Design of Optimal Controller Using Discrete Sliding Mode

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Abstract**—**In this paper, the discrete optimal control is made to have the robust property of Sliding mode controller. A augmented system with a virtual state is constructed for this objective and noble sliding surface is constructed based on this system. The sliding surface is the same as the optimal control trajectory in the original system. The states follow the optimal trajectory even if there exist uncertainties. The reaching phase problem of sliding mode control is disappear in this method.

Index Terms**—**Sliding Mode Control, Variable structure control , Robust control, Optimal control

I. INTRODUCTION

The SMC is a popular robust control method which has many good results and its applications[1][2]. Its robustness is guaranteed for uncertain system with matching condition. It is very desirable that this robustness can be with other control method. But it is not easy to achieve this object because of the dynamic property of the sliding mode control. Based on augmented system with a virtual state, Park made it possible that the robustness of SMC can be with other control method in continuous system[3].

In the discrete system, the result can be used in the same way, but more difficult to get the good result as in the continuous system because of the chattering problem. In this paper this object is achieved by using the discrete SMC which can reduced the chattering. The result of this paper is the combination of the robustness of SMC and the performance of optimal control.

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II. PROBLEM FORMULATION

Consider the n-th order system described by

$$\mathbf{x}(k+1) = (\mathbf{A}+\Delta\mathbf{A})\mathbf{x}(k) + (\mathbf{B}+\Delta\mathbf{B})\mathbf{u}_r(k) + \mathbf{D}\mathbf{f}(k) \quad (1)$$

where $\mathbf{x} \in \mathbf{R}^n, \mathbf{u}_r \in \mathbf{R}^m, \mathbf{f} \in \mathbf{R}^r$ and the bounded uncertainties $\Delta\mathbf{A}, \Delta\mathbf{B}$ and the disturbance matrix $\mathbf{D}$ satisfy the following matching condition.

$$\mathrm{rank}([\mathbf{B} : \Delta\mathbf{A} : \Delta\mathbf{B} : \mathbf{D}]) = \mathrm{rank}\mathbf{B} \quad (2)$$

With the above condition, the uncertainties and disturbance can be expressed as

$$\begin{aligned} \Delta\mathbf{A}\mathbf{x}(k) &= \mathbf{B}\Delta\mathbf{A}_1\mathbf{x}(k) \\ \Delta\mathbf{B}\mathbf{u}(k) &= \mathbf{B}\Delta\mathbf{B}_1\mathbf{u}(k) \\ \mathbf{D}\mathbf{f}(k) &= \mathbf{B}\mathbf{D}_1\mathbf{f}(k) \end{aligned} \quad (3)$$

The system (1) can be expressed as

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{u}_r(k) + \mathbf{h}(\mathbf{x},k)) \quad (4)$$

where $\mathbf{h}(\mathbf{x},k) = \Delta\mathbf{A}_1\mathbf{x}(k) + \Delta\mathbf{B}_1\mathbf{u}(k) + \mathbf{D}_1\mathbf{f}(k)$ is lumped uncertainty and bounded as follows.

$$\|\mathbf{h}(\mathbf{x},k)\| \le \rho(\mathbf{x},k) \quad (5)$$

The existing sliding mode surfaces have the following form[1].

$$\sigma(k) = \mathbf{S}\mathbf{x}(k) \quad (6)$$

where $\mathbf{S} = [c_1 \dots c_n]$ and $c_1, \dots, c_n$ are chosen so that sliding mode dynamics can be stable.

The above sliding surface has (n-1)-th order dynamics which are not the same as the original system. The reaching phase exists when the initial $\mathbf{x(t)}$ is not on the sliding surface, i.e., the initial $S(x)$ is not zero. The following condition guarantees the sliding mode[1].

$$\begin{aligned} 0 \le \sigma(k+1) \le \sigma(k), \ \sigma(k) \succ 0 \\ \sigma(k) \le \sigma(k+1) \le 0, \ \sigma(k) \prec 0 \end{aligned} \quad (7)$$

The equivalent control input is

$$\mathbf{u}_{eq} = -\mathbf{K}_{eq}\mathbf{x}(k) \quad (8)$$

where $K_{eq} = (SB)^{-1}S(A-I)$ (9)

The nonlinear control input is

$$u_{nl} = [\alpha(k) + \beta(k)]sgn[\sigma(k)] \quad (10)$$

where

$$\alpha(k) \le \eta \frac{\|\alpha(k)\|}{\|SB\|},\quad 0 \prec \eta \prec 1,\quad \beta(k) \ge h_{max},\quad \rho(x,t) \prec h_{max} \quad (11)$$

The following SMC input guarantees the condition (7).

$$u = u_{eq} + u_n \quad (12)$$

The problems considered in this paper is to propose an discrete optimal control with the robustness of SMC.

