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A NOTE ON MATRICES WITH SIGNED NULL-SPACES

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Abstract. We denote by $\mathcal{Q}(A)$ the set of all matrices with the same sign pattern as A. A matrix A has *signed null-space* provided there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $\tilde{A}$ is $\mathcal{S}$, for each $\tilde{A} \in \mathcal{Q}(A)$. Some properties of matrices with signed null-spaces are investigated.

1. Introduction

The *sign* of a real number a is defined by

$$\operatorname{sign}(a) = \begin{cases} -1 & \text{if } a < 0, \\ 0 & \text{if } a = 0, \text{ and} \\ 1 & \text{if } a > 0. \end{cases}$$

A *sign pattern* is a $(0, 1, -1)$-matrix. The *sign pattern of a matrix* A is the matrix obtained from A by replacing each entry by its sign. We denote by $\mathcal{Q}(A)$ the set of all matrices with the same sign pattern as A. The *zero pattern* of a matrix A is the $(0, 1)$-matrix obtained from A by replacing each nonzero entry by 1.

Let A be an m by n matrix and b an m by 1 vector. The linear system $Ax = b$ has *signed solutions* provided there exists a collection $\mathcal{S}$ of n by 1 sign patterns such that the set of sign patterns of the solutions to $\widetilde{A}x = \tilde{b}$ is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. This notion generalizes that of

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a sign-solvable linear system (see [1] and references therein). The linear system, $Ax = b$, is *sign-solvable* provided each linear system $\widetilde{A}x = \tilde{b}$ ($\widetilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$) has a solution and all solutions have the same sign pattern. Thus, $Ax = b$ is sign-solvable if and only if $Ax = b$ has signed solutions and the set $\mathcal{S}$ has cardinality 1.

A vector is *mixed* if it has a positive entry and a negative entry. A matrix is *row-mixed* if each of its rows is mixed. A *signing* is a nonzero, diagonal $(0, 1, -1)$-matrix. A signing is *strict* if each of its diagonal entries is nonzero. A matrix B is *(strictly) row-mixable* provided there exists a (strict) signing D such that BD is row-mixed.

The matrix A has *signed null-space* provided $Ax = 0$ has signed solutions. Thus, A has signed null-space if and only if there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $\widetilde{A}$ is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$. An *L-matrix* is a matrix, A, with the property that each matrix in $\mathcal{Q}(A)$ has linearly independent rows. A square L-matrix is a *sign-nonsingular*, or *SNS*-matrix for short. A *totally L-matrix* is an $m \times n$ matrix such that each $m \times m$ submatrix is an SNS-matrix. In particular, an m by $m+1$ totally L-matrix is S^*-matrix and a row-mixed m by $m+1$ totally L-matrix is S-matrix. It is known that totally L-matrices are matrices with signed null-spaces [3]. Hence matrices with signed null-spaces generalize totally L-matrices.

Some combinatorial properties of matrices with signed null-spaces can be found in [3, 4, 5, 6]. In this paper, we investigate some properties of matrices with signed null-spaces in a view of graph theory.

We use the following standard notations throughout the paper. If k is a positive integer, then $\langle k \rangle$ denotes the set $\{1, 2, \ldots, k\}$. Let A be an $m \times n$ matrix. If α is a subset of $\{1, 2, \ldots, m\}$ and β is a subset of $\{1, 2, \ldots, n\}$, then $A[\alpha|\beta]$ denotes the submatrix of A determined by the rows whose indices are in α and the columns whose indices are in β. We sometimes use $A[*|\beta]$ instead of $A[\langle m \rangle|\beta]$. The submatrix complementary to $A[\alpha|\beta]$ is denoted by $A(\alpha|\beta)$. Let $J_{m,n}$ denote the m by

n matrix all of whose entries are 1 and let $\mathbf{e}_i$ denote the column vector all of whose entries are 0 except for the ith entry which is 1.

