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APPROXIMATION- SOLVABILITY OF A CLASS OF *A*-MONOTONE VARIATIONAL INCLUSION PROBLEMS

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ABSTRACT. First the notion of the A-monotonicity is applied to the approximation - solvability of a class of nonlinear variational inclusion problems, and then the convergence analysis is given based on a projection-like method. Results generalize nonlinear variational inclusions involving H-monotone mappings in the Hilbert space setting.

1. Introduction and Preliminaries

Based on the notion of the *A-monotonicity,* recently the author [8] studied a new class of variational inclusion problems, including hemivariational inclusion problems applied to engineering and mechanics. The obtained results generalize some variational inclusion problems introduced and studied by Fang and Huang [2]. They solved nonlinear variational problems applying the resolvent operator technique. These notions have energized the theory of maximal monotone mappings in general. In this paper consider applications of A-monotone mappings to the approximation-solvability of a class of nonlinear variational inclusions in a Hilbert space setting. The convergence analysis for the solution is based on a projection-like method. The obtained results generalize results on general maximal monotone and H-monotone mappings, including [2]. We have established some auxiliary results as well. For more details on the generalized monotonicity, we recommend [1- 9].

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Key Words and Phrases: A-monotone mappings, Resolvent operator technique, Relaxed monotone mappings, Approximation-solvability, Projection-like methods.

Definition 1. [8] Let A: X $\rightarrow$X* be a mapping from a reflexive Banach space X into its dual X* and M: X $\rightarrow$P(X*) be another mapping from X into the power set P(X*) of X*. The map M is said to be *A-monotone* if M is m-relaxed monotone and A + ρM is maximal monotone for $\rho > 0$.

Definition 2. [2] Let H: H$\rightarrow$H and M: H$\rightarrow 2^H$ be any two mappings on H. The map M is said to be *H-monotone* if M is monotone and $(H + \rho M)(H) = H$ holds for $\rho > 0$.

This is equivalent to stating that H + ρM is maximal monotone if M is monotone and H + ρM is maximal monotone. If H is strictly monotone and M is H-monotone, then M is maximal monotone. Let the resolvent operator $J^{\rho}_{H,M}$: H $\rightarrow$H be defined by

$$J^{\rho}_{H,M}(u) = (H + \rho M)^{-1}(u) \ \forall \ u \in H.$$

On the top of that, if H is r-strongly monotone and M is H-monotone, then the resolvent operator $J^{\rho}_{H.M}$ is (1/r)-Lipschitz continuous for $r > 0$. From now on, P(H) shall denote the power set 2^H.

Definition 3. A mapping T: H $\rightarrow$H is said to be:

(i) r-strongly monotone with respect to A if there exists a positive constant r such that

$$< T(x) - T(y), A(x) - A(y) > \ \geq \|x - y\|^2 \quad \forall \ x,y \in H.$$

(ii) r-strongly monotone if there exists a positive constant r such that

$$< T(x) - T(y), x - y > \ \geq r \|x - y\|^2 \quad \forall \ x,y \in H.$$

(iii) m-relaxed monotone if there is a positive constant m such that

$$< T(x) - T(y), x - y > \ \geq (-m) \|x - y\|^2 \quad \forall \ x,y \in H.$$

(iv) (γ,s)-relaxed cocoercive with respect to A if there exist positive constants γ and s such that

$$<T(x) - T(y), A(x) - A(y)> \ \geq (-\gamma)\| T(x) - T(y)\|^2 + s \|x - y\|^2 \quad \forall \ x,y \in H$$

Lemma 1. Let A: H→ H be r-strongly monotone and M: H→ P(H) be A-monotone. Then the resolvent operator $J^{\rho}_{A,M}$(u): H → H is [1/(r -ρm)]-Lipschitz continuous for $0 < \rho < r/m$, where r, ρ and m are positive constants.

