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RELATIONSHIPS AMONG SPACES OF META-LINDELÖF PROPERTY END STAR-LINDELÖFNESS OF σ-PRODUCT SPACES

Myung Hyun Cho and Jun-Hui Kim

Abstract. In this paper, we further enrich the theory of starcompactness properties by clarifying relationships between various types of weak covering properties and starcompactness and starcompactness-like properties. We also study star-Lindeöfness of σ-product spaces.

1. Introduction

Nearly as old as general topology itself is the factorization of compactness into countable compactness and Lindelöfness. Over the years many other properties have been shown to have the capability to play the role of Lindelöfness in this factorization, e.g., paracompactness, G_δ-diagonal, symmetrizable, θ-refinable, and so on. In 1970, B. Bacon gave a name to this property, that is, a space is called *isocompact* if each of its closed countably compact subsets is compact. When the word isocompact was coined, the program of study was initiated to discover a covering property that would characterize the property of isocompactness. Many have been suggested, studied, and found to be not quite weak enough. The properties of pure-ness and ultrapure-ness are given by Arhangelskii in 1980 [1] and studied in [1] and [9]. As the search has

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gone on, the properties have become increasing technical and complicated to describe.

On the other hand, the study of star covering properties of a topological space could be dated back to 1970's or even earlier [1, 11, 12, 16], a systematic investigation of them was done first by van Douwen, G. Reed, A. Roscoe and I. Tree [10] in 1991. Since then, this area has generated substantial interest of many topologists [4, 14, 15, 24, 25]. The most influential study of this topic is Matveev's survey article [20].

Behavior of the star covering properties was studied in [10, 20, 24]. For example, at least consistently, 1-starcompactness, $1\frac{1}{2}$-starcompactness, 2-starcompactness and $2\frac{1}{2}$-starcompactness are all different in the class of first countable spaces [10]. It is consistent with **ZFC** that every $1\frac{1}{2}$-starcompact Moore space is compact [10]. By replacing "finite" with "countable" in the definition, n-starcompactness was also extended to n-star-Lindelöfness in [10, 20].

In this paper, we further enrich the theory of starcompactness properties by clarifying relationships between various types of weak covering properties and starcompactness and starcompactness-like properties. We also study star-Lindeöfness of σ-product spaces. In section 2, we study properties between countable compactness and pseudocompactness. In particular, we consider a general question: When is a countable compact space compact ? In [13], it is shown that if X is a Hausdorff acc nearly meta-Lindelöf space, then it is compact. In Theorem 2.2, one can replace "Hausdorff acc" with "regular countably compact", but it remains an open problem if it holds for "Hausdorff countably compact" [13]. However, we have a result (Theorem 2.3) that if a regular DFCC space has a discrete refinement condition on some dense set, then it is compact. Also, using the star-characterization of countable compactness in the class of Hausdorff spaces, we give another proof of a well-known theorem that every Hausdorff countably compact meta-Lindeöf space is compact. Section 3 consists of star covering properties with meta-Lindelöfness and weak covering properties. We show (Proposition 3.1) that a 1-cl-starcompact metacompact space X is absolutely 1-cl-starcompact. We observe (Theorem 3.6) that if a space X has a countable spread, then the following are equivalent: X is ultrapure, X is Lindelöf, X is star-Lindelöf and meta-Lindelöf, and X is para-Lindelöf and n-star-Lindelöf for some $n \in \widetilde{\mathbb{N}}$. In section 4, we study star-Lindelöf properties of sigma products. In [20], Matveev asked whether a σ-product (of uncountably many spaces) is star-Lindelöf unless all but countably many factors are compact. It is noted by K. Chiba [21] that the question has an affirmative answer by the following theorem of Kombarov: If any finite subproduct in the product space $X = \prod\{X_\alpha : \alpha \in I\}$ is Lindelöf, then the σ-product in X is Lindelöf (and hence star-Lindelöf). We also have a partial solution to the question with an assumption that every finite subproduct is a countably compact k-space. More precisely, we prove (Theorem 4.2) that if every finite subproduct of $\sigma(p)$ is a countably compact k-space, then $\sigma(p)$ is star-Lindelöf. Finally, we prove (Theorem 4.3) that if $\sigma(p)$ is a σ-product of $X = \prod\{X_\alpha : \alpha \in I\}$, where each X_α is a separable regular (Tychonoff) space, then σ is $2\frac{1}{2}$-star-Lindelöf (pseudo-Lindelöf).

