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MINIMAL DIGITAL PSEUDOTORUS WITH k-ADJACENCY, $k \in \{6, 18, 26\}$

SANG-EON HAN

Abstract. In this paper, three kinds of minimal digital pseudotori DT_6, DT'_{18} DT''_{26}, which are derived from the minimal simple 4- and 8-curves, MSC_4 and MSC'_8, are shown and are proved not to be digitally k-homotopy equivalent to each other, where $k \in \{6, 18, 26\}$. Furthermore, the digital topological properties of the minimal digital k-pseudotori are investigated in the digital homotopical point of view, where $k \in \{6, 18, 26\}$.

1. Introduction

Let $\mathbb{Z}$(resp. $\mathbb{N}$) represent the set of integers (resp. natural numbers) and let $\mathbb{Z}^n$ be the set of points in the Euclidean n-dimensional space with integer coordinates.

A *digital picture* is commonly represented as a quadruple $(\mathbb{Z}^n, k, \bar{k}, X)$, where $n \in \mathbb{N}$, $X \subset \mathbb{Z}^n$ is the set of finite points, k represents an adjacency relation for X, and $\bar{k}$ represents an adjacency relation for $\mathbb{Z}^n - X$ [1, 2, 8]. We say that the pair (X, k) is a *digital image*. For $a, b \in \mathbb{Z}$ with $a \lneq b$, the set $[a, b]_{\mathbb{Z}} = \{n \in \mathbb{Z} | a \leq n \leq b\}$ is called a *digital interval* with 2-adjacency [1].

The study on a digital image with a k-connectedness is an important part of discrete geometry. So far, digital images have been studied

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under the standard k-adjacency with relation to the digital (k_0, k_1)-continuity, the digital k-homotopy and the digital k-fundamental group, where $k, k_0, k_1 \in A_n := \{x \in \mathbb{N} | x = 3^n - 1 \text{ or } 2n\}$ for $n \in \mathbb{N}$, but not $n \neq 3$. For $n = 3$, $k, k_0, k_1 \in \{6, 18, 26\}$[1, 2, 8].

A digital k-fundamental group was studied in terms of the pointed digital homotopy [1] which is derived from the notion of digital continuity presented in [1, 2].

In this paper, we follow the notions of the digital continuity and the digital homotopy introduced in [1, 2].

The digital homeomorphism have come in use to the classification of digital images, to the study of a digital retract and an extension [2].

Digital images are now investigated with relation to digital (k_0, k_1)-continuity, digital (k_0, k_1)-homeomorphism [1, 2, 3, 4] and digital (k_0, k_1)-homotopy equivalence [5] with the following general adjacency relations, where $k_i \in \{3^n - 1 (n \geq 2), 18 (n = 3), 2n (n \geq 1)\}$, $i \in \{0, 1\}$.

In this paper, three kinds of minimal digital k-pseudotori in $\mathbb{Z}^3$ are studied, where $k \in \{6, 18, 26\}$. Namely, DT_6, DT'_{18} and DT''_{26} are derived from the minimal simple closed 4- and 8-curves, MSC_4 and MSC'_8 in $\mathbb{Z}^2$ [4].

Furthermore, the digital topological properties of the minimal digital 6, 18 and 26-pseudotori are investigated via their digital homotopical properties, digital k-contractibility and digital (k_0, k_1)-homotopy equivalence [5].

2. Definitions and preliminaries

The convenient digital (k_0, k_1)-continuity in terms of a digital k_i-connectedness with the standard k_i-adjacency was shown, $i \in \{0, 1\}$ [1]. Meanwhile, in order to study the pointed digital homotopy theory intensively, we need recall the digital (k_0, k_1)-continuity of [1, 2] with the general k-adjacency relations.

Definition 2.1. [1] In two digital pictures $(\mathbb{Z}^{n_0}, k_0, \overline{k}_0, X)$ and $(\mathbb{Z}^{n_1}, k_1, \overline{k}_1, Y)$, we say that a map $f : X \to Y$ is digitally (k_0, k_1)-continuous at $x \in X$ if f satisfies the following: For a given point $x \in X$ and every k_0-connected subset containing x, $O_{k_0}(x)$, $f(O_{k_0}(x))$ is k_1-connected, where $k_i \in \{3^{n_i} - 1(n_i \geq 2), 18(n_i = 3), 2n_i(n_i \geq 1)\}, i \in \{0, 1\}$[1].

