Honam Mathematical J. **26**(2004), pp. 45–59

SOLUTION AND STABILITY OF A GENERAL QUADRATIC FUNCTIONAL EQUATION IN TWO VARIABLES

EUN HWI LEE AND JO SEUNG LEE

Abstract. In this paper we obtain the general solution of the functional equation

$$a^2 f(\frac{x-2y}{a}) + f(x) + 2f(y) = 2a^2 f(\frac{x-y}{a}) + f(2y).$$

The type of the solution of this equation is $Q(x) + A(x) + C$, where $Q(x)$, $A(x)$ and C are quadratic, additive and constant, respectively. Also we prove the stability of this equation in the spirit of Hyers, Ulam, Rassias and Găvruta.

1. Introduction

In 1940 S.M. Ulam [22] raised the following question concerning the stability of homomorphisms;

Given a group G_1, a metric group G_2 with metric $d(\cdot,\cdot)$ and $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G_1$, then a homomorphism $g : G_1 \to G_2$ exists with $d(f(x), g(x)) \leq \epsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by Hyers [3] under the assumption that G_1 and G_2 are Banach spaces. Rassias [18] proved a substantial generalization of the result of Hyers and also Găvruta[2] obtained a further generalization of the Hyers-Rassias theorem. Later, many Rassias and Găvruta type theorems concerning the

Received Oct. 15, 2003. Revised Dec. 4, 2003.

2000 Mathematics Subject Classification: 39B72.

Key words and phrases: quadratic functional equation, stability.

stability of different funtional equations were obtained by numerous authers(see, for instance,[6,7,8,9,10,11,12,13,14,15, 16,17]).

In this paper we deal with a general quadratic functional equation in two variables

$$(1) \qquad a^2 f(\frac{x-2y}{a}) + f(x) + 2f(y) = 2a^2 f(\frac{x-y}{a}) + f(2y),$$

where a is a nonzero real number.

In 2002 Y. W. Lee[15] obtained the quadratic functional equation in three variables

$$\begin{aligned} & 9f(\frac{x+y+z}{3}) + f(x) + f(y) + f(z) \\ = \; & 4f[(\frac{x+y}{2}) + (\frac{y+z}{2}) + (\frac{z+x}{2})] \end{aligned}$$

which the solution is

$$f(x) = Q(x) + A(x) + c,$$

where $Q(x)$ is quadratic, $A(x)$ is additive and C is constant.

We show that the equation (1) in two variables has the same solution as that of the above equation in three variables but the method of proof is different. And then we investigate the stability of the equation (1) in the spirit of Hyers, Ulam, Rassias and Găvruta.

2. Solutions of the equation (1)

Theorem 2.1. Let X and Y be real linear spaces. A function $f : X \rightarrow Y$ satisfies (1) for all $x, y \in X$. Then there exist an element $c \in Y$, and an additive function $A : X \rightarrow Y$ and a quadratic function $Q : X \rightarrow Y$ such that

$$f(x) = Q(x) + A(x) + c$$

for all $x \in X$. In particular $c = 0$ if $a \neq \pm\sqrt{2}$, and $A = 0$ if a is an integer and $a \neq 1$.

Proof. Let $Q(x) := \frac{1}{2}\{f(x) + f(-x)\} - f(0)$, $A(x) := \frac{1}{2}\{f(x) - f(-x)\}$, and $c = f(0)$ for all $x \in X$. Then we get $A(0) = 0$, $A(-x) = -A(x)$, $Q(0) = 0$, $Q(-x) = Q(x)$,

$$a^2Q(\frac{x-2y}{a}) + Q(x) + 2Q(y) = 2a^2Q(\frac{x-y}{a}) + Q(2y) \tag{2}$$

and

$$a^2A(\frac{x-2y}{a}) + A(x) + 2A(y) = 2a^2A(\frac{x-y}{a}) + A(2y) \tag{3}$$

for all $x, y \in X$.

