Honam Mathematical J. **26**(2004), No. 3, pp. 309–330

INTUITIONISTIC FUZZY IDEALS AND BI-IDEALS

KUL HUR[a], KWANG JIN KIM[b] AND HYEONG KEE SONG[a]

Abstract. In this paper, we apply the concept of intuitionistic fuzzy sets to theory of semigroups. We give some properties of intuitionistic fuzzy ideals and intuitionistic fuzzy bi-ideals, and characterize which is left [right] simple, left [right] duo and a semilattice of left [right] simple semigroups or another type of semigroups in terms of intuitionistic fuzzy ideals and intuitionistic fuzzy bi-ideals.

0. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [19], several researchers[7,11,12,17] have applied the notion of fuzzy sets to group theory. In particular, Kuroki[11,12] have applied the concept of fuzzy sets to the theory of semigroups.

As a generalization of fuzzy sets, Atanassov[1] inturoduced the concept of intuitionistic fuzzy sets. After that time, Çoker and his colleagues [5,6,8] introduced the concept of intuitionistic fuzzy topogical spaces using intuitionistic fuzzy sets. In 1989, Biswas[3] introduced the notion of intuitionistic fuzzy subgroups and studied some of it's properties. In 2003, Banerjee and Basnet[2], Hur and his colleagues[9,10] have applied the concept of intuitionistic fuzzy sets to algebra.

In this paper, we apply this to theory of semigroups. We give some properties of intuitionistic fuzzy ideals and intuitionistic fuzzy bi-ideals,

Received July 6, 2004; Revised August 26, 2004.

2000 Mathematics Subject Classification of AMS: 20N25, 20M17, 20M18 .

Key words and phrases: intuitionistic fuzzy set, intuitionistic fuzzy semigroup,intuitionistic fuzzy ideal, intuitionistic fuzzy bi-ideal, intuitionistic fuzzy duo.

and characterize which is left [right] simple, left [right] duo and a semilattice of left [right] simple semigroups or another type of semigroups in terms of intuitionistic fuzzy ideals and intuitionistic fuzzy bi-ideals.

1. Preliminaries

We will list some concepts and results needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I.

Definition 1.1[2]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) on X if $\mu_A + \nu_A \leq 1$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2[2]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\,]A = (\mu_A, 1 - \mu_A)$, $< > A = (1 - \nu_A, \nu_A)$.

Definition 1.3[4]. Let $\{A_i\}_{i\in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.

(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4[4]. $0_\sim = (0, 1)$ and $1_\sim = (1, 0)$.

2. Intuitionistic fuzzy ideals and bi-ideals of a semigroup

Let S be a semigroup. By a *subsemigroup* of S we mean a non-empty subset A of S such that

$$A^2 \subset A$$

and by a *left* [resp. *right*] ideal of S we mean a non-empty subset A of S such that

$$SA \subset A \text{ [resp. } AS \subset A\text{]}.$$

By *two-sided ideal* or simply *ideal* we mean a subset A of S which is both a left and a right ideal of S. A semigroup S is said to be *left*[resp. *right*] *simple* if S itself is the only left [resp. right] ideal of S. S is said to be *simple* if it contains no proper ideal.

Definition 2.1[9]. Let S be a semigroup and let $A \in IFS(S)$. Then A is called an :

(1) *intuitionistic fuzzy subsemigroup* (in short, IFSG) of S if

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \text{ and } \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$$

for any $x, y \in S$.

(2) *intuitionistic fuzzy left ideal* (in short, IFLI) of S if

$$\mu_A(xy) \geq \mu_A(y) \text{ and } \nu_A(xy) \leq \nu_A(y)$$

for any $x, y \in S$.

(3) *intuitionistic fuzzy right ideal* (in short, IFRI) of S if

$$\mu_A(xy) \geq \mu_A(x) \text{ and } \nu_A(xy) \leq \nu_A(x)$$

for any $x, y \in S$.

(4) *intuitionistic fuzzy (two-sided) ideal* (in short, IFI) of S if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal of S.

We will denote the set or all IFSGs [resp. IFLIs, IFRIs and IFIs] of S as $IFSG(S)$ [resp. $IFLI(S)$, $IFRI(S)$ and $IFI(S)$].

It is clear that $A \in IFI(S)$ if and only if

$\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$

for any $x, y \in S$, and if $A \in IFLI(S)$[resp. $IFRI(S)$ and $IFI(S)$], then $A \in IFSG(S)$.

Result 2.A[9, Proposition 3.8]. Let A be a non-empty subset of a semigroup S.

(1) A is a subsemigroup of S if and only if $(\chi_A, \chi_{A^c}) \in IFSG(S)$.

(2) A is a left [resp. right] ideal of S if and only if $(\chi_A, \chi_{A^c}) \in IFLI(S)$[resp. $IFRI(S)$].

(3) A is an ideal of S if and only if $(\chi_A, \chi_{A^c}) \in IFL(S)$.

Remark 2.2. Let S be a semigroup.

(1) If μ_A is a fuzzy subsemigroup of S, then $A = (\mu_A, \mu_A^c) \in IFSG(S)$.

(2) If $A \in IFSG(S)$ [resp. $IFI(S), IFLI(S)$ and $IFRI(S)$], then μ_A and ν_A^c are fuzzy subsemigroup [resp. ideal, left ideal and right ideal] of S.

(3) If $A \in IFSG(S)$ [resp. $IFI(S), IFLI(S)$ and $IFRI(S)$], then $[\,]A, <> A \in IFSG(S)$[resp. $IFI(S), IFLI(S)$and$IFRI(S)$].

