Honam Mathematical J. **26**(2004), No. 3, pp. 299–308

ON THE HILBERT SPACE OF FORMAL POWER SERIES

B. Yousefi and R. Soltani

Abstract. Let $\{\beta(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\beta(0) = 1$. We consider the space $H^2(\beta)$ of all power series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ such that $\sum_{n=0}^{\infty} |\hat{f}(n)|^2\beta(n)^2 < \infty$. We link the ideas of subspaces of $H^2(\beta)$ and zero sets. We give some sufficient conditions for a vector in $H^2(\beta)$ to be cyclic for the multiplication operator M_z. Also we characterize the commutant of some multiplication operators acting on $H^2(\beta)$.

Introduction

Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^2 = \|f\|_\beta^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2\beta(n)^2 < \infty.$$

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ shall be used whether or not the series converges for any value of z. These are called formal power series. Let $H^2(\beta)$ denote the space of such formal power series. These are Hilbert spaces with the norm $\|\cdot\|_\beta$ ([4]). The Hardy, Bergman and Dirichlet

Received Jun 28, 2004 ; Revised August 23, 2004.

2000 Mathematical Subject Classification: Primary 47B37; Secondary 47A16.

Keywords and phrases: Hilbert space of formal power series, multiplier, cyclic vector, bounded point evaluation, polynomially bounded, Farrell-Rubel-Shields Theorem, finite codimension subspace.

spaces can be viewed in this way when respectively $\beta(n) = 1, \beta(n) = (n+1)^{-1/2}$ and $\beta(n) = (n+1)^{1/2}$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\| = \beta(k)$. Now consider M_z, the operator of multiplication by z on $H^2(\beta)$:

$$(M_z f)(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^{n+1}$$

where

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^2(\beta).$$

In other words

$$(M_z \hat{f})(n) = \begin{cases} \hat{f}(n-1) & n \geq 1 \\ 0 & n = 0 \end{cases}.$$

Clearly M_z shifts the basis $\{f_k\}_k$. The operator M_z is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded and in this case

$$\|M_z^n\| = \sup_k \frac{\beta(k+n)}{\beta(k)}, \qquad n = 0, 1, 2, \ldots.$$

Throughout this paper we suppose that M_z is bounded. The composition operator C_φ on $H^2(\beta)$ is defined by $C_\varphi f = f \circ \varphi$ for $f \in H^2(\beta)$.

We denote the set of multipliers $\{\varphi \in H^2(\beta) : \varphi H^2(\beta) \subseteq H^2(\beta)\}$ by $M(H^2(\beta))$ and the linear transformation by φ on $H^2(\beta)$ by M_φ. Here if $\varphi \in M(H^2(\beta))$ and f is in $H^2(\beta)$, then

$$\varphi f = \left(\sum_{n=0}^{\infty} \hat{\varphi}(n) z^n\right)\left(\sum_{n=0}^{\infty} \hat{f}(n) z^n\right) = \sum_{n=0}^{\infty} \hat{h}(n) z^n$$

where $\hat{h}(n) = \sum_{k=0}^{n} \hat{\varphi}(k) \hat{f}(n-k)$. Each multiplier is a bounded analytic function ([3]).

Remember that a complex number λ is said to be a bounded point evaluation on $H^2(\beta)$ if the functional of point evaluation at λ, e_λ, is bounded. Each point of the disc $\{z : |z| < \liminf \beta(n)^{1/n}\}$ is a bounded point evaluation on $H^2(\beta)$ ([4,5]).

If Ω is a bounded domain in the complex plane, then by $C(\Omega)$, $H(\Omega)$ and $H^\infty(\Omega)$ we mean respectively the set of continuous functions, analytic functions and the set of bounded analytic functions on Ω. By $\|\cdot\|_\Omega$ we denote the supremum norm on Ω.

We say that a vector x in a Banach space X is a cyclic vector of a bounded operator A on X if

$$X = span\{A^n x : n = 0, 1, 2, \ldots\}.$$

Here $span\{\cdot\}$ is the closed linear span of the set $\{\cdot\}$.

