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COMPARISON BETWEEN DIGITAL CONTINUITY AND COMPUTER CONTINUITY

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Abstract. The aim of this paper is to show the difference between the notion of digital continuity and that of computer continuity. More precisely, for digital images $(X, k_0) \subset Z^{n_0}$ and $(Y, k_1) \subset Z^{n_1}$, if $(k_0, k_1) = (3^{n_0} - 1, 3^{n_1} - 1)$, then the equivalence between digital continuity and computer continuity is proved. Meanwhile, if $(k_0, k_1) \neq (3^{n_0} - 1, 3^{n_1} - 1)$, then the difference between them is shown in terms of the uniform continuity property.

1. Introduction

A *digital picture* is commonly represented as a quadruple $(\mathbb{Z}^n, k, \bar{k}, X)$, where $n \in \mathbb{N}$: the set of natural numbers, $X \subset \mathbb{Z}^n$: the set of points in the Euclidean n-dimensional space with integer coordinates, k represents an adjacency relation for X and $\bar{k}$ represents an adjacency relation for $\mathbb{Z}^n - X$ [1, 3, 13, 14]. We say that the pair (X, k) is a *digital image* or a *discrete space* with k-adjacency. For $\{a, b\} \subset \mathbb{Z}$ with $a \lneq b$, $[a, b]_Z = \{a \leq n \leq b | n \in \mathbb{Z}\}$ is called a *digital interval* [1]. It is helpful to remind that there is some difference between *computer topology* and *digital topology*. Precisely, while computer topology needs some reasonable topological structure for the research of digital images, digital topology may not need some topology for the study of them.

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The digital continuity was originally introduced [14] and further, an advanced concept of the digital continuity was also established [1]. Recently, a generalized digital (k_0, k_1)-continuity was shown with the extended adjacency relations [3, 11]. And further, the notion of computer continuity was introduced for the establishment of computer covering map [9] and for the classification of discrete spaces with the general k-adjacency relations in the computer topological point of view. In this paper we will observe some difference between the notion of digital continuity and that of computer continuity in the digital or computer topological point of view (see Theorem 3.4).

Due to the concept of computer continuity, we can give an answer affirmatively to the open question in [2] via a further work: Is it possible to construct somewhat algebraic invariants(groups) which are associated with digital pictures in $(\mathbb{Z}^2, T^2)$?

2. Notations and terminology

In this paper, when we mention a k-adjacency, it will mean the general k-adjacency relations in $\mathbb{Z}^n$ as follows [3, 9, 11]:

$k \in \{3^n - 1(n \geq 2), 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1(2 \leq r \leq n-1, n \geq 3), 2n(n \geq 1)\} \cdots\cdots (\bigstar)$,

(C_t^n stands for the combination of n objects taken t at a time).

In the following, all discrete spaces are assumed under the above general k-adjacency relations, *i.e.*, $(\mathbb{Z}^n, k, \bar{k}, X)$, where $(k, \bar{k}) \in \{(k, 2n), (2n, \bar{k})\}$, k and $\bar{k}$ are taken from $\bigstar$ and $k \neq \bar{k}$ except $k = 2$[13].

For a space with k-adjacency $X \subset \mathbb{Z}^n$, distinct two points $x, y \in X$ are called k-connected [14] if there is a k-path $f : [0, m]_Z \to X$ which the image is a sequence $(x_0, x_1, \cdots, x_m)$ from the set of points $\{f(0) = x_0 = x, f(1) = x_1, \cdots, f(m) = x_m = y\}$ such that x_i and x_{i+1} are k-adjacent, $i \in \{0, 1, \cdots, m-1\}, m \geq 1$. And a simple closed k-curve is considered as a sequence $(x_0, x_1, \cdots, x_{m-1})$ of the k-path which x_i and x_j are k-adjacent if and only if $j = i \pm 1(\text{mod} m)$ [1, 10].

On the basis of digital (k_0, k_1)-continuity in [14], the restatement of digital (k_0, k_1)-continuity in terms of digital k_i-connectedness was shown with the standard k_i-adjacency relations, $i \in \{0, 1\}$ [1]. Namely, in two digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$, a map $f : X \to Y$ is called a digitally (k_0, k_1)-continuous map at $x \in X$ if f satisfies the following: For a point $x \in X$, the image of each k_0-connected subset containing x is k_1-connected. If f is digitally (k_0, k_1)-continuous at any point $x \in X$, then f is called a digitally (k_0, k_1)-continuous map. Furthermore, another version of digital continuity was stated in terms of the digital k-neighborhood with the general k-adjacency relations, which will be useful to study some digital topological properties via Definitions 2.1 and 2.2.