Proof. For any u,v∈H, we have from the definition of the resolvent operator that

$$J^{\rho}_{A,M}(u) = (A + \rho M)^{-1}(u)$$

$$J^{\rho}_{A,M}(v) = (A + \rho M)^{-1}(v).$$

It follows that

$$(1/\rho)[u - A(J^{\rho}_{A,M}(u))] \in M(J^{\rho}_{A,M}(u))$$

$$(1/\rho)[v - A(J^{\rho}_{A,M}(v))] \in M(J^{\rho}_{A,M}(v)).$$

Since M is A-monotone (and hence m-relaxed monotone), it implies that

$$(1/\rho)\langle u - A(J^{\rho}_{A,M}(u)) - [v - A(J^{\rho}_{A,M}(v))],\ J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\rangle$$

$$= (1/\rho)\langle u - v - [A(J^{\rho}_{A,M}(u)) - A(J^{\rho}_{A,M}(v))],\ J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\rangle$$

$$\geq (-m)\| J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\|^2.$$

As a result, we have

$$\|u - v\|\, \|J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\| \geq \langle u - v, J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\rangle$$

$$\geq \langle A(J^{\rho}_{A,M}(u)) - A(J^{\rho}_{A,M}(v)),\ J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\rangle$$

$$- \rho m\|J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\|^2$$

$$\geq r\|J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\|^2 - \rho m\,\|J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\|^2$$

$$= (r - \rho m)\,\| J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\|^2.$$

Lemma 2. Let M: H → P(H) be A-monotone. Then the resolvent operator $J^{\rho}_{I,M} := (I+\rho M)^{-1}$: H → H is [1/(1 -ρm)]-Lipschitz continuous for 0 < ρ < 1/m, where ρ and m are positive constants and I is the identity mapping.

Lemma 3. Let H: H→ H be r-strongly monotone and M: H → P(H) be H-monotone. Then the resolvent operator $J^{\rho}_{H,M}(u)$: H → H is r-cocoercive

Proof. For any u,v∈H, we have from the definition of the resolvent operator that

$$J^{\rho}_{H,M}(u) = (H+\rho M)^{-1}(u)$$

$$J^{\rho}_{H,M}(v) = (H+\rho M)^{-1}(v).$$

It follows that

$$(1/\rho)[u - H(J^{\rho}_{H,M}(u))] \in M(J^{\rho}_{H,M}(u))$$

$$(1/\rho)[v - H(J^{\rho}_{H,M}(v))] \in M(J^{\rho}_{H,M}(v)).$$

Since M is H-monotone and H *is* r-strongly monotone, it implies that

$$(1/\rho)\langle u - H(J^{\rho}_{H,M}(u)) - [v - H(J^{\rho}_{H,M}(v))],\ J^{\rho}_{H,M}(u) - J^{\rho}_{H,M}(v)\rangle$$

$$= (1/\rho)\langle u - v - [H(J^{\rho}_{H,M}(u)) - H(J^{\rho}_{H,M}(v))],\ J^{\rho}_{H,M}(u) - J^{\rho}_{H,M}(v)\rangle \geq 0.$$

As a result, we have

$$\langle u - v, J^{\rho}_{H,M}(u) - J^{\rho}_{H,M}(v)\rangle \geq \langle H(J^{\rho}_{H,M}(u)) - H(J^{\rho}_{H,M}(v)),\ J^{\rho}_{H,M}(u) - J^{\rho}_{H,M}(v)\rangle$$

$$\geq r\,\|J^{\rho}_{H,M}(u) - J^{\rho}_{H,M}(v)\|^2.$$

For $H = I$ and r ≤1, $J^{\rho}_{M}(u)=(I+\rho M)^{-1}$: H → H is 1-cocoercive.

Lemma 4. [2] Let H: H→H be r-strongly monotone and M: H → P(H) be H-monotone. Then the resolvent operator $J^{\rho}_{H,M}$: H → H is (1/r)-Lipschitz continuous for a positive constant r.