Main results

Recall that $H^2(\beta)$ has division property: if $f \in H^2(\beta)$ and $f(\lambda) = 0$, then $f/(z-\lambda)$ is in $H^2(\beta)$. The open unit disc will be denoted by $\mathbb{D}$. If $\liminf \beta(n)^{1/n} = 1$, then $H^2(\beta) \subset H(\mathbb{D})$ and if $\sum\limits_n \frac{1}{\beta(n)^2} < \infty$, then $H^2(\beta) \subset H(\mathbb{D}) \cap C(\bar{\mathbb{D}})$. Also note that the spectrum of M_z, $\sigma(M_z)$, is equal to the set $\{\lambda : |\lambda| \le r(M_z)\}$ where $r(M_z)$ is the spectral radius of M_z. Clearly $\liminf \beta(n)^{1/n} \le r(M_z) \le \|M_z\|$. For a good source of this topics see [4,5,6,7,8,9,10].

In the following theorem we link the ideas of subspaces of $H^2(\beta)$ and zero sets.

Theorem 1. Let $r(M_z) = \liminf \beta(n)^{1/n} = 1$ and $ran(M_z - \lambda)$ be dense in $H^2(\beta)$ for every $\lambda \in \partial\mathbb{D}$. Also assume that $H^2(\beta)$ has division property. If M is an invariant subspace of $H^2(\beta)$ of finite codimension, and $\Omega = \{z \in \mathbb{D} : f(z) = 0 \text{ for all } f \in M\}$, then Ω is a finite set and $M = \{f \in H^2(\beta) : f(z) = 0 \text{ for all } z \in \Omega\}$.

Proof. First note that since $\liminf \beta(n)^{1/n} = 1$, the functions of $H^2(\beta)$ are analytic on $\mathbb{D}$. Suppose that Ω is infinite and let $\{z_j\}_{j=1}^\infty$ be distinct points of Ω. Let $p_1, p_2, \ldots$ be polynomials in $H^2(\beta)$ such that

$p_i(z_j) = 0$ for $j = 1, \ldots, i-1$ and $i > 1$, also $p_i(z_i) = 1$ for all $i \geq 1$. Now if $c_1 p_1 + \cdots + c_n p_n \in M$ where $c_1, c_2, \ldots, c_n$ are in $\mathbf{C}$, then $c_1 = c_2 = \cdots = c_n = 0$. This contradicts the assumption that M has finite codimension in $H^2(\beta)$. Since each zero of an analytic function has finite multiplicity, Ω is indeed a finite set. For the second part note that if $f \in M$, then $f(z) = 0$ for all $z \in \Omega$. For the inverse inclusion suppose that $f \in H^2(\beta)$ and $f(z) = 0$ for all $z \in \Omega$. Also let $\Omega = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and put $q(z) = (z - \lambda_1)(z - \lambda_2) \ldots (z - \lambda_n)$. Since M is an invariant finite codimensional subspace of $H^2(\beta)$, by the hypothesis of the theorem $M = pH^2(\beta)$ for some polynomial p where p vanishes on Ω. Write $p = qs$. Here s is a polynomial such that $s(\lambda_i) \neq 0$ for $i = 1, \ldots, n$. Let λ be a root of s. If $\lambda \in \mathbb{D}$, then it should be $\lambda \in \Omega$ that is a contradiction since $s(\lambda_i) \neq 0$ for $i = 1, \ldots, n$. So $\lambda \in \mathbf{C} \backslash \mathbb{D}$. If $\lambda \notin \bar{\mathbb{D}}$, then $\lambda \notin \sigma(M_z)$ since $r(M_z) = 1$. So $M_z - \lambda$ is invertible which implies that $(z - \lambda)H^2(\beta) = H^2(\beta)$. If $\lambda \in \partial\mathbb{D}$, then $(z - \lambda)H^2(\beta)$ is dense in $H^2(\beta)$. Hence $M = pH^2(\beta) = \overline{qsH^2(\beta)} = qH^2(\beta)$. Since $H^2(\beta)$ contains constant functions, $q \in M$. Now if $f \in H^2(\beta)$ and $f(z) = 0$ for all $z \in \Omega = \{\lambda_1, \ldots, \lambda_n\}$, by division property we have $\frac{f}{q} \in H^2(\beta)$. So $f \in qH^2(\beta) = M$. This completes the proof. □

Now we investigate the cyclicity of the multiplication operator M_z and we give some sufficient conditions for a vector to be cyclic for the multiplication operator M_z acting on $H^2(\beta)$.

