Honam Mathematical J. **26**(2004), pp. 61–75

HOMOMORPHISMS BETWEEN POISSON BANACH ALGEBRAS AND POISSON BRACKETS

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Abstract. It is shown that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ of a unital Poisson Banach algebra $\mathcal{A}$ to a unital Poisson Banach algebra $\mathcal{B}$ is a Poisson algebra homomorphism when $h(xy) = h(x)h(y)$ holds for all $x, y \in \mathcal{A}$, and that every almost linear almost multiplicative mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson algebra homomorphism when $h(qx) = qh(x)$ for all $x \in \mathcal{A}$. Here the number q is in the functional equation given in the almost linear almost multiplicative mapping.

We prove that every almost Poisson bracket $B : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ on a Banach algebra $\mathcal{A}$ is a Poisson bracket when $B(qx, z) = B(x, qz) = qB(x, z)$ for all $x, z \in \mathcal{A}$. Here the number q is in the functional equation given in the almost Poisson bracket.

1. Introduction

Let R be a complex algebra with $\mathbb{C}$-bilinear map $\{\cdot,\cdot\} : R \times R \to R$, which is called a *Poisson bracket*, such that $(R, \{\cdot,\cdot\})$ is a complex Lie algebra and

$$\{xy, z\} = x\{y, z\} + \{x, z\}y$$

Received October 15, 2003. Revised February 5, 2004.

2000 Mathematics Subject Classification: Primary 39B52, 47B48, 17B63.

Key words and phrases: Poisson algebra homomorphism, Poisson bracket, Banach algebra, stability, Trif functional equation.

This work was supported by grant No. R05-2003-000-10006-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

holds for all $x, y, z \in R$. Then the complex algebra R is called a *Poisson algebra*. Poisson algebras have played an important role in many mathematical areas and have been studied to find sympletic leaves of the corresponding Poisson varieties. It is also important to find or construct a Poisson bracket in the theory of Poisson algebras (see [2, 4, 5, 10]).

Let X be a normed space with norm $||\cdot||$ and Y a Banach space with norm $\|\cdot\|$. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Rassias [8] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p}||x||^p$$

for all $x \in X$. Găvruta [1] generalized the Rassias' result: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in G$. C. Park [6] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra.

Recently, Trif [9] proved the following: Let $q := \frac{l(d-1)}{d-l}$, $r := -\frac{l}{d-l}$. Denote by $\varphi : X^d \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x_1, \cdots, x_d) = \sum_{j=0}^{\infty} q^{-j}\varphi(q^j x_1, \cdots, q^j x_d) < \infty$$

for all $x_1, \cdots, x_d \in X$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\begin{aligned}&\|d\ {}_{d-2}C_{l-2} f(\frac{x_1 + \cdots + x_d}{d}) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} f(x_j) \\ &\quad - l \sum_{1 \le j_1 < \cdots < j_l \le d} f(\frac{x_{j_1} + \cdots + x_{j_l}}{l})\| \\ &\le \varphi(x_1, \cdots, x_d)\end{aligned}$$

for all $x_1, \cdots, x_d \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{1}{l \cdot {}_{d-1}C_{l-1}} \widetilde{\varphi}(qx, \underbrace{rx, \cdots, rx}_{d-1 \text{ times}})$$

for all $x \in X$. And C. Park [7] applied the Trif's result to the Trif functional equation in Banach modules over a C^*-algebra.

Throughout this paper, let $q = \frac{l(d-1)}{d-l}$ and $r = -\frac{l}{d-l}$ for positive integers l, d with $2 \le l \le d-1$.

Using the stability methods of linear functional equations, we prove that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson algebra homomorphism when $h(xy) = h(x)h(y)$ holds for all $x, y \in \mathcal{A}$, and that every almost linear almost multiplicative mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson algebra homomorphism when $h(qx) = qh(x)$ for all $x \in \mathcal{A}$. We moreover prove that every almost Poisson bracket $B : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ on a Banach algebra $\mathcal{A}$ is a Poisson bracket when $B(qx, z) = B(x, qz) = qB(x, z)$ for all $x, z \in \mathcal{A}$.

2. Homomorphisms between Poisson Banach algebras

Throughout this section, let $\mathcal{A}$ be a unital Poisson Banach algebra with norm $||\cdot||$ and unit e, and $\mathcal{B}$ a unital Poisson Banach algebra with norm $\|\cdot\|$.