Result 2.B[9, Proposition 3.7]. Let S be a semigroup and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. If $A \in$ IFSG(S) [resp. IFI(S), IFLI(S) and IFRI(S)], then $A^{(\lambda,\mu)}$ is a subsemigroup [resp. ideal, left ideal and right ideal] of S.

The following result is the converse of Result 2.B:

Proposition 2.3. Let S be a semigroup and let $A \in IFS(S)$. If $A^{(\lambda,\mu)}$ is a subsemigroup [resp. ideal, left ideal and right ideal] of S for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$, then $A \in$ IFSG(S) [resp. IFI(S), IFLI(S) and IFRI(S)].

Proof. Suppose $A^{(\lambda,\mu)}$ is a semigroup of S for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. For any $x, y \in S$, let $A(x) = (\lambda_1, \mu_1)$ and let $A(y) = (\lambda_2, \mu_2)$. Then

$$\mu_A(x) = \lambda_1 \geq \lambda_1 \wedge \lambda_2,\ \nu_A(x) = \mu_1 \leq \mu_1 \vee \mu_2$$

and

$$\mu_A(y) = \lambda_2 \geq \lambda_1 \wedge \lambda_2,\ \nu_A(y) = \mu_2 \leq \mu_1 \vee \mu_2.$$

Thus $x, y \in A^{(\lambda_1 \wedge \lambda_2, \mu_1 \vee \mu_2)}$. Since $\lambda_1 \wedge \lambda_2 + \mu_1 \vee \mu_2 \leq 1$, by the hypothesis, $xy \in A^{(\lambda_1 \wedge \lambda_2, \mu_1 \vee \mu_2)}$. So

$$\mu_A(xy) \geq \lambda_1 \wedge \lambda_2 = \mu_A(x) \wedge \mu_A(y)$$

and

$$\nu_A(xy) \leq \mu_1 \vee \mu_2 \leq \nu_A(x) \vee \nu_A(y).$$

Hence $A \in IFSG(S)$.

Now suppose $A^{(\lambda,\mu)}$ is a left ideal of S for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. For each $y \in S$, let $A(y) = (\lambda, \mu)$. Then clearly $y \in A^{(\lambda,\mu)}$. Let $x \in S$. Then, by the hypothesis, $xy \in A^{(\lambda,\mu)}$. Thus $\mu_A(xy) \geq \lambda = \mu_A(y)$ and $\nu_A(xy) \leq \mu = \nu_A(y)$. Hence $A \in IFLI(S)$. Also, we easily see the rest. This completes the proof.

A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subset A$. We will denote the set of all bi-ideals of S as BI(S).

Definition 2.4. Let S be a semigroup and let $A \in IFSG(S)$. Then A is called an *intuitionistic fuzzy bi-ideal* (in short, IFBI) of S if

$$\mu_A(xyz) \geq \mu_A(x) \wedge \mu_A(z) \text{ and } \nu_A(xyz) \leq \nu_A(x) \vee \nu_A(z)$$

for any $x, y, z \in S$.

We will denote the set of all IFBI of S as $IFBI(S)$. The following result shows that the concept of an IFBI in a semigroup is an extended one of a bi-ideal:

Proposition 2.5. Let A be a non-empty subset of a semigroup S. Then A is a bi-ideal of S if and only if $(\chi_A, \chi_{A^c}) \in IFBI(S)$.

Proof.($\Rightarrow$): Suppose $A \in BI(S)$ and let $x, y, z \in S$.

Case (i): Suppose $x \in A$ and $z \in A$. Then $\chi_A(x) = \chi_A(z) = 1$ and $\chi_{A^c}(x) = \chi_{A^c}(z) = 0$. Since A is a bi-ideal of S, $xyz \in ASA \subset A$. Thus $\chi_A(xyz) = 1 = \chi_A(x) \wedge \chi_A(z)$ and $\chi_{A^c}(xyz) = 0 = \chi_{A^c}(x) \vee \chi_{A^c}(z)$.

Case (ii): Suppose $x \notin A$ or $z \notin A$. Then $\chi_A(x) = 0$ and $\chi_{A^c}(x) = 1$ or $\chi_A(z) = 0$ and $\chi_{A^c}(z) = 1$. Thus $\chi_A(xyz) \geq 0 = \chi_A(x) \wedge \chi_A(z)$ and $\chi_{A^c}(xyz) \leq 1 = \chi_{A^c}(x) \vee \chi_{A^c}(z)$. So, in either cases,

$$\chi_A(xyz) \geq \chi_A(x) \wedge \chi_A(z) \text{ and } \chi_{A^c}(xyz) \leq \chi_{A^c}(x) \vee \chi_{A^c}(z).$$

Moreover, by Result 2.A (1), $(\chi_A, \chi_{A^c}) \in IFSG(S)$. Hence $(\chi_A, \chi_{A^c}) \in IFBI(S)$.

($\Leftarrow$): Suppose $(\chi_A, \chi_{A^c}) \in IFBI(S)$. Let $t \in ASA$. Then there exist $x, z \in A$ and $y \in S$ such that $t = xyz$. Since $x, z \in A$, $\chi_A(x) = \chi_A(z) = 1$ and $\chi_{A^c}(x) = \chi_{A^c}(z) = 0$. Since $(\chi_A, \chi_{A^c}) \in IFBI(S)$,

$$\chi_A(xyz) \geq \chi_A(x) \wedge \chi_A(z) = 1 \text{ and } \chi_{A^c}(xyz) \leq \chi_{A^c}(x) \vee \chi_{A^c}(z) = 0.$$

Then $\chi_A(xyz) = 1$ and $\chi_{A^c}(xyz) = 0$. Thus $t = xyz \in A$. So $ASA \subset A$. Moreover, by Result 2.A (1), A is a subsemigroup of S. Hence $A \in BI(S)$.