Definition 2.1.[3, 4, 5, 6, 7] Let (X, k) be a digital image in $\mathbb{Z}^n$ and $\varepsilon \in \mathbb{N}$. The digital k-neighborhood of $x_0 \in X$ with radius ε is the set

$$N_k(x_0, \varepsilon) = \{x \in X | \quad l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\},$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x in X. □

The next definition characterizes digital continuity with comparison with computer continuity used later in the paper.

Definition 2.2.[3, 4, 5, 6, 7, 8] For digital images $(X, k_0) \subset \mathbb{Z}^{n_0}$ and $(Y, k_1) \subset \mathbb{Z}^{n_1}$, a function $f : X \to Y$ is said to be digitally (k_0, k_1)-continuous if and only if for every $x_0 \in X, \varepsilon \in \mathbb{N}$, and $N_{k_1}(f(x_0), \varepsilon) \subset Y$, there is a $\delta \in \mathbb{N}$ such that the corresponding $N_{k_0}(x_0, \delta) \subset X$ satisfies $f(N_{k_0}(x_0, \delta)) \subset N_{k_1}(f(x_0), \varepsilon)$. □

The notion of *digital continuity* of Definition 2.2 is an extended notion of [1, 14] with respect to the adjacency and the dimension of the related digital image and further, plays an important role in the study of a digital covering map[7, 10, 11]. Furthermore, due to the notion of digital (k_0, k_1)-continuity, digital (k_0, k_1)-homeomorphism, digital (k_0, k_1)-homotopy and digital (k_0, k_1)-covering, we can classify digital images via each of the above tools.

Meanwhile, we now remind the notion of *computer* (k_0, k_1)*-continuity*[9]. Precisely, we proceed the research of digital images or discrete objects with *relative product Khalimsky topology*[9] stemmed from Khalimsky line topology. More precisely, *Khalimsky line topology* on $\mathbb{Z}$ is induced from the subbasis $\{[2n-1, 2n+1]_{\mathbb{Z}} | n \in \mathbb{Z}\}$ [12]. Namely, the family of the subset $\{\{2n+1\}, [2m-1, 2m+1]_{\mathbb{Z}} | m, n \in \mathbb{Z}\}$, which induces open sets for $(\mathbb{Z}, T)$, is a basis of Khalimsky line topology on $\mathbb{Z}$.

We now establish the product topology on $\mathbb{Z}^n$ derived from the Khalimsky line topology on $\mathbb{Z}$ as follows.

Definition 2.3[9] Let $p_\alpha : \mathbb{Z}^n \to \mathbb{Z}$ be the α-projection map given by $p_\alpha(x_1, x_2, \cdots, x_n) = x_\alpha$.

Then we take the topology on $\mathbb{Z}^n$ having as a subbasis the following set

$\{p_\alpha^{-1}(O) | O$ is an open set in $(\mathbb{Z}, T), \alpha \in [1, n]_{\mathbb{Z}}\}$.

Then we call the topology on $\mathbb{Z}^n$ product Khalimsky topology on $\mathbb{Z}^n, n \geq 2$, and denote it $(\mathbb{Z}^n, T^n)$. □

In the following, we say that the topological space $(\mathbb{Z}^n, T^n)$ (respectively $(\mathbb{Z}, T)$) is the *digital n-space* (respectively the *digital line space*).

Since, in $(\mathbb{Z}, T)$, $O \in T$ is the union of some elements of the set $\{\{2n+1\}, [2m-1, 2m+1]_{\mathbb{Z}} | m, n \in \mathbb{Z}\}$, a closed set in $(\mathbb{Z}, T)$ is an arbitrary union of the singleton $\{2n | n \in \mathbb{Z}\}$[2,9,12].

Furthermore, we can see the following: In each of $(\mathbb{Z}, T)$ and $(\mathbb{Z}^n, T^n)$, the intersection of arbitrary open sets is also an open set[9].

For the digital n-space $(\mathbb{Z}^n, T^n)$ and the digital line space $(\mathbb{Z}, T)$, the α-projection map $p_\alpha : \mathbb{Z}^n \to \mathbb{Z}$ is an open map and is also a closed map,

$\alpha \in [0, n]_{\mathbb{Z}}$ [9].

Definition 2.4.[9] For a space $(X, k) \subset (\mathbb{Z}^n, T^n)$, we now consider the relative topology on (X, k) induced from $(\mathbb{Z}^n, T^n)$, then we denote it (X, k, T^n_X) which is called a digital n-topological space with k-adjacency.□

For a space (X, k, T^n_X), a k-neighborhood is introduced for the new notion of continuity in the computer topological point of view, which will be useful to study some computer topological properties of discrete spaces with k-adjacency relations.

For any digital n-topological space with k-adjacency (X, k, T^n_X) and $x \in X$, by the *neighborhood* $N^*(x)$ of the point x is meant the existence of some open set $O \in T^n_X$ such that $x \in O \subset N^*(x)$.