If f is digitally (k_0, k_1)-continuous at any point $x \in X$, then f is called a digitally (k_0, k_1)-continuous map. □

For a digital image X with k- adjacency and its subimage A, we call (X, A) a digital image pair with k-adjacency. In two digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, (X, A))$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, (Y, B))$, we say that $f : (X, A) \to (Y, B)$ is digitally (k_0, k_1)-continuous if $f : X \to Y$ is digitally (k_0, k_1)-continuous and $f(A) \subset B$.

In a digital image $X \subset \mathbb{Z}^n$, two distinct points $x, y \in X$ are called k-connected [8] if there is a k-path $f : [0, m]_{\mathbb{Z}} \to X$ which the image is a sequence $(x_0, x_1, \cdots, x_m)$ from the set of points $\{f(0) = x_0 = x, f(1) = x_1, \cdots, f(m) = x_m = y\}$ such that x_i and x_{i+1} are k-adjacent, $i \in [0, m-1]_{\mathbb{Z}}, m \geq 1$. The length of a k-path is the number m above [1, 6].

In [1, 2], the digital homotopy was introduced, we now define the digital relative (k_0, k_1)-homotopy on A for some subimage A as follows.

Definition 2.2. Let $(X, k_0) \subset \mathbb{Z}^{n_0}$ and $(Y, k_1) \subset \mathbb{Z}^{n_1}$ be digital images, and $A \subset X$. Let $f, g : X \to Y$ be (k_0, k_1)-continuous functions. Suppose there exist $m \in \mathbb{N}$ and a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

- for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ is $(2, k_1)$-continuous for all $t \in [0, m]_{\mathbb{Z}}$;
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ is (k_0, k_1)-continuous for all $x \in X$; and
- for all $t \in [0, m]_{\mathbb{Z}}$, $F_t(x) = x$ for $x \in A$, *i.e.* the induced map F_t on

A is fixed.

Then we call F a relative (k_0, k_1)-homotopy on A between f and g, and we say that f and g are relatively (k_0, k_1)-homotopic on A in Y. □

Especially, if $A = \{x_0\} \subset X$, then we say that F is *a pointed* (k_0, k_1)-*homotopy* at $\{x_0\}$ [1].

Roughly, for $A \subset X$, digitally continuous functions $f, g : X \to Y$ are relatively homotopic on A if there is a continuous deformation of f with A fixed in Y and finally, the deformed function coincides with g.

If the identity map 1_X is relatively (k, k)-homotopic on $\{x_0\}$ in X to a constant map with image consisting of some $x_0 \in X$, then we say that (X, x_0) is *pointed* k*-contractible* [2].

Especially, for the case of a digital (k, k)-homotopy, we call it a digital k-homotopy and use the notation: $f \simeq_{d \cdot k \cdot h} g$ instead of $f \simeq_{d \cdot (k,k) \cdot h} g$.

Furthermore, if A is a singleton set $\{p\}$ in Definition 2.2, then (X, p) is called a pointed digital image [1].

Furthermore, we say that the image X is k-contractible if $1_X \simeq_{d \cdot k \cdot h} c_{\{x_0\}}$, where $c_{\{x_0\}}$ is a constant map for some $x_0 \in X$ [2].

We say that a digitally (k_0, k_1)-continuous function $f : X \to Y$ is k_1-nullhomotopic in Y if f is digitally k_1-homotopic in Y to a constant function $c_{\{y_0\}}$, $y_0 \in Y$ [1].

Concretely, for a pointed digital image (X, p), a k-loop f based at p is a k-path in X with $f(0) = p = f(m)$, where the number m depends on the k-path above. And we put $F_1^k(X, p) = \{f | f$ is a k-loop based at p$\}$.

For maps $f, g \in F_1^k(X, p)$, *i.e.*, $f : [0, m_1]_{\mathbb{Z}} \to (X, p)$ with $f(0) = p = f(m_1)$ and $g : [0, m_2]_{\mathbb{Z}} \to (X, p)$ with $g(0) = p = g(m_2)$, we get a map $f * g : [0, m_1 + m_2]_{\mathbb{Z}} \to (X, p)$ as follows:

$f * g : [0, m_1 + m_2]_{\mathbb{Z}} \to (X, p)$ is defined by

$f * g(t) = f(t), 0 \le t \le m_1$, and $g(t - m_1), m_1 \le t \le m_1 + m_2$. Then $f * g \in F_1^k(X, p)$[7].

