c_1^2}{\|\mathbf{X} - \overline{X}\mathbf{1}\|^{2+2\alpha_1}} \text{ and } 2(K_1-1)\sum_{i=1}^{p} r_{1i}(X_i - \overline{X}) = -\frac{2c_1(p-3)}{\|\mathbf{X} - \overline{X}\mathbf{1}\|^{2+\alpha_1}}.$$

Therefore, from (2.4) one can get

$$\begin{aligned} RD(2,1) &= E\left[\frac{c_1^2}{\|\mathbf{X} - \overline{X}\mathbf{1}\|^{2+2\alpha_1}} - \frac{2c_1\alpha_1}{\|\mathbf{X} - \overline{X}\mathbf{1}\|^{2+\alpha_1}}\right] \\ &= E\left[\frac{c_1^2}{T^{1+\alpha_1}} - \frac{2c_1\alpha_1}{T^{1+\frac{\alpha_1}{2}}}\right], \end{aligned}$$

where $T = \|\mathbf{X} - \overline{X}\mathbf{1}\|^2 \sim$ noncentral $\chi^2_{p-1}(\lambda)$ with $\lambda = \|\theta - \overline{\theta}\mathbf{1}\|^2$ and $\overline{\theta} = (\theta_1 + \cdots + \theta_p)/p$. It is well known that T can be treated as a mixture of central χ^2_{p-1+2U} and $U \sim \text{Poisson}(\frac{\lambda}{2})$. Let $\beta_U = U + \frac{p-1}{2}$. Then

$$\begin{aligned} E\left[\frac{c_1^2}{T^{1+\alpha_1}}\right] &= c_1^2 \int_0^\infty t^{-(1+\alpha_1)} \sum_{u=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^u}{u!} \frac{t^{\beta_U - 1}\cdot e^{-\frac{t}{2}}}{\Gamma(\beta_U)\, 2^{\beta_U}} dt \\ &= c_1^2 \sum_{u=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^u}{u!} \int_0^\infty \frac{t^{\beta_U-(1+\alpha_1)-1}\cdot e^{-\frac{t}{2}}}{\Gamma(\beta_U - (1+\alpha_1))\cdot 2^{\beta_U-(1+\alpha_1)}} dt \\ &\quad \times \frac{\Gamma(\beta_U - (1+\alpha_1))\cdot 2^{\beta_U-(1+\alpha_1)}}{\Gamma(\beta_U)\, 2^{\beta_U}} \\ &= E_U\left[c_1^2\, 2^{-(1+\alpha_1)} \frac{\Gamma(\beta_U - (1+\alpha_1))}{\Gamma(\beta_U)}\right]. \end{aligned}$$

Similarly,

$$E\left[-\frac{2c_1\alpha_1}{T^{1+\frac{\alpha_1}{2}}}\right] = E_U\left[-2c_1\alpha_1 2^{-(1+\frac{\alpha_1}{2})} \frac{\Gamma(\beta_U - (1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U)}\right].$$

Hence,

$$RD(2,1) = E_U\left[c_1^2 2^{-(1+\alpha_1)}\frac{\Gamma(\beta_U-(1+\alpha_1))}{\Gamma(\beta_U)} - 2c_1\alpha_1 2^{-(1+\frac{\alpha_1}{2})}\frac{\Gamma(\beta_U-(1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U)}\right].$$

To make $RD(2,\ 1)\le 0$, it is sufficient to have
$c_1^2 2^{-(1+\alpha_1)}\frac{\Gamma(\beta_U-(1+\alpha_1))}{\Gamma(\beta_U)} \le 2c_1\alpha_1 2^{-(1+\frac{\alpha_1}{2})}\frac{\Gamma(\beta_U-(1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U)}$ for $U=$
$0,1,$
$2,\cdots$. Hence, the condition on c_1 is

$$0<c_1<\alpha_1 2^{(1+\frac{\alpha_1}{2})}\frac{\Gamma(\beta_U-(1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U-(1+\alpha_1))} \text{ for } U=0,1,2,\cdots.$$

Let

$$\varepsilon_1(p,\alpha_1)=\min_U\frac{\Gamma(\beta_U-(1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U-(1+\alpha_1))}. \tag{2.5}$$

Then a sufficient condition on c_1 is

$$0<c_1<\alpha_1 2^{1+\frac{\alpha_1}{2}}\varepsilon_1(p,\ \alpha_1) \tag{2.6}$$

provided $p-1>2(1+\alpha_1)$. In fact, the optimal value of c_1 which minimizes $c_1^2 2^{-(1+\alpha_1)}\frac{\Gamma(\beta_U-(1+\alpha_1))}{\Gamma(\beta_U)} - 2c_1\alpha_1 2^{-(1+\frac{\alpha_1}{2})}\frac{\Gamma(\beta_U-(1+\frac{\alpha_1}{2}))}{\Gamma(\beta_U)}$ for all U is

$$c_1^0=\alpha_1 2^{\frac{\alpha_1}{2}}\varepsilon_1(p,\ \alpha_1).$$

It can be proved as a part of a more general result that the minimum in (2.5) is attained at $U=0$, i.e., $\beta_U=\frac{(p-1)}{2}$(see Appendix). The condition that $p-1>2(1+\alpha_1)$ is necessary to ensure that all the expectation exist. The following result is obtained immediately from the above derivation.