Lemma 5. Let A: H→ H be r-strongly monotone and M: H → P(H) be A-monotone. Then the resolvent operator $J^{\rho}_{A,M}$: H → H is (r -ρm)-cocoercive for 0 < ρ < r/m, where r, ρ and m are positive constants

Proof. For any u,v∈H, we have from the definition of the resolvent operator that

$$J^{\rho}_{A,M}(u) = (A + \rho M)^{-1}(u)$$

$$J^{\rho}_{A,M}(v) = (A + \rho M)^{-1}(v).$$

It follows that

$$(1/\rho)[u - A(J^{\rho}_{A,M}(u))] \in M(J^{\rho}_{A,M}(u))$$

$$(1/\rho)[v - A(J^{\rho}_{A,M}(v))] \in M(J^{\rho}_{A,M}(v)).$$

Since M is A-monotone (and hence m-relaxed monotone), it implies that

$$(1/\rho)< u - A(J^{\rho}_{A,M}(u)) - [v - A(J^{\rho}_{A,M}(v))], \quad J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v) >$$

$$= \quad (1/\rho)< u - v - [A(J^{\rho}_{A,M}(u)) - A(J^{\rho}_{A,M}(v))], \quad J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v) >$$

$$\geq (-m) \| J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\|^2.$$

As a result, we have

$$< u - v, J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)>$$

$$\geq < A(J^{\rho}_{A,M}(u)) - A(J^{\rho}_{A,M}(v)), \quad J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)>$$

$$- \rho m \|J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\|^2$$

$$\geq r \|J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\|^2 \quad - \rho m \|J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\|^2$$

$$= (r - \rho m) \| J^{\rho}_{A,M}(u) - J^{\rho}_{A,M}(v)\|^2.$$

Example 1. [3, Lemma 7.11] Let X be a reflexive Banach space and X* its dual. Suppose that A: X →X* is m-strongly monotone and f: X→R is locally Lipschitz such that ∂f is α-relaxed monotone. Then ∂f is A-monotone (i.e., A + ∂f is maximal monotone for m -α > 0, where m, α > 0) for ρ = 1. Since A is m-strongly monotone and ∂f is α-relaxed monotone, it implies that A + ∂f is (m -α)-strongly monotone. It further follows that A + ∂f is pseudomonotone and hence A + ∂f is, in fact, maximal monotone.

Example 2. [5, Theorem 4.1] Let X be a reflexive Banach space and X* its dual. Let A: X →X* be a-strongly monotone and B: X →X* be c-strongly Lipschitz continuous. Let f: X →R be locally Lipschitz such that ∂f is relaxed α-monotone. Then ∂f is *(A -B)*-monotone (i.e. A -B + ∂f is maximal monotone for a -c -α > 0) for ρ = 1.

Let H be a real Hilbert space and let A be a nonempty closed convex subset of H. Let T: H →H be a nonlinear mapping. Let A: H→H and M: H →P(H) be any mappings. Then the problem of finding $a \in H$ such that

$$0 \in T(a) + M(a) \qquad (1)$$

is called the nonlinear variational inclusion (NVI) problem.

Let f: H →R be a locally Lipschitz continuous function and ∂f:H →P(H) be m-relaxed monotone.

Then for M = ∂f, the NVI (1) problem reduces to: find an element a∈H such that

$$0 \in T(a) + \partial f(a). \qquad (2)$$

If f: H →R is proper, convex and lower semicontinuous, and f′(x) denotes the gradient of f at x such that M(x) = ∂f(x) for all x∈H, then problem (1) reduces to: find an element a ∈A such that

$$< T(a), x - a > + < f'(a), x - a > \geq 0 \quad \forall\, x \in H, \qquad (3)$$

where A is a nonempty closed convex subset of H.