Theorem 2. Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^2(\beta)$ with $\hat{f}(0) \neq 0$. If the sequence $\{\frac{\|M_z^n\|}{\beta(n)}\}_n \in \ell^2$ and $0 < \delta = \inf_n \frac{\beta(n+1)}{\beta(n)}$, then f is a cyclic vector of M_z on $H^2(\beta)$.

Proof. Put $M = span\{M_z^n f : n = 0, 1, 2, \ldots\}$. Let $g = \sum_{n=0}^{\infty} \hat{g}(n) z^n$ be any vector in $H^2(\beta)$ such that $< M_z^n f, g >= 0$ for all $n = 0, 1, 2, \ldots$. It will suffice to show that $g \equiv 0$. We note that

$$M_z^n f = \sum_{k=0}^{\infty} \hat{f}(k) z^{n+k} = \sum_{k=n}^{\infty} \hat{f}(k-n) z^k.$$

Now the relation $< M_z^n f, g >= 0$ implies that

$$\begin{aligned} 0 &= \sum_{k=n}^{\infty} \hat{f}(k-n)\overline{\hat{g}(k)}\beta(k)^2 \\ &= \hat{f}(0)\overline{\hat{g}(n)}\beta(n)^2 + \sum_{k=n+1}^{\infty} \hat{f}(k-n)\overline{\hat{g}(k)}\beta(k)^2. \end{aligned}$$

Thus we have

$$\begin{aligned} |\overline{\hat{g}(n)}\beta(n)| &\leq \left|\frac{1}{\hat{f}(0)\beta(n)}\right| \sum_{k=n+1}^{\infty} |\hat{f}(k-n)\overline{\hat{g}(k)}\beta(k)^2| \\ &= \frac{1}{|\hat{f}(0)|} \sum_{k=n+1}^{\infty} |\hat{f}(k-n)\beta(k-n)|.|\overline{\hat{g}(k)}\beta(k)| \frac{\beta(k)}{\beta(k-n)\beta(n)}. \end{aligned}$$

Since $\|M_z^n\| = \sup_k \frac{\beta(n+k)}{\beta(k)}$ we have $c = \sup_{n,k} \frac{\beta(n+k)}{\beta(k)\beta(n)} < \infty$ and so

$$\beta(n+k) \leq c\beta(k)\beta(n)$$

for all n and k. Thus for $k \geq n+1$,

$$\beta(k) \leq c\beta(k-n-1).\beta(n+1) = c\beta(k-n).\frac{\beta(k-n-1)\beta(n+1)}{\beta(k-n)}.$$

Therefore

$$\begin{aligned} \frac{\beta(k)}{\beta(k-n)\beta(n)} &\leq c\frac{\beta(k-n-1)}{\beta(k-n)}.\frac{\beta(n+1)}{\beta(n)} \\ &\leq c\frac{\beta(n+1)}{\beta(n)}.\sup_m \frac{\beta(m)}{\beta(m+1)} \\ &\leq \frac{c}{\delta}\frac{\beta(n+1)}{\beta(n)}. \end{aligned}$$

Since $\{\frac{\beta(n+k)}{\beta(n)\beta(k)}\}_n \in \ell^2$ for all k, clearly $\{\frac{\beta(n+1)}{\beta(n)}\} \in \ell^2$. We have

$$|\overline{\hat{g}(n)}\beta(n)| \leq \frac{c\|f\|\|g\|}{\delta|\hat{f}(0)|}.\frac{\beta(n+1)}{\beta(n)}$$

for all n. Now since $\{\frac{\beta(n+1)}{\beta(n)}\}_n \in \ell^2$, we can see that $M^\perp$ is finite dimensional and so there exists an integer $m > 0$ such that $(M_z^*)^m|_{M^\perp} = 0$ and $(M_z^*)^{m-1}|_{M^\perp} \neq 0$. Therefore

$$(M_z^*)^m(g) = (M_z^*)^m(\sum_{k=0}^{\infty} \hat{g}(k)z^k) = 0$$

and so $\hat{g}(k) = 0$ for all $k \geq m$. Thus for $h = \sum_{n=0}^{\infty} \hat{h}(n)z^n$ in M, $\hat{h}(k) = 0$ for all $k < m$. Since $f \in M$ with $\hat{f}(0) \neq 0$, it follows that $m = 0$. Thus $\hat{g}(k) = 0$ for all k and so $g \equiv 0$. This completes the proof. □