Proposition 2.1 The estimator $\delta^2 = \delta^1 + \left(\frac{c_1^0}{\|\mathbf{X}-\overline{X}\mathbf{1}\|^{2+\alpha_1}}\right)(\mathbf{X}-\overline{X}\mathbf{1})$ with $\alpha_1 > 0$ dominates δ^1 (Lindley) uniformly under the quadratic loss (1.1) provided $p-1 > 2(1+\alpha_1)$ and c_1 satisfies the condition (2.6).

Remark 2.1 It is interesting to look at various choices of $\alpha_1 > 0$ in Proposition 2.1.

(a) If $0 < \alpha_1 < 0.5$, then δ^2 dominates δ^1 for $p \geq 4$.

(b) If $\alpha_1 = 1$, then $\varepsilon_1(p,\ \alpha_1) = \dfrac{\Gamma(\frac{p-4}{2})}{\Gamma(\frac{p-5}{2})}$.

Hence, $\delta^2 = \delta^1 + \dfrac{\sqrt{2}}{\|\mathbf{X}-\overline{X}\mathbf{1}\|^3}\varepsilon_1(p,\ \alpha_1)(\mathbf{X}-\overline{X}\mathbf{1})$ dominates δ^1 whenever $p > 5$.

(c) If $\alpha_1 = 2$, then $\varepsilon_1(p,\ \alpha_1) = \dfrac{p-7}{2}$. In this case, δ^1 is uniformly dominated by $\delta^2 = \overline{X}\mathbf{1} + \left(1 - \dfrac{p-3}{\|\mathbf{X}-\overline{X}\mathbf{1}\|^2} + \dfrac{2(p-7)}{\|\mathbf{X}-\overline{X}\mathbf{1}\|^4}\right)(\mathbf{X}-\overline{X}\mathbf{1})$ for $p > 7$.

2.2 The case of n = 2

To dominate $\delta^2 = \overline{X}\mathbf{1} + K_2(\mathbf{X}-\overline{X}\mathbf{1})$, where $K_2 = 1 - \dfrac{(p-3)}{\|\mathbf{X}-\overline{X}\mathbf{1}\|^2} + \dfrac{c_1}{\|\mathbf{X}-\overline{X}\mathbf{1}\|^{2+\alpha_1}}$, choose $r_2^* = \dfrac{c_2}{\|\mathbf{X}-\overline{X}\mathbf{1}\|^{2+\alpha_2}}$, where $\alpha_2 > \alpha_1 > 0$ and c_2 is suitable constant. Similar to the case $n = 1$, $RD(3,\ 2)$ can be derived form (2.4) as

$$RD(3,\ 2) = E\left[\frac{c_2^2}{T^{1+\alpha_2}} + \frac{2c_1c_2}{T^{1+\frac{(\alpha_1+\alpha_2)}{2}}} - \frac{2c_2\alpha_2}{T^{1+\frac{\alpha_2}{2}}}\right].$$

Following the earlier approach, a sufficient condition for $RD(3,2) \leq 0$ is

$0 < c_2 < \alpha_2 2^{(1+\frac{\alpha_2}{2})}\varepsilon_2(p,\alpha_1,\alpha_2)$ provided that $p-1 > 2(1+\alpha_2)$, where

$$\varepsilon_2(p,\ \alpha_1,\ \alpha_2) = \min_U \left\{ \frac{\Gamma(\beta_U - (1+\frac{\alpha_2}{2}))}{\Gamma(\beta_U - (1+\alpha_2))} \left(1 - c_1\alpha_2^{-1} 2^{\frac{-\alpha_1}{2}} \frac{\Gamma(\beta_U - (1+\frac{\alpha_1+\alpha_2}{2}))}{\Gamma(\beta_U - (1+\frac{\alpha_2}{2}))} \right) \right\}.$$

Again, the optimal value of c_2 is $c_2^0 = \alpha_2 2^{\frac{\alpha_2}{2}} \varepsilon_2(p, \alpha_1, \alpha_2)$.