III. DISCRETE SMC WITH NEW SLIDING MODE

The novel sliding surfaces are made based on the augmented system with virtual states. The virtual states are defined from the controllable canonical form of the nominal system. The following controllable canonical form is obtained from (1) by a state transformation $z = Px$.

$$z(k+1) = A_c z(k) + B_c u(k) + B_c h(k) \quad (13)$$

The nominal system for system (13) is

$$z(k+1) = A_c z(k) + B_c u(k) \quad (14)$$

where $u_0(z_0,k)$ is a nominal regulating control input and differentiable.

From the nominal system (14), a virtual nominal state z_{0v}, which is proposed in this paper, is defined as follows.

$$z_{0v}(k+1) = \begin{bmatrix} A_3P & \alpha_n \end{bmatrix} \begin{bmatrix} x_0 \\ z_{0v} \end{bmatrix} + u_z(x_0,k) \quad (15)$$

where

$$u_z(x_0,t) = u_0(x_0,k+1),\quad A_3 = [0 - \alpha_1 - \alpha_2 \cdots - \alpha_{(n-1)}] \quad (16)$$

From the above equation, the following novel virtual state z_v is defined by replacing nominal state z_0 with state z.

$$z_v(k+1) = \begin{bmatrix} A_3P & \alpha_n \end{bmatrix} \begin{bmatrix} x \\ z_v \end{bmatrix} + u_z(x,k) \quad (17)$$

With the novel virtual state, the augmented system is constructed as follows.

$$x_a(k+1) = (A_a + \Delta A_a)x_a(k) + (B_a + \Delta B_a)u(k) + B_d u_z(x,k) + D_a f(k) \quad (18)$$

where $x_a(t) = \begin{bmatrix} x \\ z_v \end{bmatrix}$, $A_a = \begin{bmatrix} A & 0 \\ A_3 & \alpha_n \end{bmatrix}$, $B_a = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $D_a = \begin{bmatrix} D \\ 0 \end{bmatrix}$,

$\Delta A_a = \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix}$, $\Delta B_a = \begin{bmatrix} \Delta B \\ 0 \end{bmatrix}$, $B_d = \begin{bmatrix} 0 \\ I \end{bmatrix}$

For the above augmented system, the noble sliding surfaces are defined as

$$\sigma(k) = \begin{bmatrix} A_4 & P & I \end{bmatrix} \begin{bmatrix} x \\ z_v \end{bmatrix} - u_0(x,k) = Sx_a - u_0(x,k) \quad (19)$$

It is noted that the nominal control inputs are used to define the noble sliding surfaces.

The following initial virtual state makes the initial value of $s(x,z_v)$ equal to zero.

$$z_v(0) = -A_4Px(0) + u_0(x(0),0) \quad (20)$$

This eliminates the reaching phase completely. Now the following theorem is obtained.

Theorem 1 If the states of the system (13) are on the novel sliding surface (19), then the states of (1) have the dynamics of nominal system of (14).

Proof) If z_v are on the noble sliding surface. Then the following equations are satisfied by (19).

$$z_v(k) + \alpha_n z_1(k) + \alpha_{(n-1)} z_2(k) + \cdots + \alpha_1 z_n(k) - u_0(x,k) = 0 \quad (21)$$

From (21), the following is obtained.

$$z_v(k+1) + \alpha_n z_1(k+1) + \alpha_{(n-1)} z_2(k+1) + \cdots + \alpha_1 z_n(k+1) - u_0(x(k+1),k+1) = 0 \quad (22)$$

From the matching condition, the following is obtained.

$$z_v(k+1) + \alpha_n z_2(k) + \alpha_{(n-1)} z_3(k) + \cdots + \alpha_1 z_n(k+1) - u_0(x(k+1),k+1) = 0 \quad (23)$$