In contrast to the digital k-neighborhood, the following k-neighborhood is considered in the computer topological point of view.

Definition 2.5.[9] In computer topology, for any space (X, k, T^n_X) and $\varepsilon \in \mathbb{N}$, the k-neighborhood of $x_0 \in X$ with radius ε, as a neighborhood of x_0, is the set

$$N^*_k(x_0, \varepsilon) = \{x \in X| \quad l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\},$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x in X. □

The current k-neighborhood is determined from two kinds of information, *i.e.*, the relative digital n-topological structure and k-connectivity. And it is different from that of Definition 2.1. More precisely, we can see the difference between Definitions 2.1 and 2.5 as follows.

Example 2.6. For the space $X = \{x_0 = (0,1), x_1 = (0,2), x_2 = (0,3), x_3 = (1,3), x_4 = (2,2)\}$ in $\mathbb{Z}^2$, let us consider $(X, 4, T^2_X)$ or $(X, 8, T^2_X)$. Then we can observe the difference between $N^*_k(p, \varepsilon)$ and $N_k(p, \varepsilon)$ for

$k \in \{4, 8\}$: More precisely, $N_4(x_1, 1) = \{x_0, x_1, x_2\}$ and $N_4^*(x_1, 1) = \phi$ owing to the non-existence of some open set O in $(X, 4, T_X^2)$ containing the point x_1 such that $x_1 \in O \subset N_4^*(x_1, 1)$. But $N_4^*(x_1, 2) = N_4(x_1, 2)$. □

Up to now, the notion of continuity in the computer topological point of view has not been established. In terms of the k-neighborhood of some point $x_0 \in X$ with some radius, Definition 2.7 characterizes a new computer continuity in a fashion used later in the paper.

Definition 2.7 [9] For any spaces $(X, k_0, T_X^{n_0})$ and $(Y, k_1, T_Y^{n_1})$, a function $f : X \to Y$ is said to be (k_0, k_1)-continuous if and only if for every $x_0 \in X, \varepsilon \in \mathbb{N}$, and $N_{k_1}^*(f(x_0), \varepsilon) \subset Y$, there is a $\delta \in \mathbb{N}$ such that the corresponding $N_{k_0}^*(x_0, \delta) \subset X$ satisfies $f(N_{k_0}^*(x_0, \delta)) \subset N_{k_1}^*(f(x_0), \varepsilon)$. □

The notion of continuity of Definition 2.7 plays an important role in the establishment of a computer (k_0, k_1)-covering map, which is different from Definition 2.2 owing to non-equivalence of the notions of $N_k(x, \varepsilon)$ and $N_k^*(x, \varepsilon)$.

3. Comparison between digital continuity and computer continuity

We can state the two types of digital[11] or computer analogs of *uniform continuity* for a given space [9]:

Definition 3.1. (1) In digital topology, let $(X, k_0) \subset \mathbb{Z}^{n_0}$ and $(Y, k_1) \subset \mathbb{Z}^{n_1}$ be digital images. We say that a function $f : X \to Y$ is uniformly (k_0, k_1)-continuous if for every $x_0 \in X, \varepsilon \in \mathbb{N}$, and $N_{k_1}(f(x_0), \varepsilon) \subset Y$, there is a constant $\delta = \delta(x_0, \varepsilon) \in \mathbb{N}$ such that the corresponding $N_{k_0}(x_0, \delta) \subset X$ satisfies $f(N_{k_0}(x_0, \delta)) \subset N_{k_1}(f(x_0), \varepsilon)$[11]; or

(2) In computer topology, for discrete spaces $(X, k_0, T_X^{n_0})$ and $(Y, k_0, T_Y^{n_1})$, we say that a function $f : X \to Y$ is uniformly (k_0, k_1)-continuous if

for every $x_0 \in X, \varepsilon \in \mathbb{N}$, and $N^*_{k_1}(f(x_0), \varepsilon) \subset Y$, there is a constant $\delta = \delta(x_0, \varepsilon) \in \mathbb{N}$ such that the corresponding $N^*_{k_0}(x_0, \delta) \subset X$ satisfies $f(N^*_{k_0}(x_0, \delta)) \subset N^*_{k_1}(f(x_0), \varepsilon)$. □

Furthermore, from Definitions 2.2 and 2.7, we get the two kinds of notions of digital (k_0, k_1)-homeomorphism and of (k_0, k_1)-homeomorphism as follows.