2.3 General case

Consider the estimator δ^n in(2.1) with

$$K_n = 1 - \frac{p-3}{\|\mathbf{X} - \overline{X}\mathbf{1}\|^2} + \sum_{j=1}^{n-1} \frac{c_j}{\|\mathbf{X} - \overline{X}\mathbf{1}\|^{2+\alpha_j}}, \tag{2.7}$$

where $\alpha_{n-1} > \alpha_{n-2} > \cdots > \alpha_1 > 0$ and $0 < c_j < \alpha_j 2^{1+\frac{\alpha_j}{2}} \varepsilon_j(p, \alpha_1, \cdots, \alpha_j)$, $j = 1, 2, \cdots, n-1$. Take $r_n^* = \dfrac{c_n}{\|\mathbf{X} - \overline{X}\mathbf{1}\|^{2+\alpha_n}}$, where $\alpha_n > \alpha_{n-1}$ and c_n is a suitable constant. Similar to the special cases $n = 1, 2$, one can get

$$\begin{aligned} RD(n+1, n) \ &= \ E_U \left[\frac{1}{\Gamma(\beta_U)} \left\{ c_n^2 2^{-(1+\alpha_n)} \Gamma(\beta_U - (1+\alpha_n)) \right.\right. \\ &\quad + 2c_n \sum_{j=1}^{n-1} c_j 2^{-(1+\frac{(\alpha_j+\alpha_n)}{2})} \Gamma\left(\beta_U - \left(1 + \frac{(\alpha_j+\alpha_n)}{2}\right)\right) \\ &\quad \left.\left. - 2c_n\alpha_n 2^{-(1+\frac{\alpha_n}{2})} \Gamma\left(\beta_U - \left(1+\frac{\alpha_n}{2}\right)\right)\right\}\right]. \end{aligned} \tag{2.8}$$

The expectation in (2.8) exist provided $p-1 > 2(1+\alpha_n)$. Define $\varepsilon_n(p, \alpha_1, \cdots, \alpha_n)$ as

$$\begin{aligned} \varepsilon_n(p, \alpha_1, \cdots, \alpha_n) &= \min_U \left[\frac{\Gamma(\beta_U - (1+\frac{\alpha_n}{2}))}{\Gamma(\beta_U - (1+\alpha_n))} \right. \\ &\quad \left. \times \left\{ 1 - \sum_{j=1}^{n-1} c_j \alpha_n^{-1} 2^{\frac{-\alpha_j}{2}} \frac{\Gamma(\beta_U - (1+\frac{\alpha_j+\alpha_n}{2}))}{\Gamma(\beta_U - (1+\frac{\alpha_n}{2}))} \right\} \right]. \end{aligned} \tag{2.9}$$

Then a sufficient condition for δ^{n+1} dominating over δ^n is

$$0 < c_n < \alpha_n 2^{1+\frac{\alpha_n}{2}} \varepsilon_n(p,\ \alpha_1, \cdots, \alpha_n), \tag{2.10}$$

and the optimal value of c_n is $c_n^0 = \alpha_n 2^{\frac{\alpha_n}{2}} \varepsilon_n(p, \alpha_1, \cdots, \alpha_n)$. The minimum in (2.9) is attained $U = 0$ and this is proved in Appendix A.2. Therefore we obtained the main theorem of this chapter from above results.

Theorem 2.1 An estimator δ^n with K_n given by (2.7) is uniformly dominated by $\delta^{n+1} = \delta^n + \dfrac{c_n}{\| \mathbf{X} - \overline{X}\mathbf{1} \|^{2+\alpha_n}}(\mathbf{X} - \overline{X}\mathbf{1})$ provided $p - 1 > 2(1 + \alpha_n)$ and c_n satisfies the condition (2.10).

Remark 2.2 Note that the functions r_{ni}, $i = 1, 2, \cdots, p$, satisfy the regularity conditions of Stein's normal identity which enables us to derive (2.8).

Remark 2.3 The limiting value of $\{c_n^0\}$ is hard to find analytically due to the complicated structure of c_n^0 ((2.9)). Hence, the problem of finding the close form of the limiting estimator of the sequence $\{\delta^n\}$ still remains open.

3. The case of unknown covariance matrices

In this section, we extend the results derived in Section 2 to the cases where the covariance matrix $\mathbf{\Sigma}$ of $\mathbf{X}$ is either completely unknown or $\mathbf{\Sigma} = \sigma^2 \mathbf{I}_p$, $\sigma^2 > 0$ unknown.