It follows from (3) that

$$< T(a), x - a > + f(x) - f(a) > \geq 0 \quad \forall\, x \in H. \qquad (4)$$

When M(x) = $\partial_A(x)$ for all x∈A, where A is a nonempty closed convex subset of and δ_A denotes the indicator function of A, the NVI (1) problem reduces to the problem: determine an element a ∈A such that

$$< T(a), x - a > \geq 0 \quad \forall\, x \in A. \qquad (5)$$

Let f: H→R∪{± ∞} be a functional on H. A functional x*∈H is a subgradient of f at u iff f(u) ≠ μ ∞ and

$$f(v) \geq f(u) + < x^*, v - u > \quad \forall\, v \in H.$$

The set of all subgradients of f at u, denoted ∂f(u), is called the subdifferential at u. If there exists no subgradients, then ∂f(u) = ∅.

A function f: H →R∪{± ∞} is said to be *one-sided directional Gâteaux- differentiable* at x* if there is the f′(x*, h) such that

$$\lim_{\mu \to 0} [f(x^* + \mu h) - f(x^*)]/\mu = f'(x^*, h) \quad \forall\, h \in H.$$

If f is convex, then f is *one-sided directional Gâteaux-differentiable* at every point $x \in H$ with $f(x) \neq \mu\, \infty$. On the top of that, we have

$$f(x) - f(u) \geq f'(u, x - u) \;\forall\, x \in H,$$

and

$$f'(u, x - u) \geq - f'(u, -(x - u)) \;\forall\, x \in H.$$

A function $f: H \to R \cup \{\pm \infty\}$ is called *locally Lipschitz* at x if a neighborhood U of x exists such that f is finite on U and

$$|f(x) - f(y)| \leq c\, \|x - y\| \;\forall\, x \in H,$$

where c is a positive constant depending on U.

Next we define the *generalized directional differential* (in the sense of Clarke) of f at x in the direction y, denoted $f^0(x,y)$, by

$$\lim_{\mu \to 0+,\, h \to 0} [f(x + h + \mu y) - f(x + h)]/\mu = f^0(x,y).$$

The corresponding generalized gradient of f at x, denoted by $\bar{\partial} f(\bar{x})$, is defined by

$$\bar{\partial} f(x) = \{x^*: x^* \in H,\; f^0(x, y - x) \geq \langle x^*, y - x \rangle \quad \forall\, y \in H\},$$

where $\bar{\partial} f: H \to 2^H$. If we set $M(x) = \partial f(x)$ in (1), then it reduces to a constrained problem: find an
element $a \in H$ such that

$$\langle T(a), x - a \rangle + f^0(a, x - a) > \;\geq 0 \quad \forall\, x \in H, \tag{6}$$

Let $B(u_0, r)$ denote the closed ball in H defined by

$$B(u_0, r) = \{v \in H: \|u_0 - v\| \leq r \text{ for } r > 0\},$$

where u_0 is the center and r is the radius. Let A be a closed and star-shaped subset of H with respect to $B(u_0, r)$. A is star-shaped with respect to $B(u_0, r)$ if

$$v \in A \Leftrightarrow \lambda v + (1 - \lambda) w \in A \text{ for any } \lambda \in [0, 1] \text{ and } w \in B(u_0, r).$$

Let $d_A : H \rightarrow R$ denote the distance function of A defined by

$$d_A(v) = \inf_{w \in A} \|v - w\| \text{ for } v \in H.$$

Further more, let $T_A(u)$ denote Clarke's tangent cone of A at u, which is defined by

$$T_A(u) = \{k \in H : \forall\, u_n \rightarrow u,\ u_n \in A,\ \forall\, \lambda_n \rightarrow 0, \text{ there exists } k_n \rightarrow k \text{ such that } u_n + \lambda_n k_n \in A\}.$$

Note that $T_A(u)$ is a closed convex cone and it always contains zero. Now if we set M(x) = $\partial\delta(T_A(x))$, where $\delta(T_A)$ denotes the indicator function of $T_A(x)$, then the NVI (1) problem reduces to: find an element $a \in A$ such that

$$< T(a), k > \ \geq 0 \quad \forall\, k \in T_A(a). \tag{7}$$

Since A is not convex, the problem (7) is called a constrained hemivariational inequality (NHI) problem. Clearly, the NHI (7) problem reduces to the NVI (5) problem when A is convex.

