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ON CHARACTERIZATIONS OF REAL HYPERSURFACES IN A COMPLEX SPACE FORM IN TERMS OF THE JACOBI OPERATORS

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Abstract. The shape operator or second fundamental tensor of a real hypersurface in $M_n(c)$ will be denoted by A, and the induced almost contact metric structure of the real hypersurface by $(\phi, <, >, \xi, \eta)$. The purpose of this paper is to prove that is no ruled real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$, who satisfies $R_\xi \phi = \phi R_\xi$ on M.

0. Introduction.

A complex n-dimensional Kaeherian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according to $c > 0$, $c = 0$ or $c < 0$. The shape operator or second fundamental tensor of a real hypersurface in $M_n(c)$ will be denoted by A, and the induced almost contact metric structure of the real hypersurface by $(\phi, <, >, \xi, \eta)$.

R. Takagi([9]) classified all homogeneous real hypersurfaces in $P_n(\mathbb{C})$ into six model spaces A_1, A_2, B, C, D and E (see also [10]). J. Berndt([2]) has completed the classification of homogeneous real hypersurfaces with principal structure vector fields ξ in $H_n(\mathbb{C})$, which are

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divided into the model spaces A_0, A_1, A_2 and B. A real hypersurface of type A_1 or A_2 in $P_n(\mathbb{C})$ or that of A_0, A_1 or A_2 in $H_n(\mathbb{C})$ is said to be *of type A* for simplicity.

On the other hand, it is well known([1] and [3]) that there exists a real hypersurface M in $M_n(c)$, $c \neq 0$ and $n \geq 2$, whose type number is equal to 2, and called it a *ruled real hypersurface.* It is also known that there is no complete ruled real hypersurface in $P_n(\mathbb{C})$ ([3]). On the other hand, B.H. Kim, I.B. Kim and R. Takagi have revealed that there is a homogeneous ruled real hypersurface in $H_n(\mathbb{C})$ ([10]).

A typical characterization for a real hypersurface M of type A in a complex space form $M_n(c)$ was given under the condition

$$\begin{aligned} &< (A\phi - \phi A)X, Y >= 0 \\ &\text{for any tangent vector fields } X \text{ and } Y \text{ on } M \end{aligned} \tag{0.1}$$

by M. Okumura([7]) for $c > 0$ and S. Montiel and A. Romero([5]) for $c < 0$. Namely,

Theorem A([5], [7]). *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If it satisfies (0.1), then M is locally congruent to a real hypersurface of type A.*

Let R be the Riemannian curvature tensor field of a real hypersurface M. We define the Jacobi operator field $R_X = R(\cdot, X)X$ with respect to a unit vector field X on M. Then we see that R_X is a self-adjoint endomorphism on the tangent space of M. It is related with the Jacobi vector fields, which are solutions of a certain second order differential equation, say the Jacobi equation, and it is well known that the notion of Jacobi vector fields involve many important geometric properties.

The purpose of this paper is to give characterizations of ruled real hypersurfaces and real hypersurfaces of type A with certain Jacobi operators. Namely we shall prove

Theorem 1. There is no ruled real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$, who satisfies $R_\xi \phi = \phi R_\xi$ on M.

Theorem 2. Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If M satisfies

$$< (A\phi - \phi A)X, Y >= 0 \qquad \text{and} \qquad < (R_\xi \phi - \phi R_\xi)X, Y >= 0$$

for any vector fields X and Y in the holomorphic distribution T_0, then M is locally congruent to a real hypersurface of type A.

1. Preliminaries.

Let M be a real hypersurface immersed in a complex space form $(M_n(c), <,>, J)$ of constant holomorphic sectional curvature c, and let N be a unit normal vector field on an open neighborhood in M. For a local tangent vector field X on the neighborhood, the images of X and N under the almost complex structure J of $M_n(c)$ can be expressed by

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where ϕ defines a linear transformation on the tangent space $T_p(M)$ of M at any point $p \in M$, and η and ξ denote a 1-form and a unit tangent vector field on the neighborhood respectively. Then, denoting the Riemannian metric on M induced from the metric on $M_n(c)$ by the same symbol $<,>$, it is easy to see that

$$< \phi X, Y > + < \phi Y, X >= 0, \qquad < \xi, X >= \eta(X)$$

for any tangent vector field X and Y on M. The collection (ϕ, $<,>$, ξ, η) is called an *almost contact metric structure* on M, and satisfies
(1.1)

$$\phi^2 X = -X + \eta(X)\xi, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1,$$
$$< \phi X, \phi Y >=< X, Y > -\eta(X)\eta(Y).$$

Let ∇ be the Riemannian connection with respect to the metric $<,>$ on M, and A be the shape operator in the direction of N on M. Then we have

$$\nabla_X \xi = \phi AX, \qquad (\nabla_X \phi)Y = \eta(Y)AX - < AX, Y > \xi. \tag{1.2}$$