3.1 The case of $\mathbf{\Sigma} = \sigma^2 \mathbf{I}_p$ ($\sigma^2 > 0$)

Let $\mathbf{X}$ and S be independent observations with $\mathbf{X} \sim N_p(\theta, \sigma^2 \mathbf{I}_p)$ and $S \sim \sigma^2 \chi_k^2$. Here we want to estimate θ under the loss function

$$L(\delta, \theta) = \frac{\| \delta - \theta \|^2}{\sigma^2}. \tag{3.1}$$

Again, the usual estimator is $\delta^0 = \mathbf{X}$ and the Lindley type estimator dominating δ^0 is

$$\delta^1 = \overline{X}\mathbf{1} + \left(1 - \frac{(p-3)S}{(k+2) \| \mathbf{X} - \overline{X}\mathbf{1} \|^2}\right)(\mathbf{X} - \overline{X}\mathbf{1}), \ p \geq 4.$$

We construct the sequence $\{\delta^n\}$ of improved estimators as follows. Let $\delta^n = \overline{X}\mathbf{1} + K_n(\mathbf{X} - \overline{X}\mathbf{1})$, where

$$K_n = 1 - \frac{(p-3)S}{(k+2) \| \mathbf{X} - \overline{X}\mathbf{1} \|^2} + \sum_{j=1}^{n-1} \frac{c_j \ S^{1+\frac{\alpha_j}{2}}}{\| \mathbf{X} - \overline{X}\mathbf{1} \|^{2+\alpha_j}},$$

$$\alpha_{n-1} > \alpha_{n-2} > \cdots > \alpha_1 > 0, \tag{3.2}$$

and

$$\delta^{n+1} = \delta^n + \mathbf{r}_n = \delta^n + r_n^*(\mathbf{X} - \overline{X}\mathbf{1}).$$

Then,

$$\begin{aligned} RD(n+1, n) &= R(\delta^{n+1}, \theta) - R(\delta^n, \theta) \\ &= \frac{1}{\sigma^2} \ E\left[\sum_{i=1}^{p} r_{ni}^2 + 2\sum_{i=1}^{p} r_{ni}(\delta_i^n - \theta_i)\right] \\ &= \frac{1}{\sigma^2} \ E\left[\sum_{i=1}^{p} \left\{r_{ni}^2 + 2(K_n - 1)r_{ni}(X_i - \overline{X}) + 2\sigma^2 \frac{\partial}{\partial X_i} \ r_{ni}\right\}\right]. \end{aligned}$$

The last expression follows from Stein's normal identity assuming that all the expectations exist. By taking $r_n^* = \dfrac{c_n \ S^{1+\frac{\alpha_n}{2}}}{\| \mathbf{X} - \overline{X}\mathbf{1} \|^{2+\alpha_n}}$, where c_n is a suitable constant and $\alpha_n > \alpha_{n-1} > \alpha_{n-2} > \cdots > \alpha_1 > 0$, one can get

$$\begin{aligned} RD(n+1,n) &= \frac{1}{\sigma^2} E\left[c_n^2 \frac{S^{2+\alpha_n}}{\| \mathbf{X} - \overline{X}\mathbf{1} \|^{2+2\alpha_n}} + 2\sigma^2 c_n (p-(3+\alpha_n)) \right. \\ & \quad \times \frac{S^{1+\frac{\alpha_n}{2}}}{\| \mathbf{X} - \overline{X}\mathbf{1} \|^{2+\alpha_n}} - 2c_n \frac{(p-3)S^{2+\frac{\alpha_n}{2}}}{(k+2) \| \mathbf{X} - \overline{X}\mathbf{1} \|^{2+\alpha_n}} \\ & \quad \left. +2c_n S^{1+\frac{\alpha_n}{2}} \sum_{j=1}^{n-1} c_j \frac{S^{1+\frac{\alpha_j}{2}}}{\| \mathbf{X} - \overline{X}\mathbf{1} \|^{2+\alpha_j+\alpha_n}} \right] \\ &= E_{T,S}\left[c_n^2 \frac{1}{T^{1+\alpha_n}} \left(\frac{S}{\sigma^2}\right)^{2+\alpha_n} \right. \\ & \quad +2c_n(p-(3+\alpha_n)) \frac{1}{T^{1+\frac{\alpha}{2}}} \left(\frac{S}{\sigma^2}\right)^{1+\frac{\alpha_n}{2}} \\ & \quad -2c_n \frac{p-3}{(k+2)T^{1+\frac{\alpha_n}{2}}} \left(\frac{S}{\sigma^2}\right)^{2+\frac{\alpha_n}{2}} \\ & \quad \left. +2c_n \sum_{j=1}^{n-1} c_j \frac{1}{T^{1+\frac{\alpha_j+\alpha_n}{2}}} \left(\frac{S}{\sigma^2}\right)^{2+\frac{\alpha_j+\alpha_n}{2}} \right], \end{aligned}$$

where $T = \dfrac{\| \mathbf{X} - \overline{X}\mathbf{1} \|^2}{\sigma^2} \sim$ noncentral $\chi^2_{p-1}(\lambda)$ with $\lambda = \dfrac{\| \theta - \overline{\theta}\mathbf{1} \|}{\sigma^2}$, $\dfrac{S}{\sigma^2} \sim \chi^2_k$, and they are independent. Similar to the previous section, a sufficient condition for $RD(n+1,n) \le 0$ is