First we claim that Q is quadratic. That is, we show $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ for all $x, y \in X$. Putting $y = 0$ in (2), we get

$$a^2Q(\frac{x}{a}) = Q(x)$$

for all $x \in X$ and

$$Q(x - 2y) + Q(x) + 2Q(y) = 2Q(x - y) + Q(2y) \tag{4}$$

for all $x, y \in X$. Replacing y by x in (4), we have $4Q(x) = Q(2x)$ for all $x \in X$.

Thus we get

$$Q(x - 2y) + Q(x) = 2Q(x - y) + 2Q(y) \tag{5}$$

for all $x, y \in X$. Replacing $x - y$ by u and y by v in (5), we have

$$Q(u + v) + Q(u - v) = 2Q(u) + 2Q(v)$$

for all $u, v \in X$. Therefore Q is quadratic. Secondly we claim that A is additive. Letting $y = 0$ in (3), $a^2A(\frac{x}{a}) = A(x)$ and replacing y by x, we have $A(2x) = 2A(x)$ for all $x \in X$. Rewriting (3) we get

$$A(x - y) + A(x) = 2A(x - y) \tag{6}$$

for all $x, y \in X$. Replacing x by $2x$ in (6) we have

$$A(x - y) + A(x) = A(2x - y) \tag{7}$$

for all $x, y \in X$. Putting $x - y = u$ and $x = v$ in (7), we have

$$A(u) + A(v) = A(u + v)$$

for all $u, v \in X$. Thus A is additive. Thus $f(x) = Q(x) + A(x) + c$ is a solution of the equation (1), where $c = f(0)$.

Suppose that a is an integer and $a \neq 1$. Since $A(2x) = 2A(x)$, we have $n^2 A(\frac{x}{n}) = A(x)$, for all x and each integer n. Thus we get $a^2 A(\frac{x}{a}) = aA(\frac{x}{a})$ for all $x \in X$. Since $a \neq 0, 1$, we get $A(x) = 0$ for all $x \in X$. Also letting $x = y = 0$ in (1) we have

$$(a^2 - 2)f(0) = 0.$$

If $a \neq \pm\sqrt{2}$, we obtain $c = f(0) = 0$.

3. Stability of the equation (1)

Throughout this section X and Y will be a real normed linear space and a real Banach space, respectively. Let $\varphi : X \times X \to X$ be a mapping satisfying one of the conditions (a), (b) and one of the conditions (c), (d);

$$\Phi_1(x, y) := \sum_{i=1}^{\infty} \frac{1}{a^{2i}} \varphi(a^i x, a^i y) < \infty, \tag{a}$$

$$\Phi_2(x, y) := \sum_{i=0}^{\infty} a^{2i} \varphi(\frac{x}{a^i}, \frac{y}{a^i}) < \infty, \tag{b}$$

$$\Phi_3(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \varphi(2^{i-1} x, 2^{i-1} y) < \infty, \tag{c}$$

$$\Phi_4(x, y) := \sum_{i=1}^{\infty} 2^{i-1} \varphi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty \tag{d}$$

for all $x, y \in X$.

One of the condition (a),(b) will be needed to derive a quadratic function and one of the condition (c),(d) will be needed to derive an additive function in the following theorem.

Theorem 3.1. If the function $f : X \to Y$ satisfies

$$\|a^2 f(\frac{x-2y}{a}) + f(x) + 2f(y) - 2a^2 f(\frac{x-y}{a}) - f(2y)\| \le \varphi(x,y)$$

for all $x, y \in X$, then there exist a unique if a is an integer, quadratic function $Q : X \to Y$, a unique additive function $A : X \to Y$, and a unique element $c \in Y$ such that

$$\|f(x) - Q(x) - A(x) - c\| \le \varepsilon(x) + \delta(x),$$

$$\|\frac{f(x)+f(-x)}{2} - Q(x) - c\| \le \varepsilon(x),$$

and

$$\|\frac{f(x)-f(-x)}{2} - A(x)\| \le \delta(x)$$

for all $x \in X$, where $\epsilon(x) = \Phi_i(x,0)$, $i = 1$ or 2, and $\delta(x) = \Phi_j(x,0) + \Phi_j(x,x) + \Phi_j(x,-x)$, $j = 3$ or 4.

