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CALCULATION OF THE HARMONIC SERIES VIA WARING'S FORMULA

JAE-YOUNG CHUNG

Abstract. We calculate the harmonic series $E_{2p} := \sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ via Waring's formula.

1. Introduction

It is well known as Euler's formula that for all natural numbers p,

$$E_{2p} := \sum_{n=1}^{\infty} \frac{1}{n^{2p}} = \frac{(-1)^{p+1} 2^{2p-1} B_{2p}}{(2p)!} \pi^{2p}. \tag{1.1}$$

Here B_{2p} are *the Bernoulli numbers* given by $B_j = f^{(j)}(0)$, where

$$f(x) = \frac{x}{e^x - 1}.$$

There are many ways to calculate the above series; for example, the Cauchy residue theorem, the Weierstrass product theorem, Fourier series or Taylor series expansions, and so on.

In this note, based on *Newton's formula* we first give a recurrence formula for the sequence E_{2p}: Let $E_{2p} = a_p \pi^{2p}$. Then a_p is computed by

$$a_p = \sum_{k=1}^{p} \frac{(-1)^{k-1} a_{p-k}}{(2k+1)!}, \quad a_0 = p. \tag{1.2}$$

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The formula (1.2) turns out to be more efficient than Euler's formula since the recurrence formula of the Bernoulli numbers B_{2p} involves $2p$-terms but the recurrence formula (1.2) involves p-terms.

Secondly, based on *Waring's formula* we show that a_p is given by

$$a_p = \sum \frac{(-1)^{k_2+k_4+\cdots}(k_1+\cdots+k_p-1)!p}{k_1!\cdots k_p!} 3!^{-k_1}5!^{-k_2}\cdots(2p+1)!^{-k_p} \tag{1.3}$$

where the summation is taken over all p-tuples $(k_1,\ldots,k_p)$ of nonnegative integers with $k_1+2k_2+\cdots+pk_p = p$.

It will be seen that the formula (1.3) is also more efficient than those which can be derived from Euler's formula.

2. Proof of the formulas

We prove the formulas (1.2) and (1.3). We denote by σ_k the k-th *elementary symmetric polynomial*

$$\sigma_k(x_1,\ldots,x_n) = \sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}\cdots x_{i_k}$$

in the indeterminates $x_1,\ldots,x_n$ over $\mathbb{R}$.

The following well known formulas are useful later which can be found in [2, Theorem 1.75, Theorem 1.76].

Proposition 1 (Newton's formula). Let $\sigma_1,\ldots,\sigma_n$ be the elementary symmetric polynomials and let $s_p = x_1^p+\cdots+x_n^p$, and $s_0 = p$. Then the equality

$$s_p - s_{p-1}\sigma_1 + s_{p-2}\sigma_2 - \cdots + (-1)^p s_0\sigma_p = 0 \tag{2.1}$$

holds for all $p = 1,2,\ldots,n$.

Proposition 2 (Waring's formula). With the same notations as in Proposition 1, we have

$$s_p = \sum \frac{(-1)^{k_2+k_4+\cdots}(k_1+\cdots+k_n-1)!p}{k_1!\cdots k_n!}\sigma_1^{k_1}\cdots\sigma_n^{k_n} \tag{2.2}$$

where the summation is taken over all n-tuples $(k_1,\cdots,k_n)$ of nonnegative integers with $k_1+2k_2+\cdots+nk_n=p$.

From De Moivre's formula

$$(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta,$$

it follows that

(2.3)
$$\sin m\theta=\sin^m\theta\left(\tbinom{m}{1}\cot^{m-1}\theta-\tbinom{m}{3}\cot^{m-3}\theta+\tbinom{m}{5}\cot^{m-5}\theta-\cdots\right).$$

Putting $m=2n+1$ and $\theta=\frac{k\pi}{2n+1}$, $k=1,2,\ldots,n$, in (2.3), it is easy to see that

$$x_k:=\cot^2\left(\frac{k\pi}{2n+1}\right),\ k=1,2,\ldots,n$$

are the solutions of the equation

$$P_n(x):=\tbinom{2n+1}{1}x^n-\tbinom{2n+1}{3}x^{n-1}+\tbinom{2n+1}{5}x^{n-2}-\ldots=0. \tag{2.4}$$