Throughout this paper, a space always means a T_1 topological space. Undefined concepts and symbols can be found in [11] and [20].

2. Properties between countable compactness and pseudocompactness

A space X is called *countably compact* if every countable open cover of X has a finite subcover. Recall that a Hausdorff space X is countably compact if and only if every infinite subset of X has a limit point. A space X is *pseudocompact* if every continuous real-valued function on X

is bounded. A space X has the *discrete finite chain condition* (henceforth abbreviated DFCC) provided every discrete family of nonempty open sets is finite.

It is well-known [11] that every pseudocompact normal space is countably compact. In fact, the property wD, which is weaker than normality, is enough to imply the countable compactness of a DFCC space. The following theorem is a characterization of DFCC in the class of regular spaces.

Theorem 2.1 [3] *A regular space X is DFCC iff every countable open cover of X has a finite subfamily whose union is dense in X.*

Recall that a space X is *meta-compact* (resp. *meta-Lindelöf*) if every open cover of X has a point-finite (resp. point-countable) open refinement. A space X is said to be *para-Lindelöf* if every open cover of X has a locally countable open refinement. A space X is *nearly meta-Lindelöf* [13] provided for every open cover $\mathcal{U}$ of X, there is a dense subspace $Y \subseteq X$ and a refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is point-countable at all points of Y.

The well-known problem closely related to isocompactness is the following: When is a countable compact space compact? The problem is well studied in [8], [9], and [26]. Every countably compact meta-Lindelöf space is compact.

On pseudocompactness, Burke and Davis showed [5] that a Tychonoff pseudocompact paraLindelöf space is compact, and it is noted in [5] that σ-paraLindelöfness is enough. A pseudocompact space with a countable network is compact (see [2]). Also, we note that the countability of character and the countability of pseudocharacter are equivalent for pseudocompact spaces [2].

Before considering a natural modification of the above problem, we recall that a space X is *absolutely countably compact*, abbreviated by *acc*, if for every open cover $\mathcal{U}$ and every dense subspace Y there exists a finite subset $F \subseteq Y$ such that $st(F,\mathcal{U}) = X$ (for the general definitions, see Definition 3.1 below). Clearly, this property is stronger than starcompactness (i.e. countable compactness).

When is an acc space compact ? So far, the only known result in this direction is the following theorem.

Theorem 2.2 [13] *If X is a Hausdorff acc nearly meta-Lindelöf space, then it is compact.*

In Theorem 2.2, one can replace "Hausdorff acc" with "regular countably compact", but it remains an open problem if it holds for "Hausdorff countably compact" [13]. However, we have a result that if a regular DFCC space has a discrete refinement condition on some dense set, then it is compact.

Theorem 2.3 *If X is a regular DFCC space such that every open cover has an open refinement which is discrete on some dense set, then X is compact.*

Proof. Suppose that $\mathcal{U}$ is an open cover of X and $\mathcal{V}$ is an open refinement of $\mathcal{U}$ such that $\mathcal{V}$ is discrete on the dense set $D \subseteq X$ and for all $V \in \mathcal{V}$ there is a $U(V) \in \mathcal{U}$ with $\overline{V} \subseteq U(V)$.