The function Q, A and the element c are given by

$$Q(x) = \begin{cases} \lim_{n\to\infty} \dfrac{f(a^n x) + f(-a^n x) - 2f(0)}{2 \cdot a^{2n}} & \text{if } \varphi \text{ satisfies (a)}, \\ \lim_{n\to\infty} \dfrac{a^{2n}}{2}\{f(\frac{x}{a^n}) + f(\frac{-x}{a^n}) - 2f(0)\} & \text{if } \varphi \text{ satisfies (b)}, \end{cases}$$

$$A(x) = \begin{cases} \lim_{n\to\infty} \dfrac{f(2^n x) - f(-2^n x)}{2 \cdot 2^n} & \text{if } \varphi \text{ satisfies (c)}, \\ \lim_{n\to\infty} \dfrac{2^n}{2}\{f(\frac{x}{2^n}) - f(\frac{-x}{2^n})\} & \text{if } \varphi \text{ satisfies (d)} \end{cases}$$

for all $x \in X$ and $f(0) = c$.

Proof. Let $f_1 : X \to Y$ be the function defined by $f_1(x) := \frac{1}{2}\{f(x) + f(-x)\} - f(0)$ for all $x \in X$. Then we get $f_1(0) = 0$, $f_1(x) = f_1(-x)$, and

$$\begin{aligned} &\|a^2 f_1(\frac{x-2y}{a}) + f_1(x) + 2f_1(y) - 2a^2 f_1(\frac{x-y}{a}) - f_1(2y)\| \\ (8) \quad &\le \frac{1}{2}\{\varphi(x,y) + \varphi(x,-y)\} \end{aligned}$$

for all $x, y \in X$. Putting $y = 0$ in (8), we have

$$(9) \qquad \|a^2 f_1(\frac{x}{a}) - f_1(x)\| \le \varphi(x,0)$$

for all $x \in X$. Replacing x by ax in (9) and dividing by a^2 we get

$$(10) \qquad \|f_1(x) - \frac{f_1(ax)}{a^2}\| \leq \frac{1}{a^2}\varphi(a^2x, 0)$$

for all $x \in X$.

Assume that φ satisfies the condition (a). Replacing x by $a^{n-1}x$ and dividing $a^{2(n-1)}$ in (10) we obtain

$$(11) \qquad \|\frac{f_1(a^{n-1}x)}{a^{2n-2}} - \frac{f_1(a^nx)}{a^{2n}}\| \leq \frac{1}{a^{2n}}\varphi(a^nx, 0)$$

for all $n \in N$ and all $x \in X$.

An induction argument implies that

$$(12) \qquad \|f_1(x) - \frac{f_1(a^nx)}{a^{2n}}\| \leq \sum_{i=1}^{n} \frac{1}{a^{2i}}\varphi(a^ix, 0)$$

for all $n \in N$ and $x \in X$.

Hence we get

$$\|\frac{f_1(a^nx)}{a^{2n}} - \frac{f_1(a^mx)}{a^{2m}}\| \leq \sum_{i=m+1}^{n} \frac{1}{a^{2i}}\varphi(a^ix, 0)$$

for all $n, m \in N$ with $n > m$ and $x \in X$. This shows that the sequence $\{\frac{f_1(a^nx)}{a^{2n}}\}$ is a Cauchy sequence for all $x \in X$, and thus converges. Therefore we can define a function $Q : X \to Y$ by