Let $H^2(\beta) \subset H(\mathbb{D})$. We say that M_z is polynomially bounded on $H^2(\beta)$ if $\|M_p\| \leq c\|p\|_{\mathbb{D}}$ for every polynomial p. In [2] we have shown that if M_z is polynomially bounded, then it is reflexive. In the following two theorems we suppose that $H^2(\beta)$ has division property, $\liminf \beta(n)^{1/n} = 1$, $\ker(M_z - \lambda)^* = \mathbf{C}e_\lambda$ for every λ in $\mathbb{D}$, M_z is polynomially bounded and the composition operator C_{-z} is bounded. Note that $(C_{-z}f)(\lambda) = f(-\lambda)$ for f in $H^2(\beta)$. In Theorems 3 and 4, under sufficient conditions we characterize the structure of an operator T when for some $n \in \mathbb{N}$, $TM_{\varphi^3} = (-1)^n M_{\varphi^3}T$, $\varphi \in M(H^2(\beta))$.

Theorem 3. Let $\sum_n \frac{1}{\beta(n)^2} < \infty$ and $\varphi \in M(H^2(\beta))$ be an odd univalent map. Also let for some $n \in \mathbb{N}$, $TM_{\varphi^3} = (-1)^n M_{\varphi^3}T$, $TM_\varphi + (-1)^{n+1}M_\varphi T$ is compact and $M_{\varphi^2}TM_\varphi = (-1)^n M_\varphi TM_{\varphi^2}$. Then there exists h in $M(H^2(\beta))$ such that $T = M_h$ if n is even and $T = M_g C_{-z}$ for some g in $M(H^2(\beta))$ if n is odd.

Proof. First let n be an odd integer number. Then we have

$$\begin{aligned}(TM_{\varphi^2} - M_{\varphi^2}T)M_\varphi &= -M_{\varphi^3}T + M_\varphi TM_{\varphi^2} \\ &= M_\varphi(TM_{\varphi^2} - M_{\varphi^2}T).\end{aligned}$$

Thus $SM_\varphi = M_\varphi S$ where $S = TM_{\varphi^2} - M_{\varphi^2}T$. Note that for $f \in H$,

$$< f, M_\varphi^* S^* e_\lambda > = < M_\varphi S f, e_\lambda > = \varphi(\lambda) < f, S^* e_\lambda > .$$

So $S^* e_\lambda \in \ker(M_\varphi - \varphi(\lambda))^*$. Now we show that $\ker(M_\varphi - \varphi(\lambda))^* = \ker(M_z - \lambda)^*$. Let $\varphi - \varphi(\lambda) = (z-\lambda)h(z)$. Since $ran(M_z - \lambda) = \ker e_\lambda$ and φ is univalent, $h \in H^2(\beta)$ and $h \neq 0$ on $\bar{\mathbb{D}}$. Also condition $\sum_n \frac{1}{\beta(n)^2} < \infty$ implies that $h \in C(\bar{\mathbb{D}})$. Thus $\frac{1}{h} \in H^\infty(\mathbb{D})$. Since $\mathbb{D}$ is simply connected, by the Farrel-Rubel-Shields Theorem [1, Theorem 5.1, p.151], there is a uniformly bounded sequence $\{p_n\}$ of polynomials converging pointwise to $\frac{1}{h}$. But M_z is polynomially bounded, thus $\|M_{p_n}\| \leq c\|p_n\|_{\mathbb{D}} \leq c_0$, by passing to a subsequence, we may assume that M_{p_n} converges to an operator A in the weak operator topology. By the same method used in the proof of Proposition 2 of [2], we can see that $A = M_{1/h}$ and so $\frac{1}{h} \in M(H^2(\beta))$. Note that $(M_z - \lambda)f = (M_\varphi - \varphi(\lambda))\frac{f}{h}$ for every f in $H^2(\beta)$. Thus indeed $ran(M_z - \lambda) = ran(M_\varphi - \varphi(\lambda))$. Now condition $\ker(M_z - \lambda)^* = \mathbf{C}e_\lambda$ implies that $S^* e_\lambda = g(\lambda)e_\lambda$ for some constant $g(\lambda)$. Therefore

$$(Sf)(\lambda) = < Sf, e_\lambda > = < f, S^* e_\lambda > = g(\lambda) f(\lambda)$$