2. The Main Results

Each S^*-matrix has signed null-space. It is easy to show that a strictly row-mixable m by $m+1$ matrix with signed null-space is S-matrix. Clearly if the m by n matrix A has signed null-space then each submatrix $A[\langle m\rangle, \beta]$ has signed null-space, and a direct sum of matrices has signed null-space if and only if each of the summands has signed null-space.

Let A be an m by n matrix. The *term rank* of A is the largest order r of a square submatrix of A which does not have an identically zero determinant. In particular, we say that A has *full term rank* if the term rank of A is equal to one of m or n.

We make use of the following results of matrices with signed null-spaces.

Theorem A([3]) *Let A be a strictly row-mixable, m by n matrix. Then the following three conditions are equivalent.*

(a) A has signed null-space

(b) A has term rank m and its mth compound is signed.

(c) AD has no mixed cycle for each strict signing such that AD is row-mixed.

Theorem B([2], [3]) *Let a strictly row-mixable matrix A has signed null-space, then there exist matrices B and C (possibly with no rows), and nonzero vectors b and c such that B and C are strictly row-mixable matrices with signed null-spaces, up to permutation of rows and columns,*

$$A = \begin{bmatrix} B & O \\ b & c \\ O & C \end{bmatrix}.$$

Whenever A is a matrix of the form in (1)*, A has signed null-space if and only if each of the matrices*

$$B, \quad C, \quad \begin{bmatrix} B \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ C \end{bmatrix}$$

has signed null-space.

Let $A = [a_{ij}]$ be an m by n $(0,1)$-matrix. The *bipartite graph* $G(A)$ of A is the graph of order $m+n$ with vertex bipartition $\{V, V'\}$ where

$$V = \{1, 2, \ldots, m\} \text{ and } \mathrm{V}' = \{1', 2', \ldots, \mathrm{n}'\},$$

and where the edges are those pairs $\{i, j'\}$ for which $a_{ij'} \neq 0$.

Theorem B implies that the graph $G(A)$ of a strictly row-mixable matrix A with signed null-space is not block.

Let A be an m by n $(0,1,-1)$-matrix. The matrix B is *conformally contractible* to A provided there exists an index k such that the rows and columns of B can be permuted so that B has the form

$$\left[\begin{array}{c|c|c} A[\langle m\rangle, \langle n\rangle \setminus \{k\}] & x & y \\ \hline 0 \;\cdots\; 0 & 1 & -1 \end{array}\right],$$

where $x = [x_1, \ldots, x_m]^T$ and $y = [y_1, \ldots, y_m]^T$ are $(0,1,-1)$-vectors such that $x_i y_i \geq 0$ for $i = 1, 2, \ldots, m$, and the sign pattern of $x + y$ is the kth column of A.

It is known that when B is conformally contractible to A, A has signed null-space if and only if B has signed null-space[3]. In investigating matrices with signed null-spaces, there is no loss in generality in restricting attention to $(0, 1, -1)$-matrices.

Proposition 1. Let A be a strictly row mixable m by n matrix with signed null-space. If A is not conformal contractible, then each row of A has at least three nonzero elements.

Proof. Suppose not. Without loss of generality, we may assume that A is row mixed and is of the form

$$A = \left[\begin{array}{cc|c} 1 & -1 & O \\ \hline 1 & -1 & B \\ \hline \multicolumn{2}{c|}{C} & D \end{array}\right].$$

If D has full term rank, A has an m by m submatrix which is not SNS-matrix but has term rank m. This is impossible by Theorem A. Hence D does not have full term rank. Therefore there exists an s by $n-1-s$ zero submatrix of D. Thus A is permutation equivalent to

$$A = \begin{bmatrix} E & O \\ M & N \end{bmatrix}$$

where E is a row-mixed square matrix. It is also impossible since N has full term rank by Theorem A. Thus we have the result. ■

Lemma 2. Let A be of the form

$$\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}.$$

If A has signed null-space and A_1 and A_3 are strictly row mixable, then A_3 have signed null-spaces.