$$(13) \qquad Q(x) := \lim_{n\to\infty} \frac{f_1(a^nx)}{a^{2n}}$$

for all $x \in X$. Note that $Q(0) = 0$, $Q(-x) = Q(x)$, and $Q(ax) = a^2Q(x)$ for all $x \in X$. By (8) we have

$$\begin{aligned} &\|a^2Q(\frac{x-2y}{a}) + Q(x) + 2Q(y) - 2a^2Q(\frac{x-y}{a}) - Q(2y)\| \\ \leq\ & \lim_{n\to\infty} \frac{1}{2a^{2n}}\{\varphi(a^nx, a^ny) + \varphi(a^nx, -a^ny)\} \\ =\ & 0 \end{aligned}$$

for all $x, y \in X$.

Since $a^2 Q(\frac{x}{a}) = Q(x)$ for all $x \in X$, we get

$$Q(x-2y) + Q(x) + 2Q(y) - 2Q(x-y) - Q(2y) = 0 \tag{14}$$

for all $x, y \in X$. By the method of proof in Theorem 2.1, Q is quadratic.

Taking the limit in (12) we obtain

$$\|f_1(x) - Q(x)\| \leq \Phi_1(x, 0) \tag{15}$$

for all $x \in X$.

If Q' is an another quadratic function satisfying (15), then we have $Q'(0) = 0, Q'(2x) = 4Q'(x)$, and $Q'(-x) = Q'(x)$ for all $x \in X$.

Replacing y by $2x$ in $Q'(x+y) + Q'(x-y) = 2Q'(x) + 2Q'(y)$ we have $Q'(3x) + Q'(-x) = 2Q'(x) + 2Q'(2x)$ and so $Q'(3x) = 9Q'(x)$ for all $x \in X$. An induction argument implies $Q'(mx) = m^2 Q'(x)$ for all $x \in X$ and $m \in N$. If a is an integer, then we have

$$\begin{aligned} \|Q(x) - Q'(x)\| &\leq \|\frac{Q(a^n x)}{a^{2n}} - \frac{f_1(a^n x)}{a^{2n}}\| + \|\frac{f_1(a^n x)}{a^{2n}} - \frac{Q'(a^n x)}{a^{2n}}\| \\ &\leq \frac{2}{a^{2n}} \Phi_1(a^n x, 0) \\ &\leq \sum_{i=n+1}^{\infty} \frac{2}{a^{2i}} \varphi(a^i x, 0) \end{aligned}$$

for all $n \in N$ and $x \in X$.

Therefore we can conclude that $Q(x) = Q'(x)$ for all $x \in X$ if a is an integer.

If φ satisfies the condition (b), then the proof is analogous to that of case (a). Indeed, replacing x by $\frac{x}{a^{n-1}}$ and multiplying by $a^{2(n-1)}$ in (9) we have

$$\|a^{2n} f_1(\frac{x}{a^n}) - a^{2n-1} f_1(\frac{x}{a^{n-1}})\| \leq a^{2n-2} \varphi(\frac{x}{a^{n-1}}, 0)$$

for all $n \in N$ and $x \in X$. An induction argument implies

$$\|a^{2n} f_1(\frac{x}{a^n}) - f_1(x)\| \leq \sum_{i=0}^{n-1} a^{2i} \varphi(\frac{x}{a^i}, 0) \tag{16}$$

for all $n \in N$ and $x \in X$.

Hence we get

$$\|a^{2n}f_1(\frac{x}{a^n}) - a^{2m}f_1(\frac{x}{a^m})\| \le \sum_{i=m}^{n-1} a^i\varphi(\frac{x}{a^i})$$

for all $n, m \in N$ with $n > m$ and $x \in X$.

This shows that the sequence $\{a^{2n}f_1(\frac{x}{a^n})\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} a^{2n}f_1(\frac{x}{a^n}) \tag{17}$$

for all $x \in X$. Then we get $Q(0) = 0$, $Q(-x) = Q(x)$, and $Q(ax) = a^2Q(x)$ for all $x \in X$. By the same proof as that of case (a), we have

$$Q(x-2y) + Q(x) + 2Q(y) - 2Q(x-y) - Q(2y) = 0$$

for all $x, y \in X$, and so

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X$.