Proposition 2.6. Let S be a semigroup. Then S is a group if and only if every $IFBI$ of S is a constant mapping.

Proof.($\Rightarrow$): Suppose S is a group with the identity e. Let $A \in IFBI(S)$ and let $a \in S$. Then

$$\begin{aligned}\mu_A(a) &= \mu_A(eae) \geq \mu_A(e) \wedge \mu_A(e) = \mu_A(e)\\ &= \mu_A(ee) = \mu_A((aa^{-1})(a^{-1}a))\\ &= \mu_A(a(a^{-1}a^{-1})a) \geq \mu_A(a) \wedge \mu_A(a)\\ &= \mu_A(a)\end{aligned}$$

and

$$\begin{aligned}\nu_A(a) &= \nu_A(eae) \leq \nu_A(e) \vee \nu_A(e) = \nu_A(e)\\ &= \nu_A(ee) = \nu_A((aa^{-1})(a^{-1}a))\\ &= \nu_A(a(a^{-1}a^{-1})a) \leq \nu_A(a) \vee \nu_A(a)\\ &= \nu_A(a).\end{aligned}$$

Thus $A(a) = A(e)$. Hence A is a constant mapping.

($\Leftarrow$): Suppose the necessary condition holds. Assume that S is not a group. Then it follows from p. 84 in [4] that S contains a proper bi-ideal A of S. Then there exists an $x \in S$ such that $x \notin A$. Let $y \in A$ with $y \neq x$. Since A is a bi-ideal of S, by Proposition 2.5, $(\chi_A, \chi_{A^c}) \in IFBI(S)$. By the hypothesis, (χ_A, χ_{A^c}) is a constant mapping. Thus $(\chi_A, \chi_{A^c})(x) = (\chi_A, \chi_{A^c})(y)$, i.e., $\chi_A(x) = \chi_A(y)$ and $\chi_{A^c}(x) = \chi_{A^c}(y)$. Since $x \notin A$ and $y \in A$, $\chi_A(x) = 0 < \chi_A(y) = 1$ and $\chi_{A^c}(x) = 1 > \chi_{A^c}(y) = 0$, i.e., $(\chi_A, \chi_{A^c})(x) = 0_\sim \neq 1_\sim = (\chi_A, \chi_{A^c})(y)$. This is a contradiction. Hence S is a group. This completes the proof.

Proposition 2.7. Every IFLI[resp. IFRI and IFI] of S is an IFBI of S.

Proof. Suppose $A \in IFLI(S)$ and let $x, y, z \in S$. Then

$$\mu_A(xyz) = \mu_A((xy)z) \geq \mu_A(z) \geq \mu_A(x) \wedge \mu_A(z)$$

and

$$\nu_A(xyz) = \nu_A((xy)z) \leq \nu_A(z) \leq \nu_A(x) \vee \nu_A(z).$$

So $A \in IFBI(S)$. Similarly, we can see that the other cases hold.

Proposition 2.8. Let S be a semigroup and let $A \in IFS(S)$. Then $A \in IFBI(S)$ if and only if $A^{(\lambda,\mu)} \in BI(S)$ for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$.

Proof. ($\Rightarrow$): Suppose $A \in IFBI(S)$ and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then, by Result 2.B, $A^{(\lambda,\mu)}$ is a subsemigroup of S. Let $t \in A^{(\lambda,\mu)}SA^{(\lambda,\mu)}$. Then there exist $x, z \in A^{(\lambda,\mu)}$ and $y \in S$ such that $t = xyz$. Since $A \in IFBI(S)$,

$$\mu_A(t) \geq \mu_A(x) \wedge \mu_A(z) \geq \lambda \text{ and } \nu_A(t) \leq \nu_A(x) \vee \nu_A(y) \leq \mu.$$

Thus $t \in A^{(\lambda,\mu)}$. So $A^{(\lambda,\mu)}SA^{(\lambda,\mu)} \subset A^{(\lambda,\mu)}$. Hence $A^{(\lambda,\mu)} \in BI(S)$.

($\Leftarrow$): Suppose the necessary condition holds. Then, by Proposition 2.3, $A \in IFSG(S)$. For any $x, z \in S$, let $A(x) = (\lambda_1, \mu_1)$ and let $A(z) = (\lambda_2, \mu_2)$. Then, by the process of the proof of Proposition 2.3, $x, z \in A^{(\lambda_1 \wedge \lambda_2, \mu_1 \vee \mu_2)}$. Let $y \in S$. Then, by the hypothesis, $xyz \in A^{(\lambda_1 \wedge \lambda_2, \mu_1 \vee \mu_2)}$. Thus

$$\mu_A(xyz) \geq \lambda_1 \wedge \lambda_2 = \mu_A(x) \wedge \mu_A(z)$$

and

$$\nu_A(xyz) \leq \mu_1 \vee \mu_2 \leq \nu_A(x) \vee \nu_A(z).$$

Hence $A \in IFBI(S)$. This completes the proof.

3. Intuitionistic fuzzy duos, ideals and bi-ideals of a regular semigroup

A semigroup S is said to be *regular* of for each $a \in S$ there exists an $x \in S$ such that $a = axa$.

A semigroup S is said to be *left*[resp. *right*] *duo* if every left [resp. right] ideal of S is a two-sided ideal of S.