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are given by

(1.3)

$$\begin{aligned} &R(X,Y)Z \\ =&\frac{c}{4}\{< Y, Z > X - < X, Z > Y + < \phi Y, Z > \phi X - < \phi X, Z > \phi Y \\ &- 2 < \phi X, Y > \phi Z\} + < AY, Z > AX - < AX, Z > AY, \end{aligned}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2 < \phi X, Y > \xi\} \tag{1.4}$$

for any tangent vector fields X, Y and Z on M, where R is the Riemannian curvature tensor of M. It follows from (1.1) and (1.3) that the Jacobi operator R_ξ is given by

$$\begin{aligned} R_\xi X =& -\{(\frac{c}{4} + \alpha^2)\eta(X) + \alpha\beta < U, X >\}\xi \\ &- \beta\{\alpha\eta(X) + \beta < U, X >\}U + \frac{c}{4}X + \alpha AX \end{aligned} \tag{1.5}$$

for any tangent vector field X on M.

The *holomorphic distribution* T_0 of a real hypersurface M in $M_n(c)$ is defined by

$$T_0(p) = \{X \in T_p(M) \mid < X, \xi >_p = 0\},$$

where $T_p(M)$ is the tangent space of M at p. It is clear that T is holomorphic for ϕ.

If the vector field $\phi\nabla_\xi\xi$ does not vanish, that is, the length β of $\phi\nabla_\xi\xi$ is not equal to zero, then it is easily seen from (1.1) and (1.2) that

$$A\xi = \alpha\xi + \beta U, \tag{1.6}$$

where $\alpha = < A\xi, \xi >$ and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi$. Therefore U is a unit tangent vector field on M and $U \in T_0$. The following is well-known and used later.

Theorem B([4] and [8]). Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. Then M is a ruled real hypersurface if and only if it satisfies

$$A\xi = \alpha\xi + \beta U,$$
$$AU = \beta\xi, \tag{1.7}$$
$$AX = 0$$

for any scalar functions α and $\beta(\neq 0)$ on M, where the vector field X is in T_0 and orthogonal to U.

2. Proof of Theorems.

In this section, we shall prove the Theorems 1 and 2 given in Introduction. First of all, it follows from (1.1) and (1.5) that

(2.1)
$$(R_\xi\phi - \phi R_\xi)X = -\alpha\beta < U, \phi X > \xi - \beta^2 < U, \phi X > U$$
$$+ \beta\{\alpha\eta(X) + \beta < U, X >\}\phi U + \alpha(A\phi - \phi A)X$$

for any tangent vector field X on M.

Proof of Theorem 1. Assume that there is a ruled real hypersurface M satisfying

$$(R_\xi\phi - \phi R_\xi)X = 0 \tag{2.2}$$

for any tangent vector field X on M. Since $\beta \neq 0$ on M, the equation (2.1) holds on M. Putting $X = \phi U$ into (2.1) and using (1.1) and (2.2), we have

$$\alpha AU + \alpha\phi A\phi U = \alpha\beta\xi + \beta^2 U. \tag{2.3}$$

Since M is a ruled real hypersurface, we have $AU = \beta\xi$ and $A\phi U = 0$ by Theorem B. Hence (2.3) is reduced to $\beta = 0$, and this is a contradiction. □

Proof of Theorem 2. We assume that there is a point p of M such that $\beta(p) \neq 0$. Then there exists an open neighborhood $\mathcal{U}$ of p such that $\beta \neq 0$ on $\mathcal{U}$. Hence the equation (2.1) holds on $\mathcal{U}$. For any vector fields X and Y in T_0, (2.1) is rewritten by

$$\begin{aligned} &< (R_\xi\phi - \phi R_\xi)X, Y > \\ &= -\beta^2\{< U, \phi X >< U, Y > - < U, X >< Y, \phi U >\} \\ &+ < (A\phi - \phi A)X, Y > . \end{aligned} \tag{2.4}$$

Under the our assumptions, it follows from (2.4) that

$$\beta^2\{< U, X >< \phi U, Y > + < \phi U, X >< U, Y >\} = 0 \tag{2.5}$$

for any vector fields X and Y in T_0. Putting $X = U$ and $Y = \phi U$ into (2.5), we obtain $\beta = 0$ on $\mathcal{U}$ and this is a contradiction.

Therefore $\beta = 0$ on the whole M, and hence the structure vector field ξ must be principal by (1.6), that is,

$$A\xi = \alpha\xi. \tag{2.6}$$

We see from (1.1) and (2.6) that $(A\phi - \phi A)\xi = 0$, which together with our assumption $< (A\phi - \phi A)X, Y >= 0$ for $X,\ Y \in T_0$ imply (0.1), that is, $A\phi = \phi A$ on M. Thus our results follows from Theorem A. □

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