Taking the limit in (16) as $n \to \infty$, we obtain

$$\|f_1(x) - Q(x)\| \le \Phi_2(x, 0)$$

for all $x \in X$.

Also we easily have that Q is unique if a is an integer.

Now let $f_2 : X \to Y$ be the function defined by $f_2(x) := \frac{1}{2}[f(x) - f(-x)]$ for all $x \in X$. Then we have $f_2(0) = 0, f_2(-x) = -f_2(x)$ and

$$\begin{aligned} &\|a^2f_2(\frac{x-2y}{a}) + f_2(x) + 2f_2(y) - 2a^2f_2(\frac{x-y}{a}) - f_2(2y)\| \\ \le\ &\frac{1}{2}\{\varphi(x,y) + \varphi(x,-y)\} \end{aligned} \tag{18}$$

for all $x, y \in X$.

Putting $y = 0$ in (18) yields

$$\|a^2f_2(\frac{x}{a}) - f_2(x)\| \le \varphi(x, 0) \tag{19}$$

for all $x \in X$.

Replacing y by x in (18) we have

$$\|-a^2 f_2(\frac{x}{a})+3f_2(x)-f_2(2x)\| \le \frac{1}{2}\{\varphi(x,x)+\varphi(x,-x)\} \tag{20}$$

for all $x \in X$. By (19) and (20), we get

$$\begin{aligned} & \|2f_2(x)-f_2(2x)\| \\ \le\ & \|a^2 f_2(\frac{x}{a})-f_2(x)\| + \|-a^2 f_2(\frac{x}{a})+3f_2(x)-f_2(2x)\| \\ \le\ & \varphi(x,0)+\frac{1}{2}\{\varphi(x,x)+\varphi(x,-x)\} \\ \le\ & \varphi(x,0)+\varphi(x,x)+\varphi(x,-x) \qquad (21) \end{aligned}$$

for all $x \in X$.

Assume that φ satisfies the condition (c). Dividing by 2 in (21), we have

$$\|f_2(x)-\frac{f_2(2x)}{2}\| \le \frac{1}{2}\{\varphi(x,0)+\varphi(x,x)+\varphi(x,-x)\} \tag{22}$$

for all $x \in X$. Replacing x by $2^{n-1}x$ and dividing 2^{n-1} in (22) we obtain

$$\begin{aligned} & \|\frac{f_2(2^{n-1}x)}{2^{n-1}}-\frac{f_2(2^n x)}{2^n}\| \\ \le\ & \frac{1}{2^n}\{\varphi(2^{n-1}x,0)+\varphi(2^{n-1}x,2^{n-1}x)+\varphi(2^{n-1}x,-2^{n-1}x)\} \qquad (23) \end{aligned}$$

for all $n \in N$ and $x \in X$.

An induction argument implies

$$\begin{aligned} & \|f_2(x)-\frac{f_2(2^n x)}{2^n}\| \\ \le\ & \sum_{i=1}^{n}\frac{1}{2^i}\{\varphi(2^{i-1}x,0)+\varphi(2^{i-1}x,2^{i-1}x)+\varphi(2^{i-1}x,-2^{i-1}x)\} \qquad (24) \end{aligned}$$

for all $n \in N$ and $x \in X$.

Hence we get

$$\begin{aligned} & \|\frac{f_2(2^n x)}{2^n}-\frac{f_2(2^m x)}{2^m}\| \\ \le\ & \sum_{i=m+1}^{n}\frac{1}{2^i}\{\varphi(2^{i-1}x,0)+\varphi(2^{i-1}x,2^{i-1}x)+\varphi(2^{i-1}x,-2^{i-1}x)\} \end{aligned}$$

for all $n, m \in N$ with $n > m$ and $x \in X$. This shows that the sequence $\{\frac{f_2(2^n x)}{2^n}\}$ is a Cauchy sequence for all $x \in X \backslash \{0\}$ and thus converges. Therefore we can define a function $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{f_2(2^n x)}{2^n}$$

for all $x \in X$. Note that $A(0) = 0$, $A(-x) = -A(x)$ and $A(2x) = 2A(x)$ for all $x \in X$.