for all λ in $\mathbb{D}$. Thus $S = M_g$ and hence $g \in M(H^2(\beta))$. Now we show that M_g is compact. Since $TM_\varphi + M_\varphi T$ is compact, $TM_{\varphi^2} + M_\varphi TM_\varphi$ and $M_\varphi TM_\varphi + M_{\varphi^2}T$ are compact. This implies that $M_g = S = TM_{\varphi^2} - M_{\varphi^2}T$ is compact and so by the Fredholm alternative theorem, $g = 0$. So $TM_{\varphi^2} = M_{\varphi^2}T$ and we have

$$M_\varphi(TM_\varphi + M_\varphi T) = M_\varphi TM_\varphi + TM_{\varphi^2} = (TM_\varphi + M_\varphi T)M_\varphi.$$

Hence $M_\varphi S = SM_\varphi$ where $S = TM_\varphi + M_\varphi T$. By the same technique used in the above argument, we can see that $S = M_g$ that is compact. So by the Fredholm alternative theorem, $g = 0$ which implies that $TM_\varphi = -M_\varphi T$. Now for $f \in H^2(\beta)$ we have

$$< f, M_\varphi^* T^* e_\lambda > = < -M_\varphi T f, e_\lambda > = -\varphi(\lambda) < f, T^* e_\lambda > .$$

Thus $T^*e_\lambda \in \ker(M_\varphi + \varphi(\lambda))^*$. Now we show that $\ker(M_\varphi + \varphi(\lambda))^* = \ker(M_z + \lambda)^*$. Let $\varphi + \varphi(\lambda) = (z+\lambda)\psi(z)$. Since $ran(M_z + \lambda) = \ker e_{-\lambda}$ and φ is univalent, $\psi \in H^2(\beta)$ and $\psi \neq 0$ on $\bar{\mathbb{D}}$. Thus $\frac{1}{\psi} \in M(H^2(\beta))$. Also note that $(M_z + \lambda)f = (M_\varphi + \varphi(\lambda))\frac{f}{\psi}$ for every f in $H^2(\beta)$. Thus indeed $ran(M_z + \lambda) = ran(M_\varphi + \varphi(\lambda))$. Since $\ker(M_z + \lambda)^* = \mathbf{C}e_{-\lambda}$ we have $\ker(M_\varphi + \varphi(\lambda))^* = \mathbf{C}e_{-\lambda}$ which implies that $T^*e_\lambda = g(\lambda)e_{-\lambda}$ for some constant $g(\lambda)$. Therefore

$$(Tf)(\lambda) =< Tf, e_\lambda >= g(\lambda) < f, e_{-\lambda} >= g(\lambda)(C_{-z}f)(\lambda)$$

for all λ in $\mathbb{D}$. Thus $T = M_gC_{-z}$ and hence $g \in M(H^2(\beta))$. Now let n be an even integer. Then $TM_{\varphi^3} = M_{\varphi^3}T$ and we have

$$(TM_{\varphi^2} - M_{\varphi^2}T)M_\varphi = -M_\varphi(TM_{\varphi^2} - M_{\varphi^2}T).$$

Hence similar to the above argument, $TM_{\varphi^2} - M_{\varphi^2}T = M_gC_{-z}$ for some g in $M(H^2(\beta))$. Since $TM_\varphi - M_\varphi T$ is compact, $TM_{\varphi^2} - M_{\varphi^2}T$ is also compact. So $M_g = M_gC_{-z} \circ C_{-z}$ is compact and by the Fredholm alternative theorem, $g = 0$ which implies that $TM_{\varphi^2} = M_{\varphi^2}T$. By continuing this way, we conclude that $TM_\varphi = M_\varphi T$ and so as we saw in the above argument, there exists h in $M(H^2(\beta))$ such that $T = M_h$. This completes the proof. □

Theorem 4. Let $\sum_n \frac{1}{\beta(n)^2} < \infty$ and $\varphi \in M(H^2(\beta))$ be an odd univalent map. Also let for some $n \in \mathbb{N}$, $TM_{\varphi^3} = (-1)^nM_{\varphi^3}T$, $TM_\varphi + (-1)^nM_\varphi T$ is compact and $M_{\varphi^2}TM_\varphi = (-1)^nM_\varphi TM_{\varphi^2}$. Then $T = M_gC_{-z}$ for some g in $M(H^2(\beta))$ if n is odd and $T = M_h$ for some h in $M(H^2(\beta))$ if n is even.