Proof. Let A_3 be an m by n matrix. Suppose that A_3 does not have signed null-space. By Theorem A, there exists an m by m submatrix S_3 of A_3 such that S_3 is not an SNS-matrix but its term rank is m. Since A_1 is a strictly row mixable matrix with signed null-space, there is a square submatrix S_1 of A_1 whose term rank is equal to the term rank of A_1. The submatrix S of A consisting of the rows and the columns of matrices S_1 and S_3 has term rank which is equal to that of A. But S is not an SNS-matrix. It is impossible by Theorem A. ■

Proposition 3. Let an m by n matrix A have signed null-space. Suppose that

(1) A is strictly row mixable

(2) A is not permutation equivalent to

$$\begin{bmatrix} S & \mathbf{s} & O \\ O & \mathbf{t} & T \end{bmatrix}$$

where $\mathbf{s}$ and $\mathbf{t}$ are nonzero vectors.

Then A is contractible or each endblock of the bipartite graph $G(A)$ of A is edge.

Proof. Suppose that the bipartite graph $G(A)$ of A has an endblock which is not edge. By (2), we may assume that A is of the form

$$\begin{bmatrix} v_1 & v_2 \\ B & O \\ O & C \end{bmatrix}$$

where B is non-vacuous, $B' = \begin{bmatrix} v_1 \\ B \end{bmatrix}$ is the matrix corresponding to the an endblock which is not an edge of $G(A)$ and $[v_1\ v_2]$ is the vector corresponding to the cut-vertex. If B' is strictly mixable, then by the first part of Theorem B, B' is decomposable in the form of (1). This implies that $G(B')$ cannot be a block. So $B'$is not strictly mixable. Then Hence B' is permutation equivalent to

$$\begin{bmatrix} B_1 & B_2 \\ O & B_3 \end{bmatrix}$$

where B_1 is strictly row mixable and ${B_3}^T$ is a non-vacuous L-matrix. Since B' has full term rank, B_3 is an SNS-matrix. Notice that the row corresponding to v_1 is one of the rows of B_3. Since B has signed null-space, the matrix S obtained from B_3 by deleting the row of v_1 has signed null-space by Lemma 2. Since the size of the matrix S is k by $k+1$ for some k, S is an S^*-matrix. S and hence A has a row with exactly two nonzero entries. By Theorem 1, A is contractible. ∎

Proposition 3 implies that matrices with signed null-spaces each of whose rows has at least three nonzero entries have a column with exactly one nonzero entry. If a graph does not consist of one block, the graph has at least two endblocks. Thus matrices with signed null-spaces each of whose rows has at least three nonzero entries are, up to permutation of rows and columns, of the form

$$(1) \qquad \begin{bmatrix} I_k & B \\ O & C \end{bmatrix}$$

where $k \geq 2$.

Notice that an m by $m+2$ totally L-matrix has at least two columns with exactly one non-zero entry. If a matrix in the form of (1) has signed null-space and C is strictly row mixable, then the matrix C has signed null-space. What can we say about a matrix A with signed null-space if C is a totally L-matrix in (1). We give an answer in the followings.

Lemma 4. Let a matrix A with signed null-space be of the form in (1) and let C be an m by $m+2$ totally L-matrix. Let $\mathbf{e}_p$ and $\mathbf{e}_q$ be columns of C. Then B has no row whose pth entry and qth entry are nonzero.

Proof. Suppose not. Let

$$M = \begin{bmatrix} 1 & \mathbf{b} \\ O & C \end{bmatrix}$$

be a submatrix of A such that $\mathbf{b}$ is a row of B with nonzero entries in the pth position and qth position. Since A has signed null-space, M has also signed null-space. We want to show that M is not totally L-matrix. If $\mathbf{b}$ has at least three nonzero entries, the first row of M has at least four nonzero entries and hence M is not totally L-matrix. Let $\mathbf{b}$ have exactly two nonzero entries. Then every row of M has exactly three nonzero entries but M is not totally L-matrix by construction of M(cf. Theorem 5.2.3 in [1]).