By (18) we have

$$a^2 A(\frac{x-2y}{a}) + A(x) + 2A(y) - 2a^2 A(\frac{x-y}{a}) - A(2y) = 0$$

for all $x, y \in X$. By the method of proof in Theorem 2.1, A is additive.

Taking the limit in (24) as $n \to \infty$, we have

$$\|f_2(x) - A(x)\| \le \Phi_3(x, 0) + \Phi_3(x, x) + \Phi_3(x, -x) \tag{25}$$

for all $x \in X$.

If A' is an another additive mapping satisfying (25), then we have

$$\begin{aligned} & \|A(x) - A'(x)\| \\ \le\ & \|\frac{A(2^n x)}{2^n} - \frac{f_2(2^n x)}{2^n}\| + \|\frac{f_2(2^n x)}{2^n} - \frac{A'(2^n x)}{2^n}\| \\ \le\ & \frac{2}{2^n}\{\Phi_3(2^n x, 0) + \Phi_3(2^n x, 2^n x) + \Phi_3(2^n x, -2^n x)\} \\ \le\ & 2\sum_{i=n+1}^{\infty} \frac{1}{2^i}\{\varphi(2^{i-1}x, 0) + \varphi(2^{i-1}x, 2^{i-1}x) + \varphi(2^{i-1}x, -2^{i-1}x)\} \end{aligned}$$

for all $n \in N$ and $x \in X$. Therefore conclude that $A(x) = A'(x)$ for all $x \in X$.

Assume that φ satisfies the condition (d). Replacing x by $\frac{x}{2}$ in (21) we get

$$\begin{aligned} \|2f_2(\frac{x}{2}) - f_2(x)\| &\le \varphi(\frac{x}{2}, 0) + \frac{1}{2}[\varphi(\frac{x}{2}, \frac{x}{2}) + \varphi(\frac{x}{2}, -\frac{x}{2})] \\ &\le \varphi(\frac{x}{2}, 0) + \varphi(\frac{x}{2}, \frac{x}{2}) + \varphi(\frac{x}{2}, -\frac{x}{2}) \end{aligned} \tag{26}$$

for all $x \in X$.

Replacing x by $\frac{x}{2^{n-1}}$ and multiplying by $2^{n-1}x$ in (26) we have

$$\begin{aligned}&\|2^n f_2(\frac{x}{2^n}) - 2^{n-1} f_2(\frac{x}{2^{n-1}})\| \\ \leq\ & 2^{n-1}[\varphi(\frac{x}{2^n}, 0) + \varphi(\frac{x}{2^n}, \frac{x}{2^n}) + \varphi(\frac{x}{2^n}, -\frac{x}{2^n})]\end{aligned}$$

for all $n \in N$ and $x \in X$.

The rest of the proof is similar to the corresponding part of the proof of the case (c). Thus there is a unique additive mapping $A : X \to Y$ such that

$$\|f_2(x) - A(x)\| \leq \Phi_4(x, 0) + \Phi_4(x, x) + \Phi_4(x, -x)$$

for all $x \in X$. Let $c = f(0)$. Since $f(x) = f_1(x) + f_2(x) + f(0)$ for all $x \in X$, it follows that

$$\begin{aligned}\|f(x) - Q(x) - A(x) - c\| &\leq \|f_1(x) - Q(x)\| + \|f_2(x) - A(x)\| \\ &\leq \Phi_i(x, 0) + \Phi_j(x, 0) + \Phi_j(x, x) + \Phi_j(x, -x) \\ &= \epsilon(x) + \delta(x)\end{aligned}$$

for all $x \in X$, $i = 1$ or 2, and $j = 3$ or 4. Thus we complete the proof.