Lemma 6. Let H be a real Hilbert space, let A: H →H be strictly monotone, and M: H → 2^H be A-monotone. Then an element $a \in H$ is a solution to the NVI (1) problem iff a satisfies

$$a = J^{\rho}_{A,M}[A(a) - \rho T(a)], \tag{8}$$

where T: H →H is any mapping on H and ρ is a positive constant.

Theorem 1. Let H be a real Hilbert space. Let A: H→H be r-strongly monotone and α-Lipschitz continuous. Let M: H → P(H) be A-monotone. Suppose that T: H →H be a mapping such that T is (s)-strongly mootone with respect to A and μ-Lipschitz continuous. If, in addition, there exists a constants ρ > 0 such that

$$\sqrt{\alpha^2 2\rho s + \rho^2 \mu^2} \quad < \quad r - \rho m,$$

then the NVI (1) problem has a unique solution.

Proof. For $u, v \in H$, let us define a mapping Λ: H →H by

$$\Lambda(u) = J^{\rho}_{A,M}(A(u) - \rho T(u)).$$

Then we have

$$\|\Lambda(u) - \Lambda(v)\| = \|J^{\rho}_{A,M}(A(u) - \rho T(u)) - J^{\rho}_{A,M}(A(u) - \rho T(u))\|$$

$\leq [1/(r - \rho m)] \, \| (A(u) - \rho T(u)) - (A(u) - \rho T(u)) \|$.

It follows that

$$\|A(u) - A(v) - \rho(T(u) - T(v))\|^2$$

$$= \|A(u) - A(v)\|^2 + \rho^2 \|T(u) - T(v))\|^2 - 2\rho < A(u) - A(v), T(u) - T(v)>$$

$$\leq \alpha^2 \|u - v\|^2 + \rho^2 \mu^2 \|u - v\|^2 - 2\rho s \|u - v)\|^2 + 2\rho\gamma \|T(u) - T(v))\|^2$$

$$\leq \alpha^2 \|u - v\|^2 + \rho^2 \mu^2 \|u - v\|^2 - 2\rho s \|u - v)\|^2$$

$$= (\alpha^2 - 2\rho s + \rho^2 \mu^2) \| u - v\|^2 .$$

Hence,

$$| \Lambda(u) - \Lambda(v) | \quad \leq [\theta/(r - \rho m)] \, | u - v | ,$$

where $\theta = \sqrt{\alpha^2 2\rho s + \rho^2 \mu^2} \quad < \quad r - \rho m$

Hence, Λ: H $\rightarrow$H is a contraction for $0 < \rho < r/m$. This implies that there exists a unique element $a \in H$ such that

$\Lambda(a) = a$,
that means,

$$a = \quad J^{\rho}_{A,M}(A(a) - \rho T(a)).$$

It follows from Lemma 6 that a is a unique solution to the NVI (1) problem.

Corollary 1. Let H be a real Hilbert space. Let A: H$\rightarrow$H be r-strongly monotone and α-Lipschitz continuous. Let ∂f: H $\rightarrow$ P(H) be A-monotone. Suppose that T: H $\rightarrow$H be a mapping such that T is (s)-strongly monotone with respect to A and μ-Lipschitz continuous. If, in addition, there exists a constants $\rho > 0$ such that

$$\sqrt{\alpha^2 2\rho s + \rho^2 \mu^2} \quad < \quad r - \rho m,$$

then the NVI (2) problem has a unique solution.