This implies that M has an $m+1$ by $m+1$ submatrix S which is not SNS-matrix. Then S contains at least one of the first column, pth column or qth column of M. The matrix obtained from S by deleting the first row and one of the first column, pth column or qth column of M which S contains is a k by k submatrix of C and hence it should be an SNS-matrix. Thus S does not have identically zero determinant. Therefore M does not have signed null-space. ■

Proposition 5. Let a matrix A with signed null-space be of the form in (1) and let C be an m by $m+2$ totally L-matrix. Then every row of B has at most two nonzero entries.

Proof. Without loss of generality we may assume that $B = \mathbf{b}$ is a 1 by $m+2$ matrix. We will prove it by induction on $m(\geq 2)$. It is easy to show that $\mathbf{b}$ has at most two nonzero entries for $m = 2$. Let $m > 2$. Since C is a totally L-matrix, at least two columns of C contain exactly one nonzero entry. By Lemma 4, there exists a column $\mathbf{c}$ of C which contains exactly one nonzero entry such that the corresponding position of $\mathbf{b}$ is zero. Deleting the row and column containing nonzero entry of

$\mathbf{c}$, we obtain a matrix

$$\begin{bmatrix} \mathbf{b}' \\ C' \end{bmatrix}$$

which satisfies the hypothesis. By induction, $\mathbf{b}'$ has at most two nonzero entries and hence $\mathbf{b}$ has at most two non-zero entries. ■

(**Another Proof.**) It is sufficient to prove it in the case that A is of the form

$$A = \begin{bmatrix} 1 & \mathbf{b} \\ O & C \end{bmatrix}.$$

If $\begin{bmatrix} \mathbf{b} \\ C \end{bmatrix}$ is strictly row mixable, then it is an $m+1$ by $m+2$ matrix with signed null-space. Hence it is an S^*-matrix. Since every row of C has three nonzero entries, $\mathbf{b}$ has exactly two nonzero entries.

If $\begin{bmatrix} \mathbf{b} \\ C \end{bmatrix}$ is not strictly row mixable, then $\begin{bmatrix} \mathbf{b} \\ C \end{bmatrix}$ is permutation equivalent to

$$\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}$$

where A_1 is strictly row mixable, and A_3^T is a non-vacuous L-matrix. The row corresponding to $\mathbf{b}$ is one of rows of A_3. If A_3 has at least one of rows of C, then the number of rows is less than the number of columns in A_3 by the construction of a totally L-matrix. This is impossible. Hence A_3 is a 1 by 1 matrix. Hence $\mathbf{b}$ is permutation equivalent to $\begin{bmatrix} O & A_3 \end{bmatrix}$ and hence $\mathbf{b}$ has exactly one nonzero entry. Thus we have the result. ■

From now on, each row of a matrix with signed null-space is assumed to have at least three nonzero entries. Let a matrix A with signed null-space be of the form in (1) and C an m by $m+2$ totally L-matrix. Then Proposition 5 implies that each row of B in (1) has exactly two non-zero entries. We can find the positions of possible nonzero entries of each row of B.

Proposition 6. Let a matrix A with signed null-space be of the form in (1) and let C be an m by $m+2$ totally L-matrix. Let $\mathbf{b}$ be a row of B such that the pth entry and the qth entry are nonzero. Then there exists a row $\mathbf{c}_i$ of C such that the zero pattern of $\begin{bmatrix} \mathbf{b} \\ \mathbf{c}_i \end{bmatrix}[*|p,q]$ is $J_{2,2}$.