Corollary 3.1. Let $p \neq 1, 2$ and θ be nonnegative real numbers. Suppose that the function $f : X \to Y$ satisfies

$$\begin{aligned}&\|a^2 f(\frac{x - 2y}{a}) + f(x) + 2f(y) - 2a^2 f(\frac{x - y}{a}) - f(2y)\| \\ \leq\ & \theta(\|x\|^p + \|y\|^p)\end{aligned}$$

for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ if a is an integer, a unique element $c \in Y$ such that

$$\|f(x) - Q(x) - A(x) - c\| \leq \theta\|x\|^p(\frac{a^p}{|a^2 - a^p|} + \frac{1}{|2^p - 2|}),$$

$$\|\frac{f(x) + f(-x)}{2} - Q(x) - c\| \leq \theta\|x\|^p \frac{a^p}{|a^2 - a^p|},$$

and

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \leq 3\theta\|x\|^p \frac{1}{|2^p - 2|}$$

for all $x \in X$.

Proof. Let $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then $\varphi(x,0) = \theta\|x\|^p$ and $\varphi(x,x) = \varphi(x,-x) = 2\theta\|x\|^p$ for all $x \in X$. If $p < 2$, then we have

$$\begin{aligned}\Phi_1(x,0) = \sum_{i=1}^{\infty} \frac{1}{a^{2i}}\varphi(a^i x, 0) &= \sum_{i=1}^{\infty} \theta\|x\|^p a^{i(p-2)} \\ &= \frac{a^p\theta\|x\|^p}{a^2 - a^p}\end{aligned}$$

for all $x \in X$.

If $p > 2$, then we get

$$\begin{aligned}\Phi_2(x,0) = \sum_{i=0}^{\infty} a^{2i}\varphi(\frac{x}{a^i}, 0) &= \sum_{i=0}^{\infty} \theta\|x\|^p a^{i(2-p)} \\ &= \frac{a^p\theta\|x\|^p}{a^p - a^2}\end{aligned}$$

for all $x \in X$.

If $p < 1$, then we have

$$\begin{aligned}&\Phi_3(x,0) + \Phi_3(x,x) + \Phi_3(x,-x) \\ = &\sum_{i=1}^{\infty} \frac{1}{2^i}[\varphi(2^{i-1}x, 0) + \varphi(2^{i-1}x, 2^{i-1}x) + \varphi(2^{i-1}x, -2^{i-1}x)] \\ = &\sum_{i=1}^{\infty} 3\theta\|x\|^p \frac{2^{i(p-1)}}{2^p} \\ = &3\theta\|x\|^p \frac{1}{2 - 2^p}\end{aligned}$$

for all $x \in X$.

If $p > 1$, then we get

$$\begin{aligned}&\Phi_4(x,0) + \Phi_4(x,x) + \Phi_4(x,-x) \\ = &\sum_{i=1}^{\infty} 2^{i-1}[\varphi(\frac{x}{2^i}, 0) + \varphi(\frac{x}{2^i}, \frac{x}{2^i}) + \varphi(\frac{x}{2^i}, -\frac{x}{2^i})] \\ = &\sum_{i=1}^{\infty} 3\theta\|x\|^p \frac{2^{i(1-p)}}{2} \\ = &3\theta\|x\|^p \frac{1}{2^p - 2}\end{aligned}$$

for all $x \in X$.