We claim that there is a finite $\mathcal{V}' \subseteq \mathcal{V}$ such that $D \subseteq \overline{\cup\mathcal{V}'} = \cup\{\overline{V} : V \in \mathcal{V}'\}$. By way of contradiction, assume that for every finite $\mathcal{V}' \subseteq \mathcal{V}$, $D \not\subseteq \overline{\cup\mathcal{V}'}$. Let $x_0 \in D - \overline{\cup\mathcal{V}'}$ and W_0 be an open neighborhood of x_0 such that $\mathcal{V}_0 = \{V \in \mathcal{V} : W_0 \cap V \neq \emptyset\}$ is at most one. Since $D \not\subseteq \overline{\cup\mathcal{V}_0}$, let $x_1 \in D - \overline{\cup\mathcal{V}_0}$ and let $W_1 \subseteq X - \overline{\cup\mathcal{V}_0}$ be an open neighborhood of x_1 (using regularity) such that $\mathcal{V}_1 = \{V \in \mathcal{V} : W_1 \cap V \neq \emptyset\}$ is at most one. Continuing this process, we construct an infinite discrete collection $\{W_i : i \in \omega\}$ of nonempty open sets of X. However, since X

is DFCC, discrete collections of open subsets of X must be finite. This is a contradiction.

Now since D is dense in X, $\{U(V) : V \in \mathcal{V}'\}$ is a finite subcover of $\mathcal{U}$ and therefore X is compact. □

A space X is *countably pracompact* if it has a dense subspace Y such that Y is relatively countably compact. It is known [20] that every countably compact space is countably pracompact and every countably pracompact space is DFCC, and every countably pracompact space is 1-cl-starcompact (and hence 2-starcompact). The Ψ-space is an example of a countably pracompact space which is not countably compact.

Just replacing DFCC by "countably pracompact" in Theorem 2.3, we get the following.

corollary 2.1 *If X is a regular countably pracompact space such that every open cover has an open refinement which is discrete on some dense set, then X is compact.*

Now we turn to star covering properties which are between countable compactness and pseudocompactness in the class of Tychonoff spaces.

Let X be a set and $\mathcal{U}$ a collection of subsets of X. For any nonempty subset A of X, let $\mathrm{st}(A,\mathcal{U}) = \mathrm{st}^1(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$ and $\mathrm{st}^{n+1}(A,\mathcal{U}) = \mathrm{st}(\mathrm{st}^n(A,\mathcal{U}),\mathcal{U})$ for all $n \in \mathbb{N}$. In particular, $\mathrm{st}^0(A,\mathcal{U}) = A$. We write $\mathrm{st}^n(x,\mathcal{U})$ instead of $\mathrm{st}^n(\{x\},\mathcal{U})$.

Definition 2.1 *Let $n \in \mathbb{N}$.*

(1) A space X is called n-starcompact *if, for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $\mathrm{st}^n(F,\mathcal{U}) = X$.*

(2) A space X is called $n\frac{1}{2}$-starcompact *if, for every open cover $\mathcal{U}$ of X, there exists a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{st}^n(\bigcup\mathcal{V},\mathcal{U}) = X$.*

(3) A space X is called n-star-Lindelöf *if, for every open cover $\mathcal{U}$ of X, there is a countable subset F of X such that* $\mathrm{st}^n(F,\mathcal{U}) = X$.

(4) A space X is called $n\frac{1}{2}$-star-Lindelöf *if, for every open cover $\mathcal{U}$ of X, there is a countable subcollection $\mathcal{V}$ of $\mathcal{U}$ such that* $\mathrm{st}^n(\bigcup\mathcal{V},\mathcal{U}) = X$.

It is well-known that countable compactness is equivalent to 1- star-compactness in the class of Hausdorff spaces [12]. Also, if $n \in \widetilde{\mathbb{N}} = \mathbb{N} \cup \{n + \frac{1}{2} : n \in \mathbb{N}\}$ and $n \geq 3$, then every n-starcompact regular space is $2\frac{1}{2}$-starcompact [20], and $2\frac{1}{2}$-starcompactness is equivalent to pseudocompactness in the class of Tychonoff spaces [20].

Using the above star-characterization of countable compactness in the class of Hausdorff spaces, we can give another proof of the following well-known theorem.