Proof. It is sufficient to prove it in the case that A is of the form

$$A = \begin{bmatrix} 1 & \mathbf{b} \\ O & C \end{bmatrix}.$$

We will prove it by induction on m, the number of rows of C. It is easy to show that we have the result for $m = 2$. Let $m > 2$. Without loss of generality, we may assume that

$$\begin{bmatrix} \mathbf{b} \\ C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ & & C & & \end{bmatrix}.$$

Since C is an m by $m+2$ totally L-matrix, by Proposition 4 there exists kth column of C which has exactly one nonzero entry at the tth position and $k \neq 1, 2$. Since $C(t|k)$ is also an $m-1$ by $m+1$ and $A(m+1|m+3)$ has signed null-space, there exists a row $\mathbf{c}_i$ of $C(t|k)$ such that the zero pattern of $\begin{bmatrix} \mathbf{b} \\ \mathbf{c}_i \end{bmatrix}[*|p,q]$ is $J_{2,2}$ by induction. Hence we have the result. ∎

Proposition 7. Let $M = \begin{bmatrix} 1 & b \\ O & C \end{bmatrix}$ have signed null-space and b have two nonzero entries. If C is an m by $m+2$ totally L-matrix, so is M.

Proof. Suppose that $N = \begin{bmatrix} b \\ C \end{bmatrix}$ is not strictly row mixable. Then there are permutation matrices P, Q such that

$$PNQ = \begin{bmatrix} N_1 & N_2 \\ O & N_3 \end{bmatrix}$$

where N_1 is strictly row mixable, and N_3^T is a k by l non-vacuous L-matrix. Since N has term rank $m+1$, A_3 is a square matrix. As shown in the another proof of Proposition 5, it is impossible since b have two nonzero entries. Thus $\begin{bmatrix} b \\ C \end{bmatrix}$ is strictly row mixable. Hence it is S^*-matrix. Since C is a totally L-matrix, every $m+1$-square submatrix of M is an SNS-matrix. Thus M is an $m+1$ by $m+3$ totally L-matrix. ∎

It is easy to show that the matrix A in (1) which has signed nullspace is a totally L-matrix if C is an m by n totally L-matrix where $n = m$ or $n = m+1$. Using Proposition 7, we have the following Corollary.

Corollary 8. Let a matrix A of the form in (1) have signed null-space. If C is an m by $m+2$ totally L-matrix, then so is A.

Proof. Let $M_r = M(1, \cdots, k-r|1, \cdots, k-r)$ for $r = 1, \cdots, k$. Then each M_r is an $m+r$ by $m+r+2$ totally L-matrix for $r = 1, \cdots, k$ by Proposition 7. Thus $M = M_k$ and we have the result. ∎

Given an m by $m+2$ totally L-matrix C, we can construct a matrix A with signed null-space of the form in (1). In this case the maximum number of rows which B can have is $m+2$ as shown in the following.

Corollary 9. Let a matrix A with signed null-space be of the form in (1) and let C be an m by $m+2$ totally L-matrix. The maximum number of rows of B is $m+2$.

Proof. An n by $n+2$ totally L-matrix A can be obtained from

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \tag{2}$$

by a sequence of single extensions and double extensions, up to row and column permutations and multiplication of rows and columns by -1(cf. Theorem 5.2.3 in [1]). From this property of an n by $n+2$ totally L-matrix A, it is easy to show that $2 \leq n(A) \leq \frac{n}{2}+1$ where $n(A)$ denote the number of columns of A which have exactly one nonzero entry. The equality on the right part holds if and only if the matrix A is obtained from the matrix in (2) by a sequence of double extensions, up to row and column permutations and multiplication of rows and columns by -1. Notice that the zero patterns of these columns are different. Thus an n by $n+2$ totally L-matrix A is permutation equivalent to a matrix of the form in (1) where the columns of C consist of the columns of A which has at least two nonzero entries. Clearly, the matrix C is an m by $m+2$ totally L-matrix for some m. Thus $\frac{n}{2}+1+m=n$ and hence $n=2m+2$. By Corollary 8 implies that the maximum value $m+1$ of $n(A)$ is equal to the maximum number of rows which B can have and hence we have the result. ■

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