$$0 < c_n < \alpha_n \frac{k+p-1}{k+2} \varepsilon_n(p, \alpha_1, \cdots, \alpha_n), \tag{3.3}$$

where

$$\begin{aligned} \varepsilon_n(p,\alpha_1,\cdots,\alpha_n) &= \frac{\Gamma\left(\frac{p-3-\alpha_n}{2}\right)\Gamma\left(\frac{k}{2}+1+\frac{\alpha_n}{2}\right)}{\Gamma\left(\frac{p-3}{2}-\alpha_n\right)\Gamma\left(\frac{k}{2}+1+\alpha_n\right)} \left\{ 1 - \frac{2(k+2)}{\alpha_n(k+p-1)} \right. \\ & \quad \left. \times \sum_{j=1}^{n-1} c_j \frac{\Gamma\left(\frac{k}{2}+2+\frac{\alpha_j+\alpha_n}{2}\right)\Gamma\left(\frac{p-3}{2}-\frac{\alpha_j+\alpha_n}{2}\right)}{\Gamma\left(\frac{k}{2}+1+\frac{\alpha_n}{2}\right)\Gamma\left(\frac{p-3-\alpha_n}{2}\right)} \right\}. \end{aligned}$$

The optimal value c_n is $c_n^0 = \alpha_n \frac{k+p-1}{2(k+2)} \varepsilon_n(p, \alpha_1, \cdots, \alpha_n)$.

Theorem 3.1 An estimator δ^n of the form (3.2) is uniformly dominated by $\delta^{n+1} = \delta^n + \left(\frac{c_n S^{1+\frac{\alpha_n}{2}}}{\| \mathbf{X} - \overline{X}\mathbf{1} \|^{2+\alpha_n}} \right) (\mathbf{X} - \overline{X}\mathbf{1})$ under the loss (3.1) provided $p - 1 > 2(1 + \alpha_n)$ and c_n satisfies the condition (3.3).

Theorem 3.1 gives us the sequence $\{\delta^n\}$ of improved estimators dominating δ^1.

3.2 The Case of Completely Unknown Covariance Matrix Σ

Let $\mathbf{X}$ and S be independent where $\mathbf{X} \sim N_p(\theta, \boldsymbol{\Sigma})$ and $S_{\text{pxp}} \sim \text{Wishart}(\boldsymbol{\Sigma}|\text{k})$. Here we estimate θ under the loss function

$$L(\delta, \theta) = (\delta - \theta)' \, \boldsymbol{\Sigma}^{-1}(\delta - \theta). \tag{3.4}$$

The Lindley type estimator dominating the usual estimator $\delta^{\mathbf{0}} = \mathbf{X}$ is

$$\delta^1 = \overline{X}\mathbf{1} + \left(1 - \frac{p-3}{(k-p+4)(\mathbf{X} - \overline{X}\mathbf{1})' \, S^{-1} \, (\mathbf{X} - \overline{X}\mathbf{1})} \right) (\mathbf{X} - \overline{X}\mathbf{1}).$$

Again, the difference in risks of δ^n and $\delta^{n+1} = \delta^n + \mathbf{r_n}$ is

$$\begin{aligned} RD(n+1, n) &= E[\mathbf{r}_n' \boldsymbol{\Sigma}^{-1} \mathbf{r}_n + 2\mathbf{r}_n' \boldsymbol{\Sigma}^{-1}(\delta^n - \theta)] \\ &= E[\mathbf{r}_n' \boldsymbol{\Sigma}^{-1} \mathbf{r}_n + 2\mathbf{r}_n' \boldsymbol{\Sigma}^{-1}(\delta^n - \mathbf{X}) + 2\mathbf{r}_n' \boldsymbol{\Sigma}^{-1}(\mathbf{X} - \theta)]. \end{aligned} \tag{3.5}$$

Let $\delta^{\mathbf{n}} = \overline{X}\mathbf{1} + K_n(\mathbf{X} - \overline{X}\mathbf{1})$, where

$$\begin{aligned} K_n &= 1 - \frac{p-3}{(k-p+4)(\mathbf{X} - \overline{X}\mathbf{1})' \, S^{-1} \, (\mathbf{X} - \overline{X}\mathbf{1})} \\ &\quad + \sum_{j=1}^{n-1} \frac{c_j}{\{(\mathbf{X} - \overline{X}\mathbf{1})' \, S^{-1} \, (\mathbf{X} - \overline{X}\mathbf{1})\}^{1+\alpha_j}} \end{aligned}$$

and

$$\begin{aligned} \mathbf{r}_n &= r_n^* \, (\mathbf{X} - \overline{X}\mathbf{1}) = \frac{c_n}{\{(\mathbf{X} - \overline{X}\mathbf{1})' \, S^{-1} \, (\mathbf{X} - \overline{X}\mathbf{1})\}^{1+\alpha_n}} \, (\mathbf{X} - \overline{X}\mathbf{1}), \\ & \alpha_{n-1} > \alpha_{n-2} > \cdots > \alpha_1 > 0. \end{aligned} \tag{3.6}$$

