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ON THE CLASS OF kTH ROOTS OF PARANORMAL OPERATORS

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Abstract. we shall study some properties of a new class $(\sqrt[k]{P})$ (defined below). Also we show that T may not be normaloid when $T \in (\sqrt[k]{P})$, and that the class $(\sqrt{H})$ may not have the translation-invariant propety.

1. Introduction

Let H be a Hilbert space and let $B(H)$ be the set of all bounded linear operators on H. The following non-normal operators have been defined as follows; An operator $T \in B(H)$ is called *hyponormal* if $T^*T - TT^* \geq 0$, or equivalently $\|Tx\| \geq \|T^*x\|$ for $x \in H$; *paranormal* or equivalently of class N if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every $x \in H$, and *normaloid* if $\|T\| = r(T)$, where $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ denotes the spectral radius of T. Let (H) and (P) denote the class of hyponormal and paranormal operators respectively.

For a positive integer k, we say that an operator $T \in B(H)$ is a *kth root of a G-operator* if T^k is a G-operator([10]). In particular, if a G-operator is paranormal, then T is called a kth root of a paranormal operator if T^k is paranormal. We denote this class by $(\sqrt[k]{P})$. In particular, if $k = 1$, the class $(\sqrt[k]{P})$ becomes the class of paranormal operators and the class $(\sqrt{P})(= (\sqrt[2]{P})$ consists of square roots of hyponormal operators.

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In this paper, we shall study some properties of a new class $(\sqrt[k]{P})$ of kth roots of paranormal operators. Also we show that T may not be normaloid when $T \in (\sqrt[k]{P})$, and that the class $(\sqrt{H})$ may not have the translation-invariant propety. We give an example of a compact operator in $(\sqrt[k]{P})$ which is not normal.

2. Properties of kth roots of a paranormal operator

Lemma 2.1([1],[3],[8]) Let T be a paranormal operator. Then

T^k is paranormal and so T is in $(\sqrt[k]{P})$ for every positive integer k.

If T is quasinilpotent, then T is a zero operator.

T is normaloid.

We give an example of a $k-$th root of a paranormal operator.

Example 2.2 Let H be k-dimensional Hilbert space. Define T on H as

$$T = (a_{ij})$$

where $a_{ij} = 0$ if $i \geq j$ and $a_{ij} = 1$ if $i < j$. Then T^n is hyponormal and so T is a kth root of a hyponormal operator. Thus $T \in (\sqrt[k]{P})$. But $TT^* \neq T^*T$. Therefore T is not hyponormal since every hyponormal operator on a finite dimensional Hilbert space is normal.

From the above Example 2.2, we can deduce that if T is any nilpotent operator of order k, i.e., $T^k = 0$, then T is a kth root of a paranormal operator, but it is not necessarily a paranormal operator. Also it is well-known ([4]) that T^2 may not be hyponormal when T is hyponormal. For example, if U is the unilateral shift on l^2 , and $T = U^* + 2U$, then

$$T^*T - TT^* = 3I - 3UU^* = 3(I - UU^*) > 0.$$

Therefore T is hyponormal, Howerver, if we take $x = (1, 0, -2, 0, \cdots)$, then

$$\|T^2x\| = 80 < 89 = \|(T^*)^2x\|^2.$$

Hence T^2 is not hyponormal.

Lemma 2.3 Let T be a weighted shift with nonzero weights $\{\alpha_n\}$ $(n = 0, 1, 2, \cdots)$. Then T is a kth root of a paranormal operator if and only if

$$|\alpha_n||\alpha_{n+1}| \cdots |\alpha_{n+k-1}| \le |\alpha_{n+k}||\alpha_{n+k+1}| \cdots |\alpha_{n+2k-1}|$$

for $n = 0, 1, 2, \cdots$

Proof. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of a Hilbert space H. Since