Now we employ the inequality

$$\cot^2 x<\frac{1}{x^2}<1+\cot^2 x,\quad 0<x<\frac{\pi}{2}. \tag{2.5}$$

Putting $x=\frac{\pi k}{2n+1}$ in (2.5), taking p-power and summing up the results we have

$$\left(\frac{\pi}{2n+1}\right)^{2p}s_p<\sum_{k=1}^{n}\frac{1}{k^{2p}}<\left(\frac{\pi}{2n+1}\right)^{2p}\left(n+\tbinom{p}{1}s_1+\cdots+\tbinom{p}{p}s_p\right),$$

where $s_p=x_1^p+\cdots+x_n^p$. It follows from (2.1) that

$$\lim_{n\to\infty}\left(\frac{\pi}{2n+1}\right)^{2p}s_k=0,\quad 1\le k\le p-1,$$

and we can write

$$a_p = \lim_{n\to\infty} \frac{s_p}{(2n+1)^{2p}}, \quad p \geq 1. \tag{2.6}$$

Now by the equation (2.1), (2.6) and the relation between the solutions of the equation $P_n(x) = 0$ and the coefficients of $P_n(x)$ we have

$$\begin{aligned} a_p &= \lim_{n\to\infty} \sum_{k=1}^{p} \frac{(-1)^{k-1} s_{p-k}}{(2n+1)^{2(p-k)}} \cdot \frac{\sigma_k}{(2n+1)^{2k}} \\ &= \sum_{k=1}^{p} \frac{(-1)^{k-1} a_{p-k}}{(2k+1)!}, \quad a_0 = p. \end{aligned}$$

Thus we obtain the recurrence formula (1.2). Now we prove the formula (1.3). In view of (2.6) and Waring's formula we can write

$$a_p = \lim_{n\to\infty} \sum_{k_1+2k_2+\cdots+pk_p=p} \frac{(-1)^{k_2+k_4+\cdots}(k_1+\cdots+k_p-1)!p}{k_1!\cdots k_p!} \cdot \frac{\sigma_1^{k_1}\ldots\sigma_p^{k_p}}{(2n+1)^{2p}}$$

On the other hand, for each $j = 1, 2, \ldots, p$, we have

$$\lim_{n\to\infty} \frac{\sigma_j}{(2n+1)^{2j}} = \frac{1}{(2j+1)!}.$$

Thus it follows that

$$\begin{aligned} \lim_{n\to\infty} \frac{\sigma_1^{k_1}\cdots\sigma_p^{k_p}}{(2n+1)^{2p}} &= \lim_{n\to\infty} \frac{\sigma_1^{k_1}\cdots\sigma_p^{k_p}}{(2n+1)^{2(k_1+2k_2+\cdots+pk_p)}} \\ &= 3!^{-k_1}5!^{-k_2}\cdots(2p+1)!^{-k_p}, \end{aligned}$$

which gives the formula (1.3).

Remark. It is not likely to obtain the formulas (1.2) and (1.3) easily from Euler's formula (1.1). Instead, if we use Faà di Bruno's formula for differentiation of composite functions we obtain a similar formula as (1.3) directly from Euler's formula. Indeed, since $B_n = (g \circ h)^{(n)}(0)$, where $g(x) = \frac{1}{x}$ and $h(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$, we can write

$$\begin{aligned} B_n &= \sum \frac{n!\, g^{(k_1+\cdots+k_n)}(h(0))}{k_1! \cdots k_n!} \left(\frac{h^{(1)}(0)}{1!}\right)^{k_1} \left(\frac{h^{(2)}(0)}{2!}\right)^{k_2} \cdots \left(\frac{h^{(n)}(0)}{n!}\right)^{k_n} \\ &= \sum \frac{n!\,(k_1 + \cdots + k_n)!(-1)^{k_1+\cdots+k_n}}{k_1! \cdots k_n!} \left(\frac{1}{2!}\right)^{k_1} \left(\frac{1}{3!}\right)^{k_2} \cdots \left(\frac{1}{n!}\right)^{k_n} \end{aligned}$$

where the summation is taken over all n-tuples $(k_1, \cdots, k_n)$ of nonnegative integers with $k_1 + 2k_2 + \cdots + nk_n = n$.

Thus we have

$$a_p = \sum \frac{(-1)^{k_1+\cdots+k_{2p}+p+1}(k_1 + \cdots + k_{2p})!2^{2p-1}}{k_1! \cdots k_{2p}!} 2!^{-k_1} 3!^{-k_2} \cdots (2p+1)!^{-k_{2p}}$$

where the summation is taken over all $2p$-tuples $(k_1, \ldots, k_{2p})$ of nonnegative integers with $k_1 + 2k_2 + \cdots + 2pk_{2p} = 2p$.

From one of the formulas (1.2) or (1.3) we obtain the followings:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}, \quad \sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450},$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{10}} = \frac{\pi^{10}}{93555}, \quad \sum_{n=1}^{\infty} \frac{1}{n^{12}} = \frac{691\pi^{12}}{638512875}, \quad \sum_{n=1}^{\infty} \frac{1}{n^{14}} = \frac{2\pi^{14}}{18243225},$$

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Jae Young Chung

Department of Mathematics,

Kunsan National University,
Kunsan 573-701,
Republic of Korea
E-mail : jychung@kunsan.ac.kr