Theorem 2.4 *[2] Every Hausdorff countably compact meta-Lindeöf space is compact.*

Proof. Suppose that X is countably compact and meta-Lindeöf. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of X. By meta-Lindelöfness of X, $\mathcal{U}$ has point-countable open refinement $\mathcal{V} = \{V_\beta : \beta \in J\}$. Since X is a Hausdorff countably compact space, there exists a finite $F \subseteq X$ such that $st(F, \mathcal{V}) = X$ by the star-characterization of countable compactness in [12]. Since $\mathcal{V}$ is point-countable and F is finite, the set $\mathcal{V}'$ of elements of $\mathcal{V}$ which intersects F is finite, that is, $\mathcal{V}' = \{V \in \mathcal{V} : V \cap F \neq \emptyset\}$ is finite. So, as $st(F, \mathcal{V}) = X$, we have that $\mathcal{V}'$ is a finite open cover of X which refines $\mathcal{U}$. So X is compact. □

3.Starcovering properties with meta-Lindelöfness and weak covering properties

Definition3.1 *[20] (1) A space X is* absolutely k-starcompact, *denoted by* (aSt^k), $(k \in \mathbb{N})$ *if the following condition holds:*

For every open cover $\mathcal{U}$ *of* X *and every dense subspace* $Y \subseteq X$, *there exists a finite subset* $A \subseteq Y$ *such that* $\mathrm{st}^k(A,\mathcal{U}) = X$.

(2) A space X is absolutely k-cl-starcompact, *denoted by* (aSt^k_{cl}), $(k \in \mathbb{N})$ *if the following condition holds:*

For every open cover $\mathcal{U}$ *of* X *and every dense subspace* $Y \subseteq X$, *there exists a finite subset* $A \subseteq Y$ *such that* $\overline{\mathrm{st}^k(A,\mathcal{U})} = X$.

It is known [20] that (countably compact) $\Rightarrow (aSt^1_{cl}) \Rightarrow (St^1_{cl})$.

The following proposition shows that the converse of the second implication is true whenever the space is metacompact.

Proposition 3.1 *A 1-cl-starcompact metacompact space X is absolutely 1-cl-starcompact.*

Proof. Let D be a dense subset of X and $\mathcal{U}$ be an open cover of X. Without loss of generality, we may assume that $\mathcal{U}$ is point-finite. Since X is 1-cl-starcompact, there is a finite $A \subseteq X$ such that $\overline{\mathrm{st}(A,\mathcal{U})} = X$. For each $a \in A$ and $U \in \mathcal{U}$ with $A \in U$, select $d_{a,U} \in U \cap D$. Then $A' = \{d_{a,U} : a \in A, a \in U \in \mathcal{U}\}$ is finite and $\overline{\mathrm{st}(A',\mathcal{U})} = X$. □

Definition 3.2 *[20] (1)A space X is* absolutely k-star-Lindelöf $(k \in \mathbb{N})$ *if the following condition holds:*

(aSt^kL) For every open cover $\mathcal{U}$ *of* X *and every dense subspace* $Y \subset X$ *there exists a countable subset* $A \subset Y$ *such that* $\mathrm{st}^k(A,\mathcal{U}) = X$.

(2) A space X is absolutely k-cl-star-Lindelöf $(k \in \mathbb{N})$ *if the following condition holds:*

$(aSt^k_{cl}L)$ For every open cover $\mathcal{U}$ *of* X *and every dense subspace* $Y \subset X$, *there exists a countable subset* $A \subset Y$ *such that* $\overline{\mathrm{st}^k(A,\mathcal{U})} = X$.

Just replacing "finite" by "countable" in the proof of Proposition 3.1, we have the following.

Proposition 3.2 *A 1-cl-starcompact meta-Lindelöf space X is absolutely 1-cl-starLindelöf.*

It is known from Example 8 in [20] that a 2-starcompact meta-Lindelöf space need not be absolutely 2-starcompact and so need not be absolutely 1-cl-starcompact. In other words, we have a pseudocompact, absolutely 2-star-Lindelöf space which is not absolutely 2-starcompact. However, every 2-starcompact metacompact space is absolutely 2-starcompact. In general, every n-starcompact metacompact space is absolutely n-starcompact.