A semigroup S is said to be *duo* if it is both left and right duo.

Definition 3.1. A semigroup S is said to be :

(1) *intuitionistic fuzzy left duo*(in short, IFLD) if every IFLI of S is an IFI of S.

(2) *intuitionistic fuzzy right duo*(in short, IFRD) if every IFRI of S is an IFI of S.

(3) *intuitionistic fuzzy duo*(in short, IFD) if it is both intuitionistic fuzzy left and intuitionistic fuzzy right duo.

Proposition 3.1. Let S be a regular semigroup. Then S is left duo if and only if S is IFLD.

Proof. ($\Rightarrow$): Suppose S is left duo. Let $A \in IFLI(S)$ and let $a, b \in S$. Then, by the process of the proof of Theorem 3.1 in [12], $ab \in (aSa)b \subset (Sa)S \subset Sa$. Thus there exists an $x \in S$ such that $ab = xa$. Since $A \in IFLI(S)$,

$$\mu_A(ab) = \mu_A(xa) \geq \mu_A(a)$$

and

$$\nu_A(ab) = \nu_A(xa) \leq \nu_A(a).$$

Then $A \in IFRI(S)$. Thus $A \in IFI(S)$. Hence S is IFLD.

($\Leftarrow$): Suppose S is IFLD and let A be any left ideal of S. Then, by Result 2.A (2), $(\chi_A, \chi_{A^c}) \in IFLI(S)$. By the assumption, $(\chi_A, \chi_{A^c}) \in IFI(S)$. Since $A \neq \emptyset$, by Result 2.A (3), A is an ideal of S. Hence S is left duo. This completes the proof.

Proposition 3.1′[The dual of Proposition 3.1]. Let S be a regular semigroup. Then S is right duo if and only if S is IFRD.

The following is the immediate result of Propositions 3.1 and 3.1′:

Proposition 3.2. Let S be a regular semigroup. Then S is duo if and only if S is IFD.

Proposition 3.3. Let S be a regular semigroup. Then every bi-ideal of S is a right ideal of S if and only if every IFBI of S is an IFRI of S.

Proof. ($\Rightarrow$): Suppose every bi-ideal of S is a right ideal of S. Let $A \in IFBI(S)$ and let $a, b \in S$. Then, by the process of proof of Theorem 3.4 in [12], $ab \in (aSa)S \subset aSa$. Thus there exists an $x \in S$ such that $ab = axa$. Since $A \in IFBI(S)$,

$$\mu_A(ab) = \mu_A(axa) \geq \mu_A(a) \wedge \mu_A(a) = \mu_A(a)$$

and

$$\nu_A(ab) = \nu_A(axa) \leq \nu_A(a) \vee \nu_A(a) = \nu_A(a).$$

Hence $A \in IFRI(S)$.

($\Rightarrow$): Suppose every IFBI of S is an IFRI of S and let A be any bi-ideal of S. Then, by Proposition 2.5, $(\chi_A, \chi_{A^c}) \in IFBI(S)$. By the assumption, $(\chi_A, \chi_{A^c}) \in IFRI(S)$. Since $A \neq \emptyset$, by Result 2.A (2), A is a right ideal of S. This completes the proof.

Result 3.A[15, Theorem 3]. Every bi-ideal of a regular left duo semigroup S is a right ideal of S.

Corollary 3.3. Let S be a regular duo semigroup. Then every IFBI of S is a IFRI of S.

Proof. By Result 3.A, every bi-ideal of S is a right ideal of S. Hence, by Proposition 3.3, it follows that every IFBI of S is an IFRI of S.

Proposition 3.3′[The dual of Proposition 3.3]. Let S be a regular semigroup. Then every bi-ideal of S is a left ideal of S if and only if every IFBI of S is an IFLI of S.

The following is the immediate result of Propositions 3.3 and 3.3′:

Proposition 3.4. Let S be a regular duo semigroup. Then every bi-ideal of S is an ideal of S if and only if every IFBI of S is an IFI of S.

A semigroup S is called a *semilattice of groups* [4] if it is the set-theorectical union of a set of mutually disjoint subgroups $G_\alpha(\alpha \in \Gamma)$, i.e., $S = \bigcup_{\alpha\in\Gamma} G_\alpha$ such that for any α, $\beta \in \Gamma$, $G_\alpha G_\beta \subset G_\gamma$ and $G_\beta G_\alpha \subset G_\gamma$ for some $\gamma \in \Gamma$.

Result 3.B[14, Theorem 4]. Every bi-ideal of a semigroup S which is a semilattice of groups, is an ideal of S.

The following is the immediate result of Result 3.B and Proposition 3.4:

Corollary 3.4. Let S be a semigroup which is a semilattice of groups. Then every IFBI of S is an IFI of S.

We denote by $L[a]$[resp. $J[a]$] the principle left [resp. two-sided] ideal of a semigroup S generated by a in S, i.e.,

$L[a] = \{a\} \cup Sa$,

$J[a] = \{a\} \cup Sa \cup aS \cup SaS$.

It is well-known [4, Lemma 2.13] that if S is a regular semigroup, then $L[a] = Sa$ for each $a \in S$.

A semigroup S is said to be *right*[resp. *left*] *zero* if $xy = y$[resp. $xy = x$] for any $x, y \in S$.

Proposition 3.5. Let S be a regular semigroup and let E_S the set of all idempotent elements of S. Then E_S forms a left zero subsemigroup of S if and only if for each $A \in IFLI(S)$, $A(e) = A(f)$ for any $e, f \in E_S$, where E_S denotes the set of all idempotent elements of S.