Also, define $\mathbf{Y} = \mathbf{\Sigma}^{-\frac{1}{2}}\ \mathbf{X}$, $\theta_* = \mathbf{\Sigma}^{-\frac{1}{2}}\ \theta$ and $S_* = \mathbf{\Sigma}^{-\frac{1}{2}} S \mathbf{\Sigma}^{-\frac{1}{2}}$. Then,

$$\begin{aligned}
&E[\mathbf{r}_n' \mathbf{\Sigma}^{-1}(\mathbf{X}-\theta)] \\
&= E\left[\frac{c_n}{\{(\mathbf{X}-\overline{X}\mathbf{1})' S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{1+\alpha_n}}(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\theta)\right] \\
&= E\left[\frac{c_n}{\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{1+\alpha_n}}(\mathbf{Y}-\overline{Y}\mathbf{1})'(\mathbf{Y}-\theta_*)\right] \\
&= \sum_{i=1}^{p} E\left[\frac{c_n(Y_i-\overline{Y})}{\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{1+\alpha_n}}(Y_i-\theta_{*i})\right],
\end{aligned} \tag{3.7}$$

where $\overline{Y} = \frac{1}{p}\sum_{i=1}^{p} Y_i$, $\overline{\theta} = \frac{1}{p}\sum_{i=1}^{p}\theta_i$ and $Y_i(\sim N(\theta_{*i}, 1))$ is the i^{th} element of $\mathbf{Y}$. Applying Stein's normal identity in (3.7), we get

$$\begin{aligned}
&E[\mathbf{r}_n' \mathbf{\Sigma}^{-1}(\mathbf{X}-\theta)] \\
&= E\left[\sum_{i=1}^{p}\frac{\partial}{\partial Y_i}\left(\frac{c_n(Y_i-\overline{Y})}{\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{1+\alpha_n}}\right)\right] \\
&= E\left[\sum_{i=1}^{p} c_n \left\{\frac{(1-\frac{1}{p})\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{1+\alpha_n}}{\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{2+2\alpha_n}}\right.\right. \\
&\quad -(Y_i-\overline{Y})(1+\alpha_n)\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{\alpha_n} \\
&\quad \left.\left.\times\frac{\frac{\partial}{\partial Y_i}(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})}{\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{2+2\alpha_n}}\right\}\right] \\
&= E\left[c_n\frac{(p-1)\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{1+\alpha_n}}{\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{2+2\alpha_n}}\right. \\
&\quad \left. - c_n\frac{2(1+\alpha_n)\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{1+\alpha_n}}{\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{2+2\alpha_n}}\right] \\
&= E\left[c_n\frac{p-1-2(1+\alpha_n)}{\{(\mathbf{Y}-\overline{Y}\mathbf{1})' S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{1+\alpha_n}}\right].
\end{aligned} \tag{3.8}$$

From (3.5) and (3.8),
$RD(n+1,n)$

$$
\begin{aligned}
&= E\Bigg[\mathbf{r}_n'\boldsymbol{\Sigma}^{-1}\mathbf{r}_n + 2\mathbf{r}_n'\boldsymbol{\Sigma}^{-1}(\delta^n - \mathbf{X}) \\
&\quad +2c_n\frac{(p-1-2(1+\alpha_n))}{\{(\mathbf{Y}-\overline{Y}\mathbf{1})'S_*^{-1}(\mathbf{Y}-\overline{Y}\mathbf{1})\}^{1+\alpha_n}}\Bigg] \\
&= E\Bigg[\frac{2c_n}{\{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{1+\alpha_n}}\Bigg\{-\frac{(p-3)/(k-p+4)}{\{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}} \\
&\quad +\sum_{j=1}^{n-1}\frac{c_j}{\{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{1+\alpha_j}}\Bigg\}(\mathbf{X}-\overline{X}\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1}) \\
&\quad +\frac{c_n^2}{\{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{2(1+\alpha_n)}}(\mathbf{X}-\overline{X}\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1}) \\
&\quad +2c_n\frac{(p-1)-2(1+\alpha_n)}{\{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{1+\alpha_n}}\Bigg] \\
&= E\Bigg[-\frac{2c_n(p-3)/(k-p+4)}{\{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{2+\alpha_n}}(\mathbf{X}-\overline{X}\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1}) \\
&\quad +2c_n\left(\sum_{j=1}^{n-1}\frac{c_j}{\{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{2+\alpha_j+\alpha_n}}\right)(\mathbf{X}-\overline{X}\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1}) \\
&\quad +c_n^2\frac{(\mathbf{X}-\overline{X}\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}{\{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{2+2\alpha_n}} \\
&\quad +2\,c_n\frac{(p-1-2(1+\alpha_n))}{\{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{1+\alpha_n}}\Bigg].
\end{aligned}
$$