We are going to investigate Poisson algebra homomorphisms between Poisson Banach algebras associated with the Trif functional equation.

Theorem 2.1. Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(xy) = h(x)h(y)$ for all $x, y \in \mathcal{A}$ for which there exists a function $\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ such that

$$\text{(2.i)} \quad \widetilde{\varphi}(x_1, \cdots, x_d, z, w) := \sum_{j=0}^{\infty} q^{-j}\varphi(q^j x_1, \cdots, q^j x_d, q^j z, q^j w) < \infty,$$

$$\text{(2.ii)} \quad \begin{aligned} &\|d\ {}_{d-2}C_{l-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{\{z, w\}}{d\ {}_{d-2}C_{l-2}}) + {}_{d-2}C_{l-1}\sum_{j=1}^{d}\mu h(x_j) \\ &-l \sum_{1 \le j_1 < \cdots < j_l \le d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) - \{h(z), h(w)\}\| \\ &\le \varphi(x_1, \cdots, x_d, z, w) \end{aligned}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x_1, \cdots, x_d, z, w \in \mathcal{A}$. Assume that (2.iii) $\lim_{n\to\infty} \frac{h(q^n e)}{q^n}$ is invertible. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson algebra homomorphism.

Proof. Put $z = w = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (2.ii). It follows from Trif Theorem [9] Theorem 3.1 that there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\text{(2.iv)} \quad \|h(x) - H(x)\| \le \frac{1}{l \cdot {}_{d-1}C_{l-1}}\widetilde{\varphi}(qx, \underbrace{rx, \cdots, rx}_{d-1 \text{ times}}, 0, 0)$$

for all $x \in \mathcal{A}$. The additive mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$\text{(2.1)} \quad H(x) = \lim_{n\to\infty} \frac{1}{q^n} h(q^n x)$$

for all $x \in \mathcal{A}$.

Put $x_1 = \cdots = x_d = x$ and $z = w = 0$ in (2.ii). Consider a Banach algebra as a vector space over $\mathbb{C}$. For each $\mu \in \mathbb{T}^1$,

$$\|d\ {}_{d-2}C_{l-2}(h(\mu x) - \mu h(x))\| \le \varphi(\underbrace{x, \cdots, x}_{d \text{ times}}, 0, 0)$$

for all $x \in \mathcal{A}$. So

$$q^{-n}\|d\ {}_{d-2}C_{l-2}(h(\mu q^n x) - \mu h(q^n x))\| \le q^{-n}\varphi(\underbrace{q^n x, \cdots, q^n x}_{d \text{ times}}, 0, 0)$$

for all $x \in \mathcal{A}$. By (2.i),

$$q^{-n}\|d\ {}_{d-2}C_{l-2}(h(\mu q^n x) - \mu h(q^n x))\| \to 0$$

as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Thus

$$q^{-n}\|h(\mu q^n x) - \mu h(q^n x)\| \to 0$$

as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Hence

$$H(\mu x) = \lim_{n\to\infty} \frac{h(q^n \mu x)}{q^n} = \lim_{n\to\infty} \frac{\mu h(q^n x)}{q^n} = \mu H(x) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [3] Theorem 1, there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. Since H is additive, $H(x) = H(3 \cdot \frac{1}{3}x) = 3H(\frac{1}{3}x)$ for all $x \in \mathcal{A}$. So $H(\frac{1}{3}x) = \frac{1}{3}H(x)$ for all $x \in \mathcal{A}$. Thus by (2.2)

$$\begin{aligned} H(\lambda x) &= H(\frac{M}{3} \cdot 3\frac{\lambda}{M}x) = M \cdot H(\frac{1}{3} \cdot 3\frac{\lambda}{M}x) = \frac{M}{3}H(3\frac{\lambda}{M}x) \\ &= \frac{M}{3}H(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)H(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $x \in \mathcal{A}$. Hence

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}(\zeta, \eta \neq 0)$ and all $x, y \in \mathcal{A}$. And $H(0x) = 0 = 0H(x)$ for all $x \in \mathcal{A}$. So the unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