Definition 3.2. (1) In digital topology, let $(X, k_0) \subset \mathbb{Z}^{n_0}$ and $(Y, k_1) \subset \mathbb{Z}^{n_1}$ be digital images. Then a map $h : X \to Y$ is called a digital (k_0, k_1)-homeomorphism if h is a digitally (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is a digitally (k_1, k_0)-continuous map. Then we use the notation, $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then h is called a digital k_0-homeomorphism[3, 4, 5, 6, 7]; or

(2) In computer topology, given spaces $(X, k_0, T^{n_0}_X)$ and $(Y, k_1, T^{n_1}_Y)$, a map $h : X \to Y$ is called a (k_0, k_1)-homeomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is a (k_1, k_0)-continuous map. Then we use the notation, $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then h is called a k_0-homeomorphism[9]. □

In fact, the two notions in Definition 3.2 are different from each other owing to non-equivalence between digital (k_1, k_0)-continuity and (k_1, k_0)-continuity.

We can observe the uniform continuity property of digital (k_0, k_1)-continuity as follows.

Theorem 3.3.[11] Let $(X, k_0) \subset \mathbb{Z}^{n_0}$ and $(Y, k_1) \subset \mathbb{Z}^{n_1}$ be digital images. Any digitally (k_0, k_1)-continuous function $f : X \to Y$ is a uniformly (k_0, k_1)-continuous map.

Proof. For any point $x \in X$, $\varepsilon \in \mathbb{N}$ and $N_{k_1}(f(x), \varepsilon) \subset Y$, there is at least the constant number $\delta = 1$ and $N_{k_0}(x, 1) \subset X$ such that $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), \varepsilon)$. Thus the assertion is completed. □

We are now in a position to tell the main theorem as follows:

Theorem 3.4. For any spaces $(X, k_0, T_X^{n_0})$ and $(Y, k_1, T_Y^{n_1})$, a computer (k_0, k_1)-continuous function $f : X \to Y$ does not hold uniform (k_0, k_1)-continuity.

Proof. (1) If $(k_0, k_1) = (3^{n_0} - 1, 3^{n_1} - 1)$, since $N^*_{k_0}(x, \delta) = N_{k_0}(x, \delta)$ and $N^*_{k_1}(f(x), \varepsilon) = N_{k_1}(f(x), \varepsilon)$ for any $x \in X$, $\varepsilon, \delta \in \mathbb{N}$, then it turns out that both of digital (k_0, k_1)-continuity and computer (k_0, k_1)-continuity are equivalent to each other. More precisely, from Theorem 3.3, we only prove the uniform (k_0, k_1)-continuity of computer (k_0, k_1)-continuity. For any point $x \in X$, $\varepsilon \in \mathbb{N}$ and $N^*_{k_1}(f(x), \varepsilon) \subset Y$, there is an $N^*_{k_0}(x, 1) \subset X$ such that $f(N^*_{k_0}(x, 1)) \subset N^*_{k_1}(f(x), \varepsilon)$, as required.

(2) If $(k_0, k_1) \neq (3^{n_0} - 1, 3^{n_1} - 1)$, while digital (k_0, k_1)-continuity has the uniform continuity property from Theorem 3.3, computer (k_0, k_1)-continuity does not hold the uniform continuity property. For example, let $SC_4^{2,8}$ be any space in $\mathbb{Z}^2$ that is 4-homeomorphic to the set

$\{w_0 = (1,1), w_1 = (1,2), w_2 = (1,3), w_3 = (2,3), w_4 = (3,3), w_5 = (3,2), w_6 = (3,1), w_7 = (2,1)\}$.

And further, the space $SC_4^{2,8,*}$ is assumed to be any space in $\mathbb{Z}^2$ that is 4-homeomorphic to the set

$\{c_0 = (0,0), c_1 = (0,1), c_2 = (-1,1),\ c_3 = (-2,1),\ c_4 = (-2,0), c_5 = (-2,-1),\ c_6 = (-1,-1), c_7 = (0,-1)\}$.

Then we can see that $SC_4^{2,8}$ is not 8-homeomorphic to $SC_4^{2,8,*}$ owing to non-8-continuity from $SC_4^{2,8,*}$ onto $SC_4^{2,8}$ at the points c_0, c_4. Actually, there are only two kinds of simple closed 4-curves with 8 elements in $\mathbb{Z}^2$ up to a 4-homeomorphism, such as $SC_4^{2,8}$ and $SC_4^{2,8,*}$. Consequently, for some point $x \in X$, $N^*_{k_0}(x, 1)$ and $N^*_{k_1}(f(x), 1)$ may not exist; or even though $N^*_{k_0}(x, \delta)$ and $N^*_{k_1}(f(x), \varepsilon)$ exist for $\varepsilon, \delta = \delta(x, \varepsilon) \in \mathbb{N}$, it may not satisfy $f(N^*_{k_0}(x, \delta) \subset N^*_{k_1}(f(x), \varepsilon)$, *e.g.*, the map $f : SC_4^{2,8,*} \to SC_4^{2,8,*}$ can be considered in terms of the rotation by 90°.□

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