According to (17), z_v has the following dynamic.

$$z_v(k+1) + \alpha_n z_2(k) + \alpha_{(n-1)} z_3(k) + \cdots + \alpha_1 z_v(k) - u_0(x(k+1),k+1) = 0$$

Comparing (23) and (17), the following m equations are obtained.

$$z_v(k) = Z_n(k+1) \quad (24)$$

Now the following is obtained from (22).

$$z_n(k+1) = -\alpha_1 z_n(k) - \alpha_2 z_{(n-1)}(k) - \cdots - \alpha_n z_1(k) + u_{0i}(e,k) \quad (25)$$

(25) is the same as the nominal controllable canonical system (14). It can be transformed to the nominal system of (1) by the state transformation $\mathbf{P}$. Therefore the states of (1) on the novel sliding mode surface (19) have the same dynamics as those of the nominal system of (1). From Theorem 1 mentioned above and SMC theory, the following result is obtained.

Theorem 2. If SMC input $u(k)$ is designed to force the states of the system onto the sliding surface (19), then the states $x(k)$ follow the trajectories of the nominal system of (1).

Proof) It is obvious from the Theorem 1 and SMC theory.

IV. OPTIMAL CONTROL USING THE NOBLE SLIDING MODE SURFACE

In this section, a robust optimal controller, which makes the error states follow the optimal trajectories in spite of parameter uncertainties, is designed. Let's consider the system (1). The performance index for the nominal system is given by

$$J = \sum_{k=0}^{\infty}(x_o{}^T Q x_o + r u_0^2)dt \tag{26}$$

The optimal control input for the nominal system is [4]

$$\begin{aligned} u_{opt}(x_0) &= -(R + B^T P B)^{-1} B^T P A x_0(t) \\ &= -K x_0(t) \end{aligned} \tag{27}$$

where $\mathbf{P}$ is the solution of the following Riccati equation.

$$A^T P A - P + Q - A^T P B(R + B^T P B)^{-1} B^T P A = 0 \tag{28}$$

$u_z(x)$ is calculated as follows. The uncertainties are not considered in this calculation.

$$\begin{aligned} u_z(x) &= -Kx(k+1) = -K[Ax(k) - BKu_{opt}(x(k))] \\ &= Lx_a(k) \end{aligned} \tag{29}$$

where $L = [-K(A-BK)\ \ 0]$, $x_a = \begin{bmatrix} x \\ z_v \end{bmatrix}$

According to (17), the virtual state z_v is defined as

$$z_v(k+1) = [A_3 P\ \ \alpha_n]x_a + Lx_a \tag{30}$$

The augmented system is constructed as follows.

$$\begin{aligned} x_a(k+1) = &(A_a + \Delta A_a)x_a(k) \\ &+ (B_a + \Delta B_a)u(k) + B_d L x_a + D_a f(k) \end{aligned} \tag{31}$$

For the above system, the proposed novel sliding surface is given by

$$\sigma_a(k) = S_a x_a(k) \tag{32}$$

where $S_a = [S_1 P\ \ I] - [K\ \ 0]$

The equivalent control input is

$$u_{eqa} = -K_{eqa} x_a \tag{33}$$

where K_{eqa} is $K_{eqa} = (S_a B_a)^{-1} S_a (A_a - I)$ (34)

The nonlinear control input is

$$u_{nla} = [\alpha_a(k) + \beta\ (k)]\mathrm{sgn}[\sigma_a(k)] \tag{35}$$

where $\alpha_a(k) \le \eta \frac{\|\sigma_a(k)\|}{\|S_a B_a\|}$, $0 \prec \eta \prec 1$, $\beta(k) \ge h_{max}$, $\rho(x,t) \prec h_{max}$

The following input guarantees the sliding mode and optimal trajectory.

$$u_a = u_{eqa} + u_{na}$$

V. CONCLUSIONS

A novel design method of sliding surfaces has been proposed for a discrete system. With the sliding surfaces, a new discrete SMC, which makes the states of the system follow the optimal trajectories is designed. Any type of controller can be a nominal controller instead of optimal controller. The reaching phase is easily eliminated by the initial virtual states chosen appropriately.

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