Since $h(xy) = h(x)h(y)$ for all $x, y \in \mathcal{A}$,

$$H(xy) = \lim_{n\to\infty} \frac{1}{q^n}h(q^n xy) = \lim_{n\to\infty} \frac{1}{q^n}h(q^n x)h(y) = H(x)h(y) \tag{2.3}$$

for all $x, y \in \mathcal{A}$. By the additivity of H and (2.3),

$$q^n H(xy) = H(q^n xy) = H(x(q^n y)) = H(x)h(q^n y)$$

for all $x, y \in \mathcal{A}$. Hence

$$H(xy) = \frac{1}{q^n} H(x)h(q^n y) = H(x)\frac{1}{q^n} h(q^n y) \tag{2.4}$$

for all $x, y \in \mathcal{A}$. Taking the limit in (2.4) as $n \to \infty$, we obtain

$$H(xy) = H(x)H(y) \tag{2.5}$$

for all $x, y \in \mathcal{A}$. So $H : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism.

By (2.3) and (2.5),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n\to\infty} \frac{h(q^n e)}{q^n} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

It follows from (2.1) that

$$H(x) = \lim_{n\to\infty} \frac{h(q^{2n} x)}{q^{2n}} \tag{2.6}$$

for all $x \in \mathcal{A}$. Let $x_1 = \cdots x_d = 0$ in (2.ii). Then we get

$$\begin{aligned}\|d\ {}_{d-2}C_{l-2} h(\frac{\{z,w\}}{d\ {}_{d-2}C_{l-2}}) - \{h(z), h(w)\}\| &= \|h(\{z,w\}) - \{h(z), h(w)\}\| \\ &\le \varphi(\underbrace{0, \cdots, 0}_{d \text{ times}}, z, w)\end{aligned}$$

for all $z, w \in \mathcal{A}$, since $h : \mathcal{A} \to \mathcal{B}$ is additive. So

$$\begin{aligned}\frac{1}{q^{2n}} \|h(\{q^n z, q^n w\}) - \{h(q^n z), h(q^n w)\}\| &\le \frac{1}{q^{2n}} \varphi(\underbrace{0, \cdots, 0}_{d \text{ times}}, q^n z, q^n w) \\ &\le \frac{1}{q^n} \varphi(\underbrace{0, \cdots, 0}_{d \text{ times}}, q^n z, q^n w) \qquad (2.7)\end{aligned}$$

for all $z, w \in \mathcal{A}$. By (2.i), (2.6), and (2.7),

$$\begin{aligned} H(\{z,w\}) &= \lim_{n\to\infty} \frac{h(q^{2n}\{z,w\})}{q^{2n}} = \lim_{n\to\infty} \frac{h(\{q^n z, q^n w\})}{q^{2n}} \\ &= \lim_{n\to\infty} \frac{1}{q^{2n}}\{h(q^n z), h(q^n w)\} = \lim_{n\to\infty}\{\frac{h(q^n z)}{q^n}, \frac{h(q^n w)}{q^n}\} \\ &= \{H(z), H(w)\} \end{aligned}$$

for all $z, w \in \mathcal{A}$.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson algebra homomorphism, as desired. □

Corollary 2.2. Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(xy) = h(x)h(y)$ for all $x, y \in \mathcal{A}$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\begin{aligned} \|d\ {}_{d-2}C_{l-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{\{z,w\}}{d\ {}_{d-2}C_{l-2}}) + {}_{d-2}C_{l-1}\sum_{j=1}^{d}\mu h(x_j) \\ -l\sum_{1\leq j_1<\cdots<j_l\leq d}\mu h(\frac{x_{j_1}+\cdots+x_{j_l}}{l}) - \{h(z), h(w)\}\| \\ \leq \theta(\sum_{j=1}^{d}||x_j||^p + ||z||^p + ||w||^p) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_d, z, w \in \mathcal{A}$. Assume that $\lim_{n\to\infty}\frac{h(q^n e)}{q^n}$ is invertible. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson algebra homomorphism.

Proof. Define $\varphi(x_1, \cdots, x_d, z, w) = \theta(\sum_{j=1}^{d}||x_j||^p + ||z||^p + ||w||^p)$, and apply Theorem 2.1. □

Theorem 2.3. Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(xy) = h(x)h(y)$ for all $x, y \in \mathcal{A}$ for which there exists a function

$\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ satisfying (2.i) and (2.iii) such that

$$\begin{aligned} &\|d\ {}_{d-2}C_{l-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{\{z,w\}}{d\ {}_{d-2}C_{l-2}}) + {}_{d-2}C_{l-1}\sum_{j=1}^{d}\mu h(x_j) \\ &-l \sum_{1\le j_1<\cdots<j_l\le d} \mu h(\frac{x_{j_1}+\cdots+x_{j_l}}{l}) - \{h(z), h(w)\}\| \\ &\le \varphi(x_1, \cdots, x_d, z, w) \end{aligned} \tag{2.v}$$

for $\mu = 1, i$, and all $x_1, \cdots, x_d, z, w \in \mathcal{A}$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson algebra homomorphism.