$$T^k e_n = \alpha_n \alpha_{n+1} \cdots \alpha_{n+(k-1)} e_{n+k}$$

and

$$T^{2k} e_n = \alpha_n \alpha_{n+1} \cdots \alpha_{n+(k-1)} \alpha_{n+k} \cdots \alpha_{n+(2k-1)} e_{n+2k}$$

for $n = 0, 1, 2, \cdots$, we know that T^k is paranormal if and only if $\|T^k e_n\|^2 \le M\|T^{2k}e_n\|$ $(n = 0, 1, 2, \cdots)$. Hence $T \in (\sqrt[k]{P})$ if and only if

$$|\alpha_n||\alpha_{n+1}| \cdots |\alpha_{n+k-1}| \le |\alpha_{n+k}||\alpha_{n+k+1}| \cdots |\alpha_{n+2k-1}|$$

for $n = 0, 1, 2, \cdots$ □

Corollary 2.4([6]) Let T be weighted shift with nonzero weights $\{\alpha_n\}_{n=0}^{\infty}$. Then $T \in (\sqrt[k]{H})$ if and only if

$$|\alpha_{n-k}| \cdots |\alpha_{n-1}| \le |\alpha_n| \cdots |\alpha_{n+k-1}|$$

for $n = k, k+1, \cdots$.

Example 2.5 Let T_x be the weighted shift with nonzero weights

$$\alpha_0 = x, \alpha_1 = \sqrt{\frac{2}{3}}, \alpha_2 = \sqrt{\frac{3}{4}}, \cdots, \alpha_n = \sqrt{\frac{n+1}{n+2}} \quad (n = 1, 2, \cdots)$$

Then by the direct calculation from above Lemma, $T_x \in (\sqrt[k]{P})$ if and only if $0 < x \leq \sqrt{\frac{(k+1)^2}{2(2k+1)}}$.

We observe that T_x is a $(k+1)$th of a paranormal operator, but is not a kth root of a paranormal operator if $\sqrt{\frac{(k+1)^2}{2(2k+1)}} < x \leq \sqrt{\frac{(k+2)^2}{2(2k+3)}}$. In particular, T_x is a kth of a paranormal operator, but is not a paranormal operator if $\sqrt{\frac{2}{3}} < x \leq \sqrt{\frac{(k+2)^2}{2(2k+1)}}$.

Next the folowing theorem generalizes some properties of an operator in $(\sqrt[k]{H})$

Theorem 2.6 Let $T \in (\sqrt[k]{P})$ be any kth root of a paranormal operator. Then

(1) $\lambda T \in (\sqrt[k]{P})$ for any $\lambda \in \mathbb{C}$.

(2) If T is quasinilpotent, then T is nilpotent.

(3) If M is invariant subspace of T, then $T|_M \in (\sqrt[k]{P})$.

(4) If T is invertible, then $T^{-1} \in (\sqrt[k]{P})$.

(5) If T is unitarily equivalent to S, then $S \in (\sqrt[k]{P})$.

Proof. (1) This follows from the fact that λT is paranormal if T is paranormal.

(2) Since T is quasinilpotent, $\sigma(T) = \{0\}$. By the spectral mapping theorem, we get that

$$\sigma(T^k) = \{\sigma(T)\}^k = \{0\}.$$

Hence T^k is quasinilpotent. Since T^k is paranormal and quasinilpotent, by Lemma 2.1, T^k is a zero operator and hence T is nilpotent.

(3) Since $(T|_M)^k = T^k|_M$ and $T^k|_M$ is paranormal, $T|_M \in (\sqrt[k]{P})$.

(4) By hypothesis. T^k is an invertible paranormal operator and so $(T^k)^{-1}$ is paranormal. Hence $(T^{-1})^k$ is paranormal i.e., T^{-1} is a kth root of a paranormal operator.

(5) Since T is unitarily equivalent to S, there exists a unitary operator U such that $S = U^*TU$. Thus $T^k = (U^*SU)^k = U^*S^kU$ and so T^k is unitarily equivalent to S^k. Since T^k is paranormal by hypothesis, S^k is paranormal and hence $S \in (\sqrt[k]{P})$. □

Corollary 2.7 ([6]) Let $T \in (\sqrt[k]{H})$ be any kth root of a hyponormal operator. Then

(1) $\lambda T \in (\sqrt[k]{H})$ for any $\lambda \in \mathbb{C}$.