I. Tree [25] constructed a pseudocompact meta-Lindelöf space which is not 2-star-Lindelöf (moreover, not 2-starcompact). However, if we assume 2-starcompactness which is stronger than pseudocompactness, then we have following theorem:

Theorem 3.1 *Every 2-starcompact meta-Lindelöf space is $1\frac{1}{2}$-star-Lindel öf.*

Proof. Let X be a 2-starcompact meta-Lindelöf space and $\mathcal{U}$ be an open cover of X. Since X is meta-Lindelöf, there exists a point-countable open refinement $\mathcal{V}$ of $\mathcal{U}$. Since X is 2-starcompact, there exists a finite subset $F \subseteq X$ such that $\mathrm{st}^2(F, \mathcal{V}) = X$. Let $\mathcal{V}_0 = \{V : A \cap V \neq \emptyset\}$. Now, for each $V_\alpha \in \mathcal{V}_0$, we chooose $U_\alpha \in \mathcal{U}$ such that $V_\alpha \subseteq U_\alpha$. Let $\mathcal{U}_0 = \{U_\alpha : V_\alpha \subseteq U_\alpha, V_\alpha \in \mathcal{V}_0\}$. Then $|\mathcal{U}_0| \leq \omega$ and $\mathrm{st}(\bigcup \mathcal{U}_0, \mathcal{U}) = X$. Therefore, X is $1\frac{1}{2}$-star-Lindelöf. □

A space X is said to be (n,k)*-starcompact* if for every open cover $\mathcal{U}$ of X there is an n-starcompact subspace A of X such that $\mathrm{st}^k(A,\mathcal{U}) = X$.

Theorem 3.2 *[17] Let X be a meta-Lindelöf space. If X is $(1,1)$-starcompact, then X is $1\frac{1}{2}$-starcompact.*

$(1,1)$-starcompactness is stronger than 2-starcompactness and is parallel to $1\frac{1}{2}$-starcompactness. By replacing $(1,1)$-starcompactness with 2-starcompactness or $1\frac{1}{2}$-starcompactness, we can ask the following.

Question 1 *Let X be a 2-starcompact (resp. $1\frac{1}{2}$-starcompact) meta-Lindelöf space. Is X $1\frac{1}{2}$-starcompact (resp. $(\frac{1}{2},1)$-starcompact)?*

In the rest of this section, we observe relations between weak covering properties and star covering properties.

A countable family $\mathcal{V} = \{\mathcal{V}_n : n \in \omega\}$ of collections of subsets of a space X is called an *interlacing* on X if $\bigcup\bigcup\mathcal{V} = X$ and for each $n \in \omega$, each $V \in \mathcal{V}_n$ is open in $\bigcup\mathcal{V}_n$. An interlacing $\mathcal{V}$ is called *suspended* (resp. *δ-suspended*) from a family $\mathcal{H}$ of subsets of a space X if for every $n \in \omega$ and $x \in \bigcup\mathcal{V}_n$, there is a finite family $\mathcal{K} \in [\mathcal{H}]^{<\omega}$ (resp. a countable family $\mathcal{K} \in [\mathcal{H}]^{\leq\omega}$) such that $st(x,\mathcal{V}_n) \cap (\bigcap\mathcal{K}) = \emptyset$. A space X is called *ultrapure* if for each free closed collection $\mathcal{F}$ on X there is an interlacing which is δ-suspended from $\mathcal{F}$. A space X is called *pure* if for each free closed ultrafilter $\mathcal{F}$ on X there is an interlacing which is δ-suspended from $\mathcal{F}$. It is easy to see that ultrapure implies pure.

Theorem 3.3 *[1] Every countably compact pure space is compact.*

Since star-Lindelöfness is a generalization of both countable compactness and Lindelöfness, we may expect a similar result that a star-Lindelöf pure space must be Lindelöf. However, it is not true in general. For example, a Ψ- space has a G_δ-diagonal and hence it is pure [1]. Also it is separable and thus star-Lindelöf, but not Lindelöf.