We denote the digital k-homotopy class of f by $[f]$. Obviously, the homotopy class $[f * g]$ depends on the homotopy classes $[f]$ and $[g]$.

Furthermore, for any $f_1, f_2, g_1, g_2 \in F_1^k(X,p)$ such that $f_1 \in [f_2], g_1 \in [g_2]$, we get the map $f_1 * g_1 \in [f_2 * g_2]$, *i.e.*, $[f_1 * g_1] = [f_2 * g_2]$ [1].

Then $\pi_1^k(X,p) = \{[f] | f \in F_1^k(X,p)\}$ is a group with an operation, $[f] \cdot [g] = [f * g]$ [7], which is called the digital k-fundamental group of a pointed digital image (X,p) [1].

Actually, if p and q belong to the same k-connected component of X, then $\phi : \pi_1^k(X,p) \to \pi_1^k(X,q)$ is an isomorphism [1].

For digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$, $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$ and a digitally (k_0, k_1)- continuous based map $h : (X,p) \to (Y,q)$, the map h induces a digital fundamental group (k_0, k_1)-homomorphism [1] as follows.

Define $\pi_1^{(k_0,k_1)}(h) = h_* : \pi_1^{k_0}(X,p) \to \pi_1^{k_1}(Y,q)$ by the equation $h_*([f_1]) = [h \circ f_1]$, where $[f_1] \in \pi_1^{k_0}(X,p)$, which is well defined. Particularly, if $k_0 = k_1$, we use the following notation, $\pi_1^{k_0}(h)$ [1]. If X is k-contractible, then $\pi_1^k(X,p)$ is trivial [1].

3. Minimal simple closed k-curves and digital 26-pseudotori

For classifying digital images, we need special relations among digital images with k-adjacency relations. One of them is a digital (k_0, k_1)-homeomorphism as follows: For digital images X with k_0- adjacency, Y with k_1-adjacency, a map $h : X \to Y$ is called a *digital* (k_0, k_1)-*homeomorphism* if h is digitally (k_0, k_1)-continuous and bijective and further $h^{-1} : Y \to X$ is digitally (k_1, k_0)-continuous [3, 4]. Then we denote it by $X \approx_{d \cdot (k_0, k_1) \cdot h} Y$. If $k_0 = k_1$, we say that it a digital homeomorphism [1, 2].

For a digital image $X \subset \mathbb{Z}^n$, distinct two points $x, y \in X$ are called k-connected [8] if there is a k-path $f : [0, m]_{\mathbb{Z}} \to X$ whose image is a sequence $(x_0, x_1, \cdots, x_m)$ from the set of points $\{f(0) = x_0 = x, f(1) = x_1, \cdots, f(m) = x_m = y\}$ such that x_i and x_{i+1} are k-adjacent, $i \in [0, m-1]_{\mathbb{Z}}, m \geq 1$. The length of a k-path is the number m [8]. And a simple k-curve is considered as a sequence $(x_0, x_1, \cdots, x_m)$ of an image of the k-path such that x_i and x_j are k-adjacent if and only if $j = i + 1$ or $j = i - 1$ [1].

For one of the general k-adjacency relations on $\mathbb{Z}^n$, a *simple closed k-curve* in X [1] is the image of a $(2, k)$-continuous function $f : [0, m - 1]_{\mathbb{Z}} \to X$ such that $f(i)$ and $f(j)$ are k-adjacent if and only if either $j = i + 1(\text{mod} m)$ or $i = j + 1(\text{mod} m)$. And a *closed k-curve* in X is the image of a $(2, k)$-continuous function $f : [0, m - 1]_{\mathbb{Z}} \to X$ such that $f(i)$ and $f(j)$ are k-adjacent if either $j = i + 1(\text{mod} m)$ or $i = j + 1(\text{mod} m)$.

Now we introduce the minimal simple closed curves in $\mathbb{Z}^2$. For clarifying the digital k-connectedness, we use the subscript k for the denotation of the minimal simple closed k-curves by MSC_k or MSC'_k according to $k \in \{4, 8\}$, *i.e.*, MSC_8, MSC_4 and MSC'_8 [4]:

(1) Let MSC_8 be the set which is digitally homeomorphic to the image,

$\{(0, 0), (-1, 1), (-2, 0), (-2, -1), (-1, -2), (0, -1)\}$ [3, 4].