Proof. Put $z = w = 0$ and $\mu = 1$ in (2.v). By the same reasoning as in the proof of Theorem 2.1, there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (2.iv). The additive mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{q^n} h(q^n x)$$

for all $x \in \mathcal{A}$. By the same reasoning as in the proof of [8] Theorem, the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $x_1 = \cdots = x_d = x$, $z = w = 0$ and $\mu = i$ in (2.v). We get

$$\|d\ {}_{d-2}C_{l-2}(h(ix) - ih(x))\| \le \varphi(\underbrace{x, \cdots, x}_{d \text{ times}}, 0, 0)$$

for all $x \in \mathcal{A}$. By the same method as in the proof of Theorem 2.1, one can obtain that

$$H(ix) = \lim_{n\to\infty} \frac{h(q^n ix)}{q^n} = \lim_{n\to\infty} \frac{ih(q^n x)}{q^n} = iH(x)$$

for all $x \in \mathcal{A}$. For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) \\ &= sH(x) + itH(x) = (s + it)H(x) = \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 2.1. □

Theorem 2.4. Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(qx) = qh(x)$ for all $x \in \mathcal{A}$ for which there exists a function $\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ satisfying (2.i), (2.ii), and (2.iii) such that

$$\|h(xy) - h(x)h(y)\| \leq \varphi(x, y, \underbrace{0, \cdots, 0}_{d \text{ times}}) \tag{2.vi}$$

for all $x, y \in \mathcal{A}$. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (2.iv).

By (2.vi) and the assumption that $h(qx) = qh(x)$ for all $x \in \mathcal{A}$,

$$\begin{aligned} \|h(xy) - h(x)h(y)\| &= \frac{1}{q^{2n}} \|h(q^n x \cdot q^n y) - h(q^n x)h(q^n y)\| \\ &\leq \frac{1}{q^{2n}} \varphi(q^n x, q^n y, \underbrace{0, \cdots, 0}_{d \text{ times}}) \\ &\leq \frac{1}{q^n} \varphi(q^n x, q^n y, \underbrace{0, \cdots, 0}_{d \text{ times}}), \end{aligned}$$

which tends to zero as $n \to \infty$ by (2.i). So

$$h(xy) = h(x)h(y)$$

for all $x, y \in \mathcal{A}$. But by (2.1),

$$H(x) = \lim_{n \to \infty} \frac{1}{q^n} h(q^n x) = h(x)$$

for all $x \in \mathcal{A}$.

The rest of the proof is the same as in the proof of Theorem 2.1. □

3. Poisson brackets on Banach algebras

Throughout this section, let $\mathcal{A}$ be a Banach algebra with norm $\|\cdot\|$. We are going to investigate Poisson brackets on Banach algebras associated with the Trif functional equation.

Theorem 3.1. Let $B : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ be a mapping satisfying $B(qx, z) = B(x, qz) = qB(x, z)$ for all $x, z \in \mathcal{A}$ for which there exists a function $\varphi : \mathcal{A}^{d+1} \to [0, \infty)$ such that

$$\sum_{j=0}^{\infty} \frac{1}{q^{2j}} \varphi(q^j x_1, \cdots, q^j x_d, q^j z) < \infty, \tag{3.i}$$

$$\begin{aligned} &\|d\ {}_{d-2}C_{l-2} B(\frac{\mu x_1 + \cdots + \mu x_d}{d}, z) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu B(x_j, z) \\ &-l \sum_{1\le j_1<\cdots<j_l\le d} \mu B(\frac{x_{j_1} + \cdots + x_{j_l}}{l}, z)\| \le \varphi(x_1, \cdots, x_d, z), \end{aligned} \tag{3.ii}$$

$$\|B(x, z) + B(z, x)\| \le \varphi(x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}}, z), \tag{3.iii}$$

$$\|B(xy, z) - xB(y, z) - B(x, z)y\| \le \varphi(x, y, \underbrace{0, \cdots, 0}_{d-2 \text{ times}}, z), \tag{3.iv}$$

(3.v)
$$\|B(x, B(y, z)) + B(y, B(z, x)) + B(z, B(x, y))\| \le \varphi(x, y, \underbrace{0, \cdots, 0}_{d-2 \text{ times}}, z)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_d, x, y, z \in \mathcal{A}$. Then the mapping $B : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a Poisson bracket.