Theorem 3.4 *[20] If a T_1-space X has a countable extent, then X is star-Lindelöf.*

The converse is not true since a Ψ-space has its extent $\geq 2^\omega$. So we have a natural question.

Question 2 *[20] Must a pure space with countable extent be Lindelöf?*

Question is negative under CH (see Example 2 in [8]). On the other hand, if we strengthen pure to ultrapure, but if we weaken countable extent to countable spread, then we have the following:

Theorem 3.5 *[9] Every ultrapure space X with countable spread is Lindelöf.*

Theorem 3.6 *Let X be a space with countable spread. Then the following are equivalent.*

(1) X is ultrapure

(2) X is Lindelöf

(3) X is star-Lindelöf and meta-Lindelöf

(4) X is para-Lindelöf and n-star-Lindelöf for some $n \in \widetilde{\mathbb{N}}$

Proof. (1) $\Rightarrow$ (2): Theorem 3.5. This is the only place where countable spread is used.

(2) $\Rightarrow$ (1): It is always true.

(2) $\Leftrightarrow$ (3): Theorem 1.2 in [4].

(2) $\Leftrightarrow$ (4): Corollary 2.2.3 in [18]. □

Question 3 *Must a pure space with countable extent be meta-Lindelöf?*

If the answer to Question 3 is affirmative, then the answer to Question 2 will be affirmative by Theorem 3.6 and Theorem 3.4.

4.Star-Lindelöf properties of sigma products

Let $X = \prod\{X_\alpha : \alpha \in I\}$ be a product space and p be a fixed point of X which is called the base point. We define Σ-product, denoted by $\Sigma(p)$, as

$$\Sigma(p) = \{x \in X : |\{\alpha : x(\alpha) \neq p(\alpha)\}| \leq \omega\}.$$

We also define a σ-product, denoted by $\sigma(p)$, as

$$\sigma(p) = \{x \in X : |\{\alpha : x(\alpha) \neq p(\alpha)\}| < \omega\}$$

A σ-product $\sigma(p)$ (resp. Σ-product $\Sigma(p)$) is called *non-trivial* if $\sigma(p) \neq X$ (resp. $\Sigma(p) \neq X$).

Let $\sigma_n(p) = \{x \in X : |\{\alpha : x(\alpha) \neq p(\alpha)\}| \leq n\}$ for each $n \in \omega$. Clearly, $\sigma(p) = \bigcup\{\sigma_n(p) : n \in \omega\}$.

It is well-known that every non-trivial σ-product is not countably compact. K. Chiba [7] recently showed that every non-trivial σ-product is not even 1-starcompact(and hence not countably compact). It is also well-known that a σ-product of compact spaces is σ-compact and hence star-Lindelöf. It is reasonable to ask whether compactness of the factors is a necessary condition.

Question 4 *(Question 54 in* [20]*) Can a σ-product (of uncountably many spaces) be star-Lindelöf unless all but countably many factors are compact?*

It is noted by K. Chiba [21] that the question has an affirmative answer by the following theorem of Kombarov:

Theorem 4.1 *If any finite subproduct in the product space $X = \prod\{X_\alpha : \alpha \in I\}$ is Lindelöf, then the σ-product in X is Lindelöf (and hence star-Lindelöf).*

It is also noted [21] that another possible solution is the following: If the product space X is countably compact, then the σ-product is σ-(countably compact) and hence star-Lindelöf. So, the question should be asked in a much weaker form. From this point of view, we have a partial solution to the question with an assumption that every finite subproduct is a countably compact k-space.

Recall that a Hausdorff space X is a *k-space* if for each $A \subseteq X$, A is closed in X provided that the intersection A with any compact subspace Z of X is closed in Z.