Thus we obtain

$$\varepsilon(x)+\delta(x)=\begin{cases}\theta\|x\|^p(\frac{a^p}{a^p-a^2}+\frac{3}{2^p-2}) & \text{if } p>2\\ \theta\|x\|^p(\frac{a^p}{a^2-a^p}+\frac{3}{2^p-2}) & \text{if } 1<p<2\\ \theta\|x\|^p(\frac{a^p}{a^2-a^p}+\frac{3}{2-2^p}) & \text{if } p<1\end{cases}$$

Corollary 3.2. Let $\theta > 0$ be a real number. If the function $f : X \to Y$ satisfies

$$\|a^2 f(\frac{x-2y}{a})+f(x)+2f(y)-2a^2 f(\frac{x-y}{a})-f(2y)\| \leq \theta$$

for all $x, y \in X$. Then there exist a (unique if a is an integer) quadratic function $Q : X \to Y$, a unique additive mapping $A : X \to Y$, and a unique element $c \in Y$ such that

$$\|f(x)-Q(x)-A(x)-c\| \leq \frac{\theta}{2}(\frac{1}{|a^2-2|}-3),$$

$$\|\frac{f(x)+f(-x)}{2}-Q(x)-c\| \leq \frac{\theta}{2|a^2-2|}$$

and

$$\|\frac{f(x)-f(-x)}{2}-A(x)\| \leq \frac{3}{2}\theta$$

for all $x \in X$.

References

[1] G. L. Forti, *Hyers-Ulam stability of functional eqautions in several variables*, Aequationes math. 50(1995), 146-190.

[2] D. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 184(1994), 431-436.

[3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. 27 (1941), 222-224.

[4] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser/ Boston/Basel/Berlin (1998).

[5] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. 44(1992), 125-153.

[6] K. W. Jun, G. H. Kim and Y. W. Lee, *Stability of generalized gamma and beta functional equations*, Aequationes Math. 60(2000), 15-24.

[7] S. M. Jung, *Hyers-Ulam-Rassias stability of functional equations*, Dynam. Systems Appl. 6(1997), 541-566.

[8] ———, *On the Hyers-Ulam stability of functional equations that have the quadratic property*, J. Math. Anal. Appl. 222(1998), 126-137.

[9] ———, *On the Hyers-Ulam-Rassias stability of a quadratic functional equation*, J. Math. Anal. Appl. 232(1999), 384-393.

[10] G. H. Kim, *On the stability of the quadratic mapping in normed space*, IJMMS. 25 No.4(2001), 217-229.

[11] E. H. Lee, Y. W. Lee and S. H. Park, *Stability of a Jensen type functional equations*, J. Appl. Math and comp. 10(1-2)(2002), 283-295.

[12] E. H. Lee, *On the solution of stability of the quadratic type functional equations*, Commu.(2003).

[13] I. S. Chang, E. H. Lee and H. M. Kim, *On Hyers-Ulam-Rassias stability of a quadratic functional equation*, Math. Ineq. and Appl. vol.6 No. 1 (2003), 87-95.

[14] Y. H. Lee and K. W. Jun, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation*, J. Math. Anal. Appl. 238(1999), 305-315.

[15] Y. W. Lee, *On the stability of a quadratic Jensen type functional equation*, J. Math. Anal. Appl. 270(2002), 596-601.

[16] K. W. Jun, H. M. Kim and D. O. Lee, *On stability of a quadratic functional equation*, J. Chungcheong Math. Soc. Vol.15 No.2(2002), 73-84.

[17] S. H. Lee and K. W. Jun, *Hyers-Ulam-Rassias stability of a quadratic type functional equation*, Bull. Korean. Math. Soc. 40(2003), 183-193.

[18] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978), 297-300.

[19] T. Trif, *Hyers-Ulam-Rassias stability of a quadratic functional equation*, Bull. Korean. Math. Soc.40(2003), 253-267.

[20] Th. M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. 251(2000), 264-284.

[21] ———, *On the stability of functional equations and a problem of Ulam*, Acta Applications Math. 62(2000), 23-130.

[22] S. M. Ulam, *Problems in Modern Mathematics*, Proc. Chapter VI. Wiley, New York (1964).

School of Natural Science,
Jeonju University,
Jeonju, 560-759
E-mail: ehl@jj.ac.kr