Proof. ($\Rightarrow$): Suppose E_S forms a left zero subsemigroup of S. Let $A \in IFLI(S)$ and let $e, f \in E_S$. Then, by the hypothesis, $ef = e$ and $fe = f$. Since $A \in IFLI(S)$,

$$\mu_A(e) = \mu_A(ef) \geq \mu(f) = \mu_A(fe) \geq \mu_A(e)$$

and

$$\nu_A(e) = \nu_A(ef) \leq \nu(f) = \nu_A(fe) \leq \nu_A(e).$$

Hence $A(e) = A(f)$.

($\Leftarrow$): Suppose the necessary condition holds. Since S is regular, $E_S \neq \emptyset$. Let $e, f \in E_S$. Then, by Result 2.A(2), $(\chi_{L[f]}, \chi_{L[f]^c}) \in IFLI(S)$. Thus $\chi_{L[f]}(e) = \chi_{L[f]}(f) = 1$ and $\chi_{L[f]^c}(e) = \chi_{L[f]^c}(f) = 0$. So $e \in L[f] = Sf$. Then there exists an $x \in S$ such that $e = xf = xff = ef$. Hence E_S is a left zero semigroup. This completes the proof.

Corollary 3.5. Let S be an idempotent semigroup. Then S is left zero if and only if for each $A \in IFLI(S)$, $A(e) = A(f)$ for any $e, f \in S$.

Proposition 3.5′[The dual of Proposition 3.5]. Let S be a regular semi group. Then E_S forms a right zero subsemigroup of S if and only if for each $A \in IFRI(S)$, $A(e) = A(f)$ for any $e, f \in E_S$.

Corollary 3.5′[The dual of Corollary 3.5]. Let S be an semigroup. Then S is right zero if and only if for each $A \in IFRI(S)$, $A(e) = A(f)$ for any $e, f \in S$.

Proposition 3.6. Let S be a regular semigroup. Then S is a group if and only if for each $A \in IFBI(S)$, $A(e) = A(f)$ for any $e, f \in E_S$.

Proof ($\Rightarrow$): Suppose S is a group. Let $A \in IFBI(S)$. Then, by Proposition 2.5, A is a constant mapping. Hence $A(e) = A(f)$ for any $e, f \in E_S$.

($\Leftarrow$): Suppose the necessary condition holds. Let $e, f \in E_S$. Let $B[x]$ denote the principal bi-ideal of S generated by x in S, i.e., $B[x] = \{x\} \cup \{x^2\} \cup xSx$[4, p. 84]. Moreover, if S is regular, then $B[x] = xSx$ for each $x \in S$. Then, by Proposition 2.5, $(\chi_{B[f]}, \chi_{B[f]^c}) \in IFBI(S)$. Since $f \in B[f]$,

$$\chi_{B[f]}(e) = \chi_{B[f]}(f) = 1 \text{ and } \chi_{B[f]^c}(e) = \chi_{B[f]^c}(f) = 0.$$

Then $e \in B[f] = fSf$. Thus, by the process of the proof of Theorem 3.14 in [12], $e = f$. Since S is regular, $E_s \neq \emptyset$ and S contains exactly one idempotent. So it follows from [4, p. 33(Ex. 4)] that S is a group. This completes the proof.

4. Intra-regular semigroups

A semigroup S is said to be *intra-regular* if for each $a \in S$, there exist $x, y \in S$ such that $a = xa^2y$. For characterization of such a semigroup, see[4, Theorem 4.4] and [16, II.4.5 Theorem].

Proposition 4.1. Let S be a semigroup. Then S is intra-regular if and only if for each $A \in IFI(S)$, $A(a) = A(a^2)$ for each $a \in S$.

Proof.($\Rightarrow$): Suppose S is intra-regular. Let $A \in IFI(S)$ and let $a \in S$. Then, by the hypothesis, there exist $x, y \in S$ such that $a = xa^2y$. Since $A \in IFI(S)$,

$$\mu_A(a) = \mu_A(xa^2y) \geq \mu_A(xa^2) \geq \mu_A(a^2) \geq \mu_A(a)$$

and

$$\nu_A(a) = \nu_A(xa^2y) \leq \nu_A(xa^2) \leq \nu_A(a^2) \leq \nu_A(a).$$

Hence $A(a) = A(a^2)$ for each $a \in S$.

($\Leftarrow$): Suppose the necessary condition holds and let $a \in S$. Then, by Result 2.A(3), $(\chi_{J[a^2]}, \chi_{J[a^2]^c}) \in IFI(S)$. Since $a^2 \in J[a^2]$,

$$\chi_{J[a^2]}(a) = \chi_{J[a^2]}(a^2) = 1 \text{ and } \chi_{J[a^2]^2}(a) = \chi_{J[a^2]^2}(a^2) = 0.$$

Thus $a \in J[a^2] = \{a\} \cup Sa^2 \cup a^2S \cup Sa^2S$. So we can easily see that S is intra-regular. This completes the proof.

Proposition 4.2. Let S be an intra-regular semigroup. Then for each $A \in IFI(S)$, $A(ab) = A(ba)$ for any $a, b \in S$.

Proof. Let $A \in IFI(S)$ and let $a, b \in S$. Then, by Proposition 4.1,

$$\mu_A(ab) = \mu_A((ab)^2) = \mu_A(a(ba)b) \geq \mu_A(ba)$$
$$= \mu_A((ba)^2) = \mu_A(b(ab)a) \geq \mu_A(ab)$$

and

$$\nu_A(ab) = \nu_A((ab)^2) = \nu_A(a(ba)b) \leq \nu_A(ba)$$
$$= \nu_A((ba)^2) = \nu_A(b(ab)a) \leq \nu_A(ab).$$

Thus $A(ab) = A(ba)$. This completes the proof.