Theorem 4.2 *If every finite subproduct of $\sigma(p)$ is a countably compact k-space, then $\sigma(p)$ is star-Lindelöf.*

Proof. It suffices to prove that every $\sigma_n(p)$ is countably compact. We will show it by induction. It is obvious that $\sigma_0(p) = \{p\}$ is countably compact. Suppose that $\sigma_k(p)$ is countably compact for all $k \leq n$. Let $\mathcal{U}$ be a basic open cover of $\sigma_{n+1}(p)$ and let $U \in \mathcal{U}$ such that $p \in U$. Then there exist a finite subset $F \subseteq I$ and open neighborhoods V_α of $p(\alpha)$ in X_α such that $U = [\prod\{V_\alpha : \alpha \in F\} \times \prod\{X_\alpha : \alpha \in I \setminus F\}] \cap \sigma_{n+1}(p)$. For each $\beta \in F$, we define $G_\beta = [(X_\beta \setminus V_\beta) \times \prod\{X_\alpha : \alpha \neq \beta\}] \cap \sigma_{n+1}(p)$. Then G_β is homeomorphic to $(X_\beta \setminus V_\beta) \times \sigma_n(p)$. Since $X_\beta \setminus V_\beta$ is a countably compact k-space and $\sigma_n(p)$ is countably compact, G_β is countably compact. It follows from $\sigma_{n+1}(p) = U \cup \bigcup\{G_\beta : \beta \in F\}$ that $\sigma_{n+1}(p)$ is countably compact. Therefore, $\sigma(p)$ is star-Lindelöf. □

It was proved in [4] that every star-Lindelöf space with countable tightness is absolutely star-Lindelöf. So, we have the following corollary.

Corollary 4.1 *Every σ-product of countably compact k-spaces with countable tightness is absolutely star-Lindelöf.*

Theorem 4.3 *Let $\sigma(p)$ be a σ-product of $X = \prod\{X_\alpha : \alpha \in I\}$ where each X_α is a separable regular (Tychonoff) space. Then $\sigma(p)$ is $2\frac{1}{2}$-star-Lindelöf (pseudo-Lindelöf).*

Proof. First, we will prove that $\sigma_1(p)$ is star-Lindelöf. Let $\mathcal{U}$ be a basic open cover of $\sigma_1(p)$ and let $U \in \mathcal{U}$ such that $p \in U$. Then there exist a finite subset $F \subseteq I$ and open neighborhoods V_α of $p(\alpha)$ in X_α such that $U = [\prod\{V_\alpha : \alpha \in F\} \times \prod\{X_\alpha : \alpha \in I \setminus F\}] \cap \sigma_1(p)$. For each $\beta \in F$, we define $G_\beta = [(X_\beta \setminus \{p(\beta)\}) \times \prod\{X_\alpha : \alpha \neq \beta\}] \cap \sigma_1(p)$. Because G_β is homeomorphic to $(X_\beta \setminus \{p(\beta)\}) \times \sigma_1(p)$ and each X_β is a T_1 space, G_β is separable. It follows from $\sigma_1(p) = U \cup \bigcup\{G_\beta : \beta \in F\}$ that $\sigma_1(p)$ is star-Lindelöf.

By following the same way, we can prove $\sigma_n(p)$ is n-star-Lindelöf. Since n-star-Lindelöfness is equivalent to $2\frac{1}{2}$-star-Lindelöfness if $n \geq 3$ in the class of regular spaces, $\sigma(p)$ can be represented by the union of countably many $2\frac{1}{2}$-star-Lindelöf spaces. Therefore, $\sigma(p)$ is $2\frac{1}{2}$-star-Lindelöf. Moreover, if each X_α is Tychonoff, then $\sigma(p)$ is pseudo-Lindelöf. □

Similar to results on σ-products, we may ask the following question of Σ-products:

Question 5 *When is a Σ-product star-Lindelöf under the condition that every finite subproduct is countably compact?*

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Myung Hyun Cho
Division of Mathematics and Informational Statistics
Wonkwang University
Iksan 570-749, Korea
mhcho@wonkwang.ac.kr

Jun-Hui Kim
Division of Mathematics and Informational Statistics
Wonkwang University Iksan 570-749, Korea
junhikim@wonkwang.ac.kr