Proof. First let n be an odd integer. Note that since $TM_\varphi - M_\varphi T$ is compact, $TM_{\varphi^2} - M_{\varphi^2}T$ is also compact. We have

$$\begin{aligned} M_\varphi(TM_{\varphi^2} - M_{\varphi^2}T) &= -M_{\varphi^2}TM_\varphi + TM_{\varphi^3} \\ &= (TM_{\varphi^2} - M_{\varphi^2}T)M_\varphi. \end{aligned}$$

Thus $M_\varphi S = SM_\varphi$ where $S = TM_{\varphi^2} - M_{\varphi^2}T$ that is compact. Now by a similar method used in the proof of Theorem 3, we can see that $S = M_g$ for some g in $M(H^2(\beta))$. Since S is compact, $g = 0$ which implies that $TM_{\varphi^2} = M_{\varphi^2}T$. Again by the proof of Theorem 3, $T = M_h$ for some h in $M(H^2(\beta))$. Now let n be an even integer. Then since $TM_\varphi + M_\varphi T$ is compact, we can see that $TM_{\varphi^2} - M_{\varphi^2}T$ is also compact. Now we have

$$M_\varphi(TM_{\varphi^2} - M_{\varphi^2}T) = M_{\varphi^2}TM_\varphi - TM_{\varphi^3} = -(TM_{\varphi^2} - M_{\varphi^2}T)M_\varphi.$$

Thus $M_\varphi S = -SM_\varphi$ where $S = TM_{\varphi^2} - M_{\varphi^2}T$. So there exists g in $M(H^2(\beta))$ such that $S = M_g C_{-z}$. Since S is compact, $M_g = S \circ C_{-z}$ is also compact and so $g = 0$. Thus $S = 0$ which implies that $TM_{\varphi^2} = M_{\varphi^2}T$ and $M_\varphi(TM_\varphi + M_\varphi T) = (TM_\varphi + M_\varphi T)M_\varphi$. By continuing as above, we see that $TM_\varphi = -M_\varphi T$ which implies that there exists g in $M(H^2(\beta))$ such that $T = M_g C_{-z}$. This completes the proof. □

Theorems 3 and 4 can be extended easily for the case $TM_{\varphi^m} = (-1)^n M_{\varphi^m}T$, $m \geq 3$.

References

[1] T. Gamelin, Uniform algebras, Chelsea, New York, 1984.

[2] K. Seddighi and B. Yousefi, On the reflexivity of operators on function spaces, Proc. Amer. Math. Soc., 116 (1992), 45-52.

[3] A. Shields and L. Wallen, The commutants of certain Hilbert space operators, Indiana Univ. Math. J., 20 (1971), 777-788.

[4] A. Shields, Weighted shift operators and analytic function theory, Math. Survey, Amer. Math. Soc. Providence, 13 (1974), 49-128.

[5] B. Yousefi, On the space $\ell^p(\beta)$, Rend. Circ. Mat. Palermo, Serie II, Tomo XLIX (2000), 115-120.

[6] B. Yousefi, On the commutant of certain multiplication operators on some Banach spaces of analytic functions, Far East Journal of Mathematical Sciences, 3 (2001), 313-324.

[7] B. Yousefi, Bounded analytic structure of the Banach space of formal power series, Rend. Circ. Mat. Palermo, 51 (2002), 403-410.

[8] B. Yousefi, Strictly cyclic algebra of operators acting on Banach spaces of formal power series, Southeast Asian Bulletin of Mathematics, 27 (2003), 575-582.

[9] B. Yousefi, Strictly cyclic algebra of operators acting on Banach spaces $H^p(\beta)$, Czechoslovak Mathematical Journal, 54(129) (2004), 261-266.

[10] N. Zorboska, Cyclic composition operators on smooth weighted Hardy spaces, Rocky Mountain J. of Math., 29 (1999), 725-740.

Bahman Yousefi
Department of Mathematics
College of Sciences
Shiraz University
Shiraz 71454, Iran
E-mail: byousefi@hafez.shirazu.ac.ir

Rahmat Soltani
Department of Mathematics
Shiraz Payam-e-noor University
Shiraz, Iran
E-mail: Rsoltani@spnu.ac.ir