Proof. Put $\mu = 1 \in \mathbb{T}^1$ in (3.ii). By (3.ii) and the assumption that $B(qx, z) = B(x, qz) = qB(x, z)$ for all $x, z \in \mathcal{A}$,

$$\begin{aligned} &\|d\ {}_{d-2}C_{l-2} B(\frac{x_1 + \cdots + x_d}{d}, z) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} B(x_j, z) \\ &\quad - l \sum_{1\le j_1<\cdots<j_l\le d} B(\frac{x_{j_1} + \cdots + x_{j_l}}{l}, z)\| \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q^{2n}} \| d\ {}_{d-2}C_{l-2} B(\frac{q^n x_1 + \cdots + q^n x_d}{d}, q^n z) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} B(q^n x_j, q^n z) \\
&\quad - l \sum_{1 \le j_1 < \cdots < j_l \le d} B(\frac{q^n x_{j_1} + \cdots + q^n x_{j_l}}{l}, q^n z) \| \\
&\le \frac{1}{q^{2n}} \varphi(q^n x_1, \cdots, q^n x_d, q^n z),
\end{aligned}
$$

which tends to zero as $n \to \infty$ by (3.i). So

$$
\begin{aligned}
d\ {}_{d-2}C_{l-2} B(\frac{x_1 + \cdots + x_d}{d}, z) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} B(x_j, z) \\
- l \sum_{1 \le j_1 < \cdots < j_l \le d} B(\frac{x_{j_1} + \cdots + x_{j_l}}{l}, z) = 0
\end{aligned}
$$

for all $x_1, \cdots, x_d, z \in \mathcal{A}$. By [9] Theorem 2.1, $B(x, z)$ is additive in the first variable. That is,

$$
B(x + y, z) = B(x, z) + B(y, z)
$$

for all $x, y, z \in \mathcal{A}$.

Put $x_1 = \cdots = x_d = x$ in (3.ii). By (3.ii) and the assumption that $B(qx, z) = B(x, qz) = qB(x, z)$ for all $x, z \in \mathcal{A}$,

$$
\begin{aligned}
&\| d\ {}_{d-2}C_{l-2}(B(\mu x, z) - \mu B(x, z)) \| \\
&\qquad = d\ {}_{d-2}C_{l-2} \frac{1}{q^{2n}} \| B(q^n \mu x, q^n z) - \mu B(q^n x, q^n z) \| \\
&\qquad \le \frac{1}{q^{2n}} \varphi(\underbrace{q^n x, \cdots, q^n x}_{d \text{ times}}, q^n z),
\end{aligned}
$$

which tends to zero as $n \to \infty$ by (3.i). So

$$
B(\mu x, z) = \mu B(x, z)
$$

for all $\mu \in \mathbb{T}^1$ and $x, z \in \mathcal{A}$. By the same reasoning as in the proof of Theorem 2.1, $B(x, z)$ is $\mathbb{C}$-linear in the first variable.

By (3.iii) and the assumption that $B(qx,z) = B(x,qz) = qB(x,z)$ for all $x, z \in \mathcal{A}$,

$$\begin{aligned}\|B(x,z)+B(z,x)\| &= \frac{1}{q^{2n}}\|q^{2n}B(x,z)+q^{2n}B(z,x)\| \\ &= \frac{1}{q^{2n}}\|B(q^nx,q^nz)+B(q^nz,q^nx)\| \\ &\le \frac{1}{q^{2n}}\varphi(q^nx,\underbrace{0,\cdots,0}_{d-1\text{ times}},q^nz),\end{aligned}$$

which tends to zero as $n \to \infty$ by (3.i). So

$$B(x,z) = -B(z,x)$$

for all $x, z \in \mathcal{A}$. Hence $B(x,z)$ is $\mathbb{C}$-linear in the second variable.