Corollary 2. Let H be a real Hilbert space. Let H: H$\rightarrow$H be r-strongly monotone and α-Lipschitz continuous. Let M: H $\rightarrow$ P(H) be H-monotone. Suppose that T: H $\rightarrow$H be a mapping such that T is

(s)-strongly monotone with respect to H and μ-Lipschitz continuous. If, in addition, there exists a constants $\rho > 0$ such that

$$\sqrt{\alpha^2 2\rho s + \rho^2 \mu^2} < r,$$

then the NVI (1) problem has a unique solution.

2. Convergence Analysis

In this section, we apply a projection-type iterative algorithm to approximate the unique solution to the NVI (1) problem.

Algorithm 1. For an arbitrarily chosen initial point $a^0 \in H$, compute the sequence $\{a^k\}$ such that

$$a^{k+1} = (1 - \alpha^k)a^k + \alpha^k J_{A,M}[A(a^k) - \rho T(a^k)] \quad \text{for } k \geq 0,$$

where the sequence $\{\alpha^k\}$ satisfies

$$0 \leq \alpha^k < 1 \text{ and } \Sigma^{\infty}{}_{k=0}\, \alpha^k = \infty.$$

Theorem 2. Let H be a real Hilbert space. Let A: H→H be r-strongly monotone with respect to A and α-Lipschitz continuous. Let M: H → P(H) be A-monotone. Suppose that T: H →H be a mapping such that T is (s)-strongly monotone with respect to A and μ-Lipschitz continuous. If, in addition, there exists a constants $\rho > 0$ such that

$$\sqrt{\alpha^2 2\rho s + \rho^2 \mu^2} < r - \rho m \text{ for } \rho < r/m,$$

and the sequence $\{a^k\}$ is generated by Algorithm 1, then the sequence $\{a^k\}$ converges to a unique solution to the NVI (1) problem.

Proof. Since in Theorem 1, it is shown that an element $a \in H$ is the unique solution to the, NVI (1) problem, we have

$$\|a^{k+1} - a\| = \| (1-\alpha^k)a^k + \alpha^k J_{A,M}[A(a^k) - \rho T(a^k)] - (1 - \alpha^k)a - \alpha^k) J_{A,M}[A(a) - \rho T(a)] \|$$

$$\leq (1 - \alpha^k) \| (a^k - a) \| + \alpha^k \| J_{A,M}[A(a^k) - \rho T(a^k)] - J_{A,M}[A(a) - \rho T(a) \|$$

$$\leq (1 - \alpha^k) \| (a^k - a) \| + \alpha^k /(r - \rho m) \| A(a^k) - A(a) - \rho(T(a^k) - T(a)) \|$$

$$\leq (1 - \alpha^k) \| (a^k - a) \| + \{\alpha^k/(r - \rho m)\} \sqrt{\alpha^2 2\rho s + \rho^2 \mu^2} \| (a^k - a) \|$$
$$= \{1 - \alpha^k + [\alpha^k/(r - \rho m)]\theta\} \| (a^k - a) \|$$

$$= [1 - \alpha^k + (\alpha^k/(r - \rho m))\theta] \| (a^k - a) \|$$

$$= \{1 - (1 - \Theta)\alpha^k\} \| (a^k - a) \|$$

$$\leq \Pi^k{}_{j=0} \{1 - (1 - \Theta)\alpha^j\} \| (a^0 - a) \|, \quad (9)$$

where $\Theta < 1$ for $\Theta = \theta/(r - \rho m)$ and for

$$\sqrt{\alpha^2 2\rho s + \rho^2 \mu^2} < r - \rho m.$$

Since $\Theta < 1$ and $\Sigma^\infty{}_{k=0}\, \alpha^k$ is divergent, it implies from [9] that

$$\lim_{k\to\infty} \Pi^k{}_{j=0} \{1 - (1 - \Theta)\alpha^j\} = 0.$$

Now it follows from (5) that the sequence $\{a^k\}$ converges to a, the unique solution to the NVI (1) problem.

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