(2) Let MSC_4 be the set which is digitally homeomorphic to the image,

$\{(0, 0), (0, 1), (-1, 1), (-2, 1), (-2, 0), (-2, -1), (-1, -1), (0, -1)\}$ [3, 4].

(3) Let MSC'_8 be the set which is digitally homeomorphic to the image,

$\{(0, 0), (-1, 1), (-2, 0), (-1, -1)\}$ [3, 4].

Actually, MSC_8 is not 8-contractible [4] and MSC_4 and MSC'_8 are not 4-contractible either [4]. But MSC_4 and MSC'_8 are 8-contractible (Theorem 3.1).

Theorem 3.1 [1] The minimal simple closed 4-curve, MSC_4 is 8-contractible.

The minimal simple closed k-curves. MSC_8, MSC_4 and MSC'_8 above are distinct up to a digital homeomorphism [3, 4].

For the digital images X with k_1-adjacency and Y with k_2-adjacency, the product digital image $X \times Y = \{(x, y)|x \in X, y \in Y\}$ with k_3-adjacency is taken [6]. The k_3-adjacency depends on the k_1- and k_2-adjacency relations [6].

Actually, $X \times Y$ is digitally homeomorphic to $Y \times X$ with the k_t-adjacency [6] above.

Furthermore, from the minimal simple closed k-curves, MSC_4 and MSC'_8, the following product images are established [6]:

$(MSC_4 \times MSC_4, 32) \subset \mathbb{Z}^4$,

$(MSC_4 \times MSC'_8, 64) \subset \mathbb{Z}^4$ and

$(MSC'_8 \times MSC'_8, 80) \subset \mathbb{Z}^4$.

Moreover, we get the following minimal digital k-pseudotori in $\mathbb{Z}^3$ with relation to the digital homeomorphism, where $k \in \{6, 18, 26\}$, *i.e.*,

(4) $MSC_4 \times MSC_4 \approx_{d\cdot(32,6)\cdot h} DT_6$ in $(\mathbb{Z}^3, 6, 26, DT_6)$,

(5) $MSC_4 \times MSC'_8 \approx_{d\cdot(64,18)\cdot h} DT'_{18}$ in $(\mathbb{Z}^3, 18, 6, DT'_{18})$,

(6) $MSC'_8 \times MSC'_8 \approx_{d\cdot(80,26)\cdot h} DT''_{26}$ in $(\mathbb{Z}^3, 26, 6, DT''_{26})$,

For clarifying the digital k-connectivity of the minimal digital pseudotorus in $\mathbb{Z}^3$, where $k \in \{6, 18, 26\}$, we use the subscript k like DT_6, DT'_{18} and DT''_{26}.

We prove that the digital k-pseudotori in $\mathbb{Z}^3$, DT_6, DT'_{18} and DT''_{26} are not digitally 6-, 18-, or 26-homotopy equivalent to each other in section 5.

5. Digital topological properties of the digital 26-pseodotori

The notion of digital (k_0, k_1)-homotopy equivalence is now introduced in order to classify digital images.

Definition 5.1.[5] Given two digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$, if there are a digitally (k_0, k_1)-continuous map $h : X \to Y$ and a digitally (k_1, k_0)-continuous map $l : Y \to X$ such that $l \circ h \simeq_{d\cdot k_0 \cdot h} 1_X$

and $h \circ l \simeq_{d \cdot k_1 \cdot h} 1_Y$, then the map $h : X \to Y$ is called a digital (k_0, k_1)-homotopy equivalence. And we use the notation, $X \simeq_{d \cdot (k_0,k_1) \cdot h \cdot e} Y$. Furthermore, if $k_0 = k_1$, we call h a digital k_0-homotopy equivalence and denote it by $X \simeq_{d \cdot k_0 \cdot h \cdot e} Y$. □

Theorem 5.2 The minimal simple closed k-curves MSC_4, MSC_8 and MSC_8' are distinct up to the digital k-homotopy equivalence, $k \in \{4, 8\}$ except that $MSC_4 \simeq_{d \cdot 8 \cdot h \cdot e} MSC_8'$.

Proof. We can easily see the following cases: MSC_4 is not digitally 4- or 8-homotopy equivalent to MSC_8', MSC_4 is not digitally 4- or 8-homotopy equivalent to MSC_8 either, MSC_8 is not digitally 8-homotopy equivalent to MSC_8', and finally MSC_8 is not digitally 8-homotopy equivalent to MSC_4 either.