(2) If T is quasinilpotent, then T is nilpotent.

(3) If M is invariant subspace of T, then $T|_M \in (\sqrt[k]{H})$.

(4) If T is invertible, then $T^{-1} \in (\sqrt[k]{H})$.

(5) If T is unitarily equivalent to S, then $S \in (\sqrt[k]{H})$.

We recall that the smallest positive integer n, for which $N(T^n) = N(T^{n+1})$ is *the ascent of* T, where $N(T)$ denotes the null space of T. It is well known that the ascent of a normal operator is 0 or 1. I.H. Sheth([9, Theorem 1]) has proved that if T is hyponormal then the ascent of T is 0 or 1. We generalize this result to paranormal operators.

Lemma 2.8 If T is paranormal, then the ascent of T is 0 or 1.

Proof. Let x be any vector in $N(T^2)$. Then $T^2x = 0$. Since T is paranormal, we have $\|Tx\|^2 \leq \|T^2x\|\|x\| = 0$ and so $Tx = 0$ i.e., $x \in N(T)$. Hence $N(T^2) \subseteq N(T) \subseteq N(T^2)$. This complete the result. □

Theorem 2.9 The class $(\sqrt[k]{P})$ is a proper subclass of $B(H)$.

Proof. Let T be any operator in $(\sqrt[k]{P})$. Then T^k is a paranormal operator. By Lemma 2.8, we get $\ker T^k = \ker T^{2k}$, and so $\ker T^k = \ker T^{k+1}$ since $\ker T^k \subseteq \ker T^{k+1} \subseteq \cdots \subseteq \ker T^{2k}$.

Let U^* be any unilateral backward shift on $l^2(\mathbb{N})$. Since $\ker(U^*)^k \neq \ker(U^*)^{k+1}$ for any $k \in \mathbb{N}$, $(U^*)^k$ is not paranormal. Therefore U^* is not a $n-$th root of a paranormal operator. □

Theorem 2.10 If $T \in B(H)$ is a kth root of a paranormal operator, then $T^n \in (\sqrt[k]{P})$ for every positive integer n.

Proof. By hypothesis T^k is paranormal and so $(T^n)^k = (T^k)^n$ is paranormal for every positive integer k. Hence T^n is a $k-$th root of a paranormal operator for every positive integer n. □

Next we characterize a matrix on two dimensional complex Hilbert space which is $(\sqrt[k]{H})$. Since every matrix on finite dimensional complex Hilbert space is unitarily equivalent to a upper triangular matrix and a kth root of a hyponormal operator is unitarily invariant by Corollary 2.7(5), it suffices to characterize a upper triangular matrix T. From the direct calculation, we get the following characterization.

Theorem 2.11 For $k \geq 2$ we have

$$T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} x \in (\sqrt[k]{H}) \Longleftrightarrow b(a^{k-1} + a^{k-2}c + \cdots + c^{k-1}) = 0.$$

We observe that $(\sqrt[k]{H})$ is not necessarily normal on a finite dimensional space.

Example 2.12 If $k = 3$ in the above Theorem 2.11, then

$$T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in (\sqrt[3]{H}) \Longleftrightarrow b(a^2 + ac + c^2) = 0.$$

Take $a = 2, b = 1,$ and$c = -1 + \sqrt{3}i$. Then

$$T = \begin{pmatrix} 2 & 1 \\ 0 & -1 + \sqrt{3}i \end{pmatrix} \in (\sqrt[3]{H}),$$

but T is not a normal operator.

By [8], we know that the set of all hyponormal operators on H is closed in the norm topology.

Theorem 2.13 The class $(\sqrt[k]{H})$ is closed in the norm topology.