Note that given $\mathbf{X}$, the conditional distribution of $\dfrac{(\mathbf{X}-\overline{X}\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})} = \dfrac{\mathbf{X}'(I-\frac{1}{p}\mathbf{1}\mathbf{1}')\boldsymbol{\Sigma}^{-1}(I-\frac{1}{p}\mathbf{1}\mathbf{1}')\mathbf{X}}{\mathbf{X}'(I-\frac{1}{p}\mathbf{1}\mathbf{1}')S^{-1}(I-\frac{1}{p}\mathbf{1}\mathbf{1}')\mathbf{X}}$ is χ^2_{k-p+2} which is free from $\mathbf{X}$ (Graybill(1976) and Rao(1983)). So unconditionally $\dfrac{(\mathbf{X}-\overline{X}\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}$

is χ^2_{k-p+2} and this is independent of $(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})$. Therefore,

$$\begin{aligned} RD(n+1,n) &= E\Bigg[-2c_n\frac{p-3}{(k-p+4)}\left(\frac{(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}\right)^{2+\alpha_n} \\ &\cdot\{(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{-(1+\alpha_n)} \\ &+2c_n\sum_{j=1}^{n-1}c_j\left\{\frac{(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}\right\}^{2+\alpha_j+\alpha_n} \\ &\cdot\{(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{-(1+\alpha_j+\alpha_n)} \\ &+c_n^2\left\{\frac{(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}\right\}^{2+2\alpha_n} \\ &\cdot\{(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{-(1+2\alpha_n)} \\ &+2c_n(p-1-2(1+\alpha_n))\left\{\frac{(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}{(\mathbf{X}-\overline{X}\mathbf{1})'S^{-1}(\mathbf{X}-\overline{X}\mathbf{1})}\right\}^{1+\alpha_n} \\ &\cdot\{(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{-(1+\alpha_n)}\Bigg]. \end{aligned}$$

Let

$$\begin{aligned} A_1 &= 2\frac{p-3}{k-p+4}2^{2+\alpha_n}\frac{\Gamma((k-p+6+2\alpha_n)/2)}{\Gamma((k-p+2)/2)}, \\ A_2 &= 2^{2+2\alpha_n}\frac{\Gamma((k-p+6+4\alpha_n)/2)}{\Gamma((k-p+2)/2)}, \\ A_3 &= 2(p-3-2\alpha_n)2^{1+\alpha_n}\frac{\Gamma((k-p+4+2\alpha_n)/2)}{\Gamma((k-p+2)/2)} \end{aligned} \tag{3.9}$$

and

$$B_j = 2c_j 2^{2+\alpha_j+\alpha_n}\frac{\Gamma((k-p+6+2\alpha_j+2\alpha_n)/2)}{\Gamma((k-p+2)/2)}.$$

Then

$$\begin{aligned} RD(n+1,n) &= E[c_n^2A_2\{(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{-(1+2\alpha_n)} \\ &-c_n(A_1-A_3)\{(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{-(1+\alpha_n)} \\ &+c_n\sum_{j=1}^{n-1}B_j\{(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1})\}^{-(1+\alpha_j+\alpha_n)}]. \end{aligned}$$

Since $(\mathbf{X}-\overline{X}\mathbf{1})'\mathbf{\Sigma}^{-1}(\mathbf{X}-\overline{X}\mathbf{1}) \sim$ noncentral $\chi^2_{p-1}(\lambda)$ with $\lambda = (\theta - \overline{\theta}\mathbf{1})'\mathbf{\Sigma}^{-1}(\theta-\overline{\theta}\mathbf{1})$, we choose c_n by using earlier technique as

$$0 < c_n < \min_U \left\{ 2^{\alpha_n}\left(\frac{A_1 - A_3}{A_2}\right)\frac{\Gamma(\beta_U - (1+\alpha_n))}{\Gamma(\beta_U - (1+2\alpha_n))} - \sum_{j=1}^{n-1} 2^{\alpha_n - \alpha_j}\frac{B_j\Gamma(\beta_U - (1+\alpha_j+\alpha_n))}{A_2\Gamma(\beta_U-(1+2\alpha_n))} \right\},$$

where A_1, A_2, A_3 and B_j's are given in (3.9). The following result summarizes the above derivation.

Theorem 3.2 Assume $p-1 > 2(1+2\alpha_n)$. The estimator $\delta^{n+1} = \delta^n + \mathbf{r}_n((3.6))$ dominates δ^n uniformly under the loss (3.4). As a result, this gives a sequence $\{\delta^n\}$ of improved estimators dominating δ^1.