By (3.iv) and the assumption that $B(qx,z) = B(x,qz) = qB(x,z)$ for all $x, z \in \mathcal{A}$,

$$\begin{aligned}&\|B(xy,z) - xB(y,z) - B(x,z)y\| \\ &= \frac{1}{q^{3n}}\|B(q^nx\cdot q^ny,q^nz) - q^nxB(q^ny,q^nz) - B(q^nx,q^nz)\cdot q^ny\| \\ &\le \frac{1}{q^{3n}}\varphi(q^nx,q^ny,\underbrace{0,\cdots,0}_{d-2\text{ times}},q^nz) \\ &\le \frac{1}{q^{2n}}\varphi(q^nx,q^ny,\underbrace{0,\cdots,0}_{d-2\text{ times}},q^nz),\end{aligned}$$

which tends to zero as $n \to \infty$ by (3.i). So

$$B(xy,z) = xB(y,z) + B(x,z)y$$

for all $x, y, z \in \mathcal{A}$.

By (3.v) and the assumption that $B(qx,z) = B(x,qz) = qB(x,z)$ for all $x, z \in \mathcal{A}$,

$$\begin{aligned}&\|B(x,B(y,z)) + B(y,B(z,x)) + B(z,B(x,y))\| \\ &= \frac{1}{q^{3n}}\|B(q^nx,B(q^ny,q^nz)) + B(q^ny,B(q^nz,q^nx)) \\ &\quad + B(q^nz,B(q^nx,q^ny))\|\end{aligned}$$

$$\leq \frac{1}{q^{3n}}\varphi(q^n x, q^n y, \underbrace{0, \cdots, 0}_{d-2 \text{ times}}, q^n z) \leq \frac{1}{q^{2n}}\varphi(q^n x, q^n y, \underbrace{0, \cdots, 0}_{d-2 \text{ times}}, q^n z),$$

which tends to zero as $n \to \infty$ by (3.i). So

$$B(x, B(y,z)) + B(y, B(z,x)) + B(z, B(x,y)) = 0$$

for all $x, y, z \in \mathcal{A}$.

Therefore, the mapping $B : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a Poisson bracket. □

Corollary 3.2. Let $B : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $B(qx, z) = B(x, qz) = qB(x,z)$ for all $x, z \in \mathcal{A}$ for which there exist constants $\theta \geq 0$ and $p \in [0, 2)$ such that

$$\|d\ {}_{d-2}C_{l-2} B(\frac{\mu x_1 + \cdots + \mu x_d}{d}, z) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu B(x_j, z)$$

$$- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu B(\frac{x_{j_1} + \cdots + x_{j_l}}{l}, z)\| \leq \theta(\sum_{j=1}^{d} \|x_j\|^p + \|z\|^p),$$

$$\|B(x,z) + B(z,x)\| \leq \theta(||x||^p + ||z||^p),$$

$$\|B(xy, z) - xB(y,z) - B(x,z)y\| \leq \theta(||x||^p + ||y||^p + ||z||^p),$$

$$\|B(x, B(y,z)) + B(y, B(z,x)) + B(z, B(x,y))\|$$

$$\leq \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_d, x, y, z \in \mathcal{A}$. Then the mapping $B : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a Poisson bracket.

Proof. Define $\varphi(x_1, \cdots, x_d, z) = \theta(\sum_{j=1}^{d} \|x_j\|^p + \|z\|^p)$, and apply Theorem 3.1. □

Theorem 3.3. Let $B : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $B(qx, z) = B(x, qz) = qB(x,z)$ for all $x, z \in \mathcal{A}$ for which there exists a function $\varphi : \mathcal{A}^{d+1} \to [0, \infty)$ satisfying (3.i), (3.iii), (3.iv), and (3.v) such that

$$\|d\ {}_{d-2}C_{l-2} B(\frac{\mu x_1 + \cdots + \mu x_d}{d}, z) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu B(x_j, z)$$

$$-l \sum_{1\le j_1<\cdots<j_l\le d} \mu B(\frac{x_{j_1}+\cdots+x_{j_l}}{l}, z)\| \le \varphi(x_1,\cdots,x_d,z)$$

for $\mu = 1, i$ and all $x_1, \cdots, x_d, z \in \mathcal{A}$. If $B(tx, z)$ is continuous in $t \in \mathbb{R}$ for each fixed $x, z \in \mathcal{A}$, then the mapping $B : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a Poisson bracket.

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.1. □

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