5. Completely regular semigroups

A semigroup S is said to be *completely regular* if for each $a \in S$, there exists an $x \in S$ such that

$$a = axa \text{ and } ax = xa.$$

A semigroup S is said to be *left*[resp. *right*]*regular* if for each $a \in S$, there exists an $x \in S$ such that

$$a = xa^2[\text{resp. } a = a^2x].$$

For characterizations of such a semigroup, see [4, Theorem 4.2.]. It is well-known[4, Theorem 4.3.] that S is completely regular if and only if it is left and right regular.

Result 5.A[16, p. 105]. Let S be a semigroup. Then the followings are equivalent:

(1) S is completely regular.

(2) S is a union of groups.

(3) $a \in a^2Sa^2$ for each $a \in S$.

Proposition 5.1. Let S be a semigroup. Then S is left regular if and only if for each A in $IFLI(S)$, $A(a) = A(a^2)$ for each $a \in S$.

Proof.($\Rightarrow$): Suppose S is left regular. Let $A \in IFLI(S)$ and let $a \in S$. Then, by the hypothesis, there exists an $x \in S$ such that $a = xa^2$. Since $A \in IFLI(S)$,

$$\mu_A(a) = \mu_A(xa^2) \geq \mu_A(a^2) \geq \mu_A(a)$$

and

$$\nu_A(a) = \nu_A(xa^2) \leq \nu_A(a^2) \leq \nu_A(a)$$

Hence $A(a) = A(a^2)$, for each $a \in S$.

($\Leftarrow$): Suppose the necessary condition holds. Let $a \in S$. Then, by Result 2.A(2), $(\chi_{L[a^2]}, \chi_{L[a^2]^c}) \in IFLI(S)$. Since $a^2 \in L[a^2]$,

$$(\chi_{L[a^2]}(a) = \chi_{L[a^2]}(a^2) = 1 \text{ and } (\chi_{L[a^2]^c}(a) = \chi_{L[a^2]^c}(a^2) = 0.$$

Then $a \in L[a^2] = \{a^2\} \cup Sa^2$. Hence S is left regular. This completes the proof.

Proposition 5.1′[The dual of Proposition 5.1]. Let S be a semigroup. Then S is right regular if and only if for each $A \in IFRI(S)$, $A(a) = A(a^2)$ for each $a \in S$.

Now we give another characterization of a completely regular semigroup by intuitionistic fuzzy bi-ideals:

Proposition 5.2. Let S be a semigroup. Then the followings are equivalent:

(1) S is completely regular.

(2) For each $A \in IFBI(S)$, $A(a) = A(a^2)$ for each $a \in S$.

(3) For each $B \in IFLI(S)$ and each $C \in IFRI(S)$,
$B(a) = B(a^2)$ and $C(a) = C(a^2)$

for each $a \in S$.

Proof. It is clear that (1)⇔(3) by Propositions 5.1 and 5.1′. Thus it is sufficient to show that (1)⇔(2).

(1) ⇒ (2): Suppose the condition (1) holds. Let $A \in IFBI(S)$ and let $a \in S$. Then, by Result 5.A(3), there exists an $x \in S$ such that $a = a^2xa^2$. Since $A \in IFBI(S)$,

$$\mu_A(a) = \mu_A(a^2xa^2) \geq \mu_A(a^2) \wedge \mu_A(a^2)$$
$$= \mu_A(a^2) \geq \mu_A(a) \wedge \mu_A(a) = \mu_A(a)$$

and

$$\nu_A(a) = \nu_A(a^2xa^2) \leq \nu_A(a^2) \vee \nu_A(a^2)$$
$$= \nu_A(a^2) \leq \nu_A(a) \vee \nu_A(a) = \nu_A(a).$$

Hence $A(a) = A(a^2)$.

(2) ⇒ (1): Suppose the condition (2) holds. For each $x \in S$, let $B[x]$ denote the principal bi-ideal of S generated by x, i.e.,

$$B[x] = \{x\} \cup \{x^2\} \cup xSx.$$

Let $a \in S$. Then, by Proposition 2.5, $(\chi_{B[a^2]}, \chi_{B[a^2]^c}) \in IFBI(S)$. Since $a^2 \in B[a^2]$, $\chi_{B[a^2]}(a) = \chi_{B[a^2]}(a^2) = 1$ and $\chi_{B[a^2]^c}(a) = \chi_{B[a^2]^c}(a^2) = 0$. Thus $a \in B[a^2] = \{a^2\} \cup \{a^4\} \cup a^2Sa^2$. Hence S is completely regular. This completes the proof.

Result 5.B[13, Theorem 1]. Let S be a semigroup. Then S is a semilattice of groups if and only if $BI(S)$ is a semilattice under the multiplication of subsets.

Proposition 5.3. Let S be a semigroup. Then S is a semilattice of groups if and only if for each $A \in IFBI(S)$, $A(a) = A(a^2)$ and $A(ab) = A(ba)$ for any $a, b \in S$.