Appendix

Simplification of $\varepsilon_n(p, \alpha_1, \cdots, \alpha_n)$ in (2.9).

Let $l > \alpha \geq 1$. Define $R(l,\alpha) = \Gamma(l-\alpha)/\Gamma(l)$, where $\Gamma(\cdot)$ is the usual gamma function. We first study the function $R(l,\alpha)$ where $l \to \infty$ over the set $L_\beta = \{\beta, \beta+1, \beta+2, \cdots\}$ for some real $\beta > \alpha$.

Lemma 1 a) $R(l,\alpha)$ is decreasing as l increases over L_β, and
b) $R(l,\alpha) \to 0$ as $l \to \infty$.

Proof. a) Since $R(l+1,\alpha)/R(l,\alpha) = \Gamma(l-\alpha+1)\Gamma(l)/\Gamma(l+1)\Gamma(l-\alpha) = 1 - \frac{\alpha}{l} < 1$, $R(l+1,\alpha) < R(l,\alpha)$ for any $l \in L_\beta$.
b) We show that $\lim_{l\to\infty} R(l,\alpha) = 0$, where $l \to \infty$ over L_β. It is known that

$$\frac{\Gamma(l-\alpha)\Gamma(\alpha)}{\Gamma(l)} = \text{Beta}(l-\alpha,\alpha) = \int_0^1 x^{l-\alpha-1}(1-x)^{\alpha-1}dx.$$

Let $g_l(x) = x^{l-\alpha-1}(1-x)^{\alpha-1}$. Since $R(l,\alpha) = (\Gamma(\alpha))^{-1}\int_0^1 g_l(x)dx$, it is enough to show that $\lim_{l\to\infty}\int_0^1 g_l(x)dx = 0$. Observe that for any

$0 < \varepsilon < 1$,

$$\int_0^1 g_l(x)dx = \int_0^\varepsilon g_l(x)dx + \int_\varepsilon^{1-\varepsilon} g_l(x)dx + \int_{1-\varepsilon}^1 g_l(x)dx.$$

Let $A_1 = \int_0^\varepsilon g_l(x)dx$, $A_2 = \int_\varepsilon^{1-\varepsilon} g_l(x)dx$ and $A_3 = \int_{1-\varepsilon}^1 g_l(x)dx$.
Then $A_1 = \int_0^\varepsilon x^{l-\alpha-1}(1-x)^{\alpha-1}dx < \int_0^\varepsilon x^{l-\alpha-1} \cdot 1dx = \dfrac{1}{l-\alpha}\varepsilon^{l-\alpha} < \varepsilon$
$A_3 = \int_{1-\varepsilon}^1 x^{l-\alpha-1}(1-x)^{\alpha-1}dx < \int_{1-\varepsilon}^1 1 \cdot (1-x)^{\alpha-1}dx = \dfrac{1}{\alpha}\varepsilon^\alpha < \varepsilon.$
If $\varepsilon < x < 1-\varepsilon$, $\varepsilon < 1-x < 1-\varepsilon$. Hence

$$\begin{aligned} A_2 &\leq \int_\varepsilon^{1-\varepsilon} (1-\varepsilon)^{l-\alpha-1}(1-\varepsilon)^{\alpha-1}dx. \\ &= \int_\varepsilon^{1-\varepsilon} (1-\varepsilon)^{l-2}dx < (1-\varepsilon)^{l-2}(1-2\varepsilon) < (1-\varepsilon)^{l-1} \end{aligned}$$

Since $0 < 1-\varepsilon < 1$, there exists $l_0 \in L_\beta(l_0 \geq \beta)$ such that $(1-\varepsilon)^{l-1} < \varepsilon$ for $l \geq l_0$. Therefore, $\int_0^1 g_l(x)dx < 3\varepsilon$ for $l \geq l_0$.
Since $\varepsilon > 0$ is arbitrary, the result follows immediately.
Using the above lemma, it follows that

$$\frac{\Gamma(\beta_U - (1+\alpha_n/2))}{\Gamma(\beta_U - (1+\alpha_n))} \uparrow \infty$$

and

$$\frac{\Gamma(\beta_U - (1+(\alpha_j+\alpha_n)/2)}{\Gamma(\beta_U - (1+\alpha_n/2))} \downarrow 0 \text{ as } \beta_U \to \infty$$

over the set $\{\frac{p-1}{2}, \frac{p-1}{2}+1, \frac{p-1}{2}+2, \cdots\}$. In expression (2.9), the minimum is attained at $U = 0$, i.e., $\beta_U = \frac{p-1}{2}$. Therefore we can simplify $\varepsilon_n(p, \alpha_1, \cdots, \alpha_n)$ in (2.9).

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