Proof. Let T_n be a kth root of a hyponormal operator for each positive integer n and let $\{T_n\}$ converge to an operator T in norm. Then $\{T_n^k\}$ converge to an operator T^k in norm. Since the set of all hyponormal operators is closed in the norm topology and T_n^k are hyponormal, T^k is hyponormal and hence $T \in (\sqrt[k]{H})$ □

Every hyponormal operator has translation-invariant property, i.e., If T is hyponormal, then $T - \lambda$ is hyponormal for every complex number λ. But the class $(\sqrt{H})$ of square roots of hyponormal operators may not have the translation-invariant property. For example, if

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then $T^2 = 0$ is hyponormal, and so T is the square root of a hyponormal operator. But $(T-\lambda)^{*2}(T-\lambda)^2 - (T-\lambda)^2(T-\lambda)^{*2}$ is not positive by the direct calculation. Hence $(T-\lambda)^2$ is not necessarily hyponorml and so $T - \lambda$ is not necessarily the square root of a hyponormal operator.

By [2], we recall that an operator T is paranormal if and only if $T^{*2}T^2 + 2\lambda T^*T + \lambda^2 I \geq 0$ for all real λ.

Theorem 2.14 If T is a kth root of a paranormal operator and commutes with an isometric operator S, then TS is also a kth root of paranormal

Proof. If $A = (TS)^k = T^kS^k$, then using $TS = ST, T^*S^* = S^*T^*$ and $S^*S = I$, we get

$$A^{*2}A^2 + 2\lambda A^*A + \lambda^2 I = {T^k}^{*2}{T^k}^2 + 2\lambda {T^k}^* T^k + \lambda^2 I \geq 0$$

for any real λ since T^k is paranormal. Hence $A = (TS)^k$ is paranormal and so $TS \in (\sqrt[k]{P})$. □

If T is paranormal, then T is normaloid i.e., $\|T^n\| = \|T\|^n$ for each natural number n. But the converse is not true. This is not true in the case of a kth root of a paranormal operator. This can be seen as follows; Let T be the operator on a k dimensional Hilbert space H in Example 2.2. Then T^k is hyponormal and so $T \in (\sqrt[k]{H})$. Hence $T \in (\sqrt[k]{P})$ is a kth root of a paranormal operator and $\|T^k\| = 0$. But $\|T\|^k = 1$. Hence $\|T\|^k = 1 \neq \|T^k\| = 0$ i.e., T is not normaloid.

It is known that every hyponormal and compact operator is normal. But we observe that a square root of a hyponormal which is compact, is not necessarily a normal operator. For example, $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a square root of a hyponormal operator(and so $T \in (\sqrt{P})$), and T is a compact operator, but T is not necessarily a normal operator.

References

[1] T. Ando, Operators with a norm condition , Acta Sci. Math. 33(1972), 169-178

[2] S. C. Arora and R. Kumar, M-hyponormal operators. Publications De L'Institut Mathematique, 29(43)(1981) ,5-13

[3] T. Furuta, On the class of paranormal operators, Proc. Japan Acad. 43 (1967), 594-598

[4] P. R. Halmos, A Hilbert space problem book, Springer-Verlag, New York (1984)

[5] V. Istratescu, T. Saito and T. Yoshino, On a class of operators, Tohoku Math. Journ. 18(4)(1966), 410-413

[6] E. Ko, Properties of kth root of a hyponormal operator. Bull. Korean Math. Soc. 40(4)(2003), 685-692

[7] M. Martin and M. Putinar, Lectures on hyponormal operators, Operator Theory, Birkhauser-Verlag, (1989)

[8] N.C. Shah and I.H. Sheth, On paranormal operators, Mathematica Japonicae, 19(1)(1974), 79-81

[9] I.H. Sheth, Some results on hyponormal operators, Rev. Roum. Math. Pures, et Appl. 15(1970), 305-308

[10] Y. Yang and D. Kim, On generalized roots of hyponormal operators, Far East J. math. Sci(FJMS), 11(1)(2003) ,75-83

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