Finally, we only prove the following: $MSC_4 \simeq_{d \cdot 8 \cdot h \cdot e} MSC_8'$. Meanwhile, we can assume MSC_8' to be a subimage of MSC_4. Let us consider $MSC_4 = \{(0,0), (0,1), (-1,1), (-2,1), (-2,0), (-2,-1), (-1,-1), (0,-1)\}$ and assume $MSC_8' = \{(0,0), (-1,1), (-2,0), (-1,-1)\}$. Then we consider two digital continuous maps, $l : MSC_8' \to MSC_4$ as the inclusion and $h : MSC_4 \to MSC_8'$ is mapped as follows:

$h((0,1)) = (0,0)$, $h((-2,1)) = (-1,1)$, $h((-2,-1)) = (-2,0)$, $h((0,-1)) = (-1,-1)$ and for all point $p \in \{(0,0), (-1,1), (-2,0), (-1,-1)\}$, $h(p) = p$. Then we get $MSC_4 \simeq_{d \cdot 8 \cdot h \cdot e} MSC_8'$, as required. □

For three kinds of minimal digital k-pseudotori, where $k \in \{6, 18, 26\}$, DT_6, DT_{18}', and DT_{26}'', we get the digital k-fundamental groups of them. And we now prove that DT_6, DT_{18}' and DT_{26}'' are not digitally k-homotopy equivalent to each other, where $k \in \{6, 18, 26\}$.

Theorem 5.3 The group $\pi_1^k(DT''_{26}, t_0)$ is trivial, where $t_0 \in DT''_{26}$ and $k \in \{6, 18, 26\}$.

Proof. Since DT''_{26} is assumed to be 26-homeomorphic to $\cup_{i \in M} T_i$ below, where $M = [1, 4]_{\mathbb{Z}}$

$T_1 = \{t_0 = (0, 0, 0), (1, 0, 1), (2, 0, 0), (1, 0, -1)\}$,

$T_2 = \{(-1, 1, 0), (-1, 2, 1), (-1, 3, 0), (-1, 2, -1)\}$,

$T_3 = \{(-2, 0, 0), (-3, 0, 1), (-4, 0, 0), (-3, 0, -1)\}$ and

$T_4 = \{(-1, -1, 0), (-1, -2, 1), (-1, -3, 0), (-1, -2, -1)\}$,

DT''_{26} is proved to be 26-contractible from the similar method as proof of Theorem 3.1. And further, each point in DT''_{26} is distinct from each other with respect to the k-connectedness, where $k \in \{18, 6\}$. Then we get easily that $\pi_1^k(DT''_{26}, t_0)$ is group isomorphic to the trivial group, where $k \in \{18, 6\}$. □

Similarly, we observe that $\pi_1^k(DT_6, p_1)$ is a trivial group, where $k \in \{18, 26\}$, but $\pi_1^6(DT_6, p_1)$ is not abelian group for $p_1 \in DT_6[6]$.

Theorem 5.4 The minimal digital pseudotori, DT_6, DT'_{18} and DT''_{26}, are different from each other up to the digital k-homotopy equivalence, $k \in \{6, 18, 26\}$.

Proof. The digital (k_0, k_1)-homotopy equivalence preserves the digital k_0-contractibiltiy into k_1-contractibiltiy [5]. More precisely, DT''_{26} is 26-contractible, but DT''_{26} is not k-contractible, $k \in \{18, 6\}$. Further, DT''_{26} can not be digitally 26- or 6-homotopy equivalent to DT_6. Similarly, DT'_{18} must not be digitally 18- or 26-homotopy equivalent to DT''_{26} either, and DT'_{18} is not be digitally 18- or 6-homotopy equivalent to DT_6.

Moreover, since the digital (k_0, k_1)-homotopy equivalence preserves the digital k_0-fundamental group into the digital k_1-fundamental group

[5] we can see that DT_6, DT'_{18} and DT''_{26} are distinct from each other Theorems 5.3 and 5.4. □

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Department of Computer and Applied Mathematics,
College of Natural Science, Honam University,
Gwangju, 506 - 714, Korea
e-mail:sehan@honam.ac.kr,
Tel: 82-62-940-5421,
Fax: 82-62-940-5644