Proof.(⇒): Suppose S is a semilattice of groups. Then S is a union of groups. By Result 5.A, S is completely regular. Let $A \in IFBI(S)$ and let $a \in S$. Then, by Proposition 5.2, $A(a) = A(a^2)$. Now let $a, b \in S$. Then, by the process of the proof of Theorem 6 in [11], there exists an $x \in S$ such that $(ab)^3 = (ba)x(ba)$. Thus

$$\mu_A(ab) = \mu_A((ab)^3) = \mu_A((ba)x(ba))$$
$$\geq \mu_A(ba) \wedge \mu_A(ba) = \mu_A(ba)$$

and

$$\nu_A(ab) = \nu_A((ab)^3) = \nu_A((ba)x(ba))$$
$$\leq \nu_A(ba) \vee \nu_A(ba) = \nu_A(ba).$$

Similarly, we can see that $\mu_A(ba) \geq \mu_A(ab)$ and $\nu_A(ba) \leq \nu_A(ab)$. So $A(ab) = A(ba)$. Hence the necessary conditions hold.

($\Leftarrow$): Suppose the necessary conditions hold. Then, by the first condition and Proposition 5.2, S is completely regular. Thus it is easily shown that A is idempotent for each $A \in BI(S)$. Let $A, B \in BI(S)$ and let $t \in BA$. Then there exist $a \in A$ and $b \in B$ such that $t = ab$. Moreover $B[t] = B[ab] \in BI(S)$. By Proposition 2.5, $(\chi_{B[ab]}, \chi_{B[ab]^c}) \in IFBI(S)$. By the hypothesis, $(\chi_{B[ab]}, \chi_{B[ab]^c})(ab) = (\chi_{B[ab]}, \chi_{B[ab]^c})(ba)$. Since $ab \in B[ab]$,

$$\chi_{B[ab]}(ab) = \chi_{B[ab]}(ba) = 1 \text{ and } \chi_{B[ab]^c}(ab) = \chi_{B[ab]^c}(ba) = 0.$$

Then $ba \in B[ab] = \{ab\} \cup \{abab\} \cup abSab$. It follows from the process of the proof of Theorem 6 in [11] that $BA = AB$. So $(BI(S), \cdot)$ is a commutative idempotent semigroup. Hence, by Result 5.B, S is a semilattice of groups. This completes the proof.

Corollary 5.3. Let S be an idempotent semigroup. Then S is commutative if and only if for each $A \in IFBI(S)$, $A(ab) = A(ba)$ for any $a, b \in S$.

6. Semigroups that are semilattices of left [resp. right] simple semigroups

Result 6.A[13, Theorem 7 and 18, Theorem]. Let S be a semigroup. Then the followings are equivalent :

(1) S is a semilattice of left simple semigroups.

(2) S is left regular and $AB = BA$ for any two left ideals A and B of S.

(3) S is left regular and every left ideal of it is an ideal of S.

The following result can be proved in a similar way as in the proof of Propositions 3.1 and 3.1′.

Proposition 6.1. Let S be a left [resp. right] regular semigroup. Then S is left [resp. right] duo if and only if S is IFLD [resp. IFRD].

The characterization of a semigroup that is a semilattice of left simple semigroups can be founded in [16, Theorem II.4.9].

Proposition 6.2. Let S be a semigroup. Then S is a semilattice of left simple semigroups if and only if for each $A \in IFLI(S)$, $A(a) = A(a^2)$ and $A(ab) = A(ba)$ for any $a, b \in S$.

Proof. ($\Rightarrow$): Suppose S is a semilattice of left simple semigroups. Let $A \in IFLI(S)$ and let $a, b \in S$. Then, by Result 6.A, S is left regular. By Proposition 5.1, $A(a) = A(a^2)$. By the hypothesis and Result 6.A, S is left duo. By Proposition 6.1, S is IFLD. Then $A \in IFI(S)$. Thus

$$\mu_A(ab) = \mu_A((ab^2)) = \mu_A(a(ba)b) \geq \mu_A(ba)$$

and

$$\nu_A(ab) = \nu_A((ab^2)) = \nu_A(a(ba)b) \leq \nu_A(ba).$$

By the similar arguments, we have

$$\mu_A(ba) \geq \mu_A(ab) \text{ and } \nu_A(ba) \leq \nu_A(ab).$$

Hence $A(ab) = A(ba)$ for any $a, b \in S$.

($\Leftarrow$): Suppose the necessary conditions hold. Then, by the first condition and Proposition 5.1, S is left regular. Let A and B be any left ideals of S and let $x \in AB$. Then there exist $a \in A$ and $b \in B$ such that $x = ab$. By Result 2.A(2), $(\chi_{L[ba]}, \chi_{L[ba]^c}) \in IFLI(S)$. Since $ba \in L[ba]$,

$$\chi_{L[ba]}(ab) = \chi_{L[ba]}(ba) = 1 \text{ and } \chi_{L[ba]^c}(ab) = \chi_{L[ba]^c}(ba) = 0.$$

Thus $ab \in L[ba] = \{ba\} \cup Sba \subset BA \cup SBA \subset BA$. So ,by the process of the proof of Theorem 6.3 in [12], we have $AB = BA$. Hence, by Result 6.A, S is a semilattice of left simple semigroups. This completes the proof.

Proposition 6.2′[The dual of Proposition 6.2]. Let S be a semigroup. Then the S is a semilattice of right simple semigroups if and only if for each $A \in IFRI(S)$, $A(a) = A(a^2)$ and $A(ab) = A(b)$ for any $a, b \in S$.

7. Left [resp. right] simple semigroups

Definition 7.1. A semigroup S is said to be *intuitionistic fuzzy left*[resp. *right*]*simple* if every IFLI [resp. IFRI] of S is a constant mapping and is said to be *intuitionistic fuzzy simple* if every IFI of S is a constant mapping.

Proposition 7.2. Let S be a semigroup. Then S is left simple if and only if S is intuitionistic fuzzy left simple.

Proof. ($\Rightarrow$): Suppose S is left simple. Let $A \in IFLI(S)$ and let $a, b \in S$. Since S is left simple, from [4, p. 6], there exist $x, y \in S$ such that $b = xa$ and $a = yb$. Since $A \in IFLI(S)$,

$$\mu_A(a) = \mu_A(yb) \geq \mu_A(b) = \mu_A(xa) \geq \mu_A(a)$$

and

$\nu_A(a) = \nu_A(yb) \leq \nu_A(b) = \nu_A(xa) \leq \nu_A(a)$.

Thus $A(a) = A(b)$. So A is a constant mapping. Hence S is intuitionistic fuzzy left simple.

($\Leftarrow$): Suppose the necessary condition holds. Let A be any left ideal of S. By Result 2.A(2), $(\chi_A, \chi_{A^c}) \in IFLI(S)$. By the hypothesis, (χ_A, χ_{A^c}) is a constant mapping. Since $A \neq \emptyset$, $(\chi_A, \chi_{A^c}) = 1_{\sim}$. Then $\chi_A(a) = 1$ and $\chi_{A^c}(a) = 0$ for each $a \in S$. Thus $a \in A$ for each $a \in S$, i.e., $S \subset A$. Hence S is left simple. This completes the proof.

The following two results can be seen in a similar way as in the proof of Proposition 7.2:

Proposition 7.2′[The dual of Proposition 7.2]. Let S be a semigroups. Then S is right simple if and only if S is intuitionistic fuzzy simple.

Proposition 7.3. Let S be a semigroup. Then S is simple if and only if S is intuitionistic fuzzy simple.

It is well-known that a semigroup S is a group if and only if it is left and right simple. Thus from this and Propositions 7.2 and 7.2′, we obtain the following result :

Proposition 7.4. Let S be a semigroup. Then S is a group if and only if S is both intuitionistic fuzzy left and intuitionistic fuzzy right simple.

Proposition 7.5. Let S be a left simple semigroup. Then every IFBI of S is an IFRI of S.

Proof. Let $A \in IFBI(S)$ and let $a, b \in S$. Since S is left simple, there exists an $x \in S$ such that $b = xa$. Since $A \in IFBI(S)$,

$$\mu_A(ab) = \mu_A(axa) \geq \mu_A(a) \wedge \mu_A(a) = \mu_A(a)$$

and

$$\nu_A(ab) = \nu_A(axa) \leq \nu_A(a) \vee \nu_A(a) = \nu_A(a).$$

Hence $A \in IFRI(S)$. This completes the proof.

Corollary 7.5. Let S be a left simple semigroup. Then every bi-ideal of S is a right ideal of S.

References

[1] K.Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1986), 87-96.

[2] B. Banerjee and D. Kr. Basnet, Intuitionistic fuzzy subrings and ideals, J. Fuzzy Mathematics 11(1)(2003), 139-155

[3] R.Biswas, Intuitionistic fuzzy subgroups, Mathematical Forum x(1989), 37-46

[4] A.H.Clifford and G.B.Preston, The Algebraic Theory of semigroups, Vol.1, Math. Surveys No. 7, Amer. Soc. Providence, R.I.(1961).

[5] D. Çoker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88(1997), 81-89.

[6] D. Çoker and A.Haydar Es, On fuzzy compactness in intuitionistic fuzzy topological spaces, J. Fuzzy Math. 3(1995), 899-909.

[7] P.S.Das, Fuzzy groups and level subgroups, J. Math. Anal. Appl. 84(1981), 264-269.

[8] H.Gürçay, D. Çoker and A.Haydar Es, On fuzzy continuity in intuitionistic fuzzy topological spaces, J. Fuzzy Math. 5(1997), 365-378.

[9] K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy subgroupoids, International Journal of Fuzzy Logic and Intelligent Systems 3(1)(2003), 72-77.

[10] K.Hur, H.W.Kang and H.K.Song, Intuitionistic fuzzy subgroups and subrings, Honam Math. J. 25(1) (2003), 19-41.

[11] N.Kuroki, Fuzzy bi-ideals in semigroups, Comment. Math. Univ. St. Pauli XXVIII-1(1979), 17-21.

[12] _, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets and Systems 5(1981), 203-215.

[13] S.Lajos, A note on semilattice of groups, Acta Sei, Math. Szeged 33(1972), 315-317.

[14] _, On (m, n)-ideals in regular duo semigroups, Acta Sei, Math. Szeged 28(1967), 261-265.

[15] _, Theorems on $(1, 1)$-ideals in semigroups I-II, K. Marx Univ. Economics Dept. Math., Budapest(1972).

[16] M.Petrich, Introduction to Semigroups, Columbus, Ohio(1973).

[17] A.Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35(1971), 512-517.

[18] T.Saitō, On semigroups which are semilattices of left simple semigroups, Math. Japan 18 (1972), 95-97.

[19] L.A.Zadeh, Fuzzy sets, Inform. and Control 8(1965), 338-353.

[a] Division of Mathematics and Informational
Statistics
Wonkwang University Iksan, Chonbuk, Korea 579-792
E-mail:kulhur@wonkwang.ac.kr
E-mail:songh0ki@wonkwang.ac.kr

[b] Dept. of Mathematics Education Woosuk University
Hujong-Ri Samrae-Eup, Wanju-kun Chonbuk, Korea 565-701
E-mail: khwon@woosuk.ac.kr