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NEUTRAL INTEGRODIFFERENTIAL CONTROL SYSTEMS WITH INFINITE DELAY IN BANACH SPACES

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ABSTRACT. Sufficient conditions for the controllability of neutral functional integrodifferential systems with infinite delay in Banach space are established by means of the Schaefer fixed point theorem.

1. INTRODUCTION

Several authors have investigated the neutral functional differential equations in abstract spaces [3,5,7,9,10,12]. These type of equations occur in the study of heat conduction in materials with memory and many other physical phenomena. So it is interesting to study the controllability problem for such systems. There are several papers appeared on the controllability of nonlinear systems in infinite dimensional spaces [2]. Balachandran and Anandhi [1] discussed the controllability of neutral functional integrodifferential systems in abstract phase space with the help of Schauder's fixed point theorem. Recently Fu [6] studied the same problem in abstract phase space by utilizing the Sadovskii fixed point theorem. Wang and Wang [14] discussed the controllability of abstract neutral functional differential systems with infinite delay using the Schaefer fixed point theorem. The purpose of this paper is to study the controllability of neutral functional integrodifferential systems with infinite delay by using the Schaefer fixed point theorem.

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2. PRELIMINARIES

Consider the following neutral functional integrodifferential systems with infinite delay

$$
\begin{aligned}
(1)\ \frac{d}{dt}\Big[x(t)-g(t,x_t)\Big] &= Ax(t)+Bu(t)+f\Big(t,x_t,\int_0^t q(t,s,x_s)ds\Big),\ t\in[0,b]=J,\\
x_0 &= \phi\in\mathcal{B},
\end{aligned}
$$

where the state $x(\cdot)$ takes values in Banach space X endowed with norm $|\cdot|$, the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. A is the infinitesimal generator of a C_0 semigroup of bounded linear operators $T(t),\ t\geq 0$ on X, B is a bounded linear operator from U into X, where $q: J\times J\times\mathcal{B}\to X$, $f: J\times\mathcal{B}\times X\to X$ and $g: J\times\mathcal{B}\to X$ are appropriate functions. The histories $x_t: (-\infty,0]\to X$, $x_t(\theta)=x(t+\theta)$, $\theta\leq 0$, belong to some abstract phase space $\mathcal{B}$, that is, a linear space of functions mapping $(-\infty,0]$ into X endowed with a semi norm $\|\cdot\|_{\mathcal{B}}$ in $\mathcal{B}$ [8,9,11]. Throughout this paper, we assume that $\mathcal{B}$ satisfies the following axioms:

: (A_1) If $x:(-\infty,\sigma+a)\to X$, $a>0$, is continuous on $[\sigma,\sigma+a)$ and $x_\sigma\in\mathcal{B}$, then for every t in $[\sigma,\sigma+a)$ the following conditions hold:
 : (i) x_t is in $\mathcal{B}$;
 : (ii) $|x(t)|\leq H\|x_t\|_{\mathcal{B}}$;
 : (iii) $\|x_t\|_{\mathcal{B}}\leq K(t-\sigma)\sup\{|x(s)|:\sigma\leq s\leq t\}+M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,
 where $H\geq 0$ is a constant; $K:[0,\infty)$ is continuous, $M:[0,\infty)\to[0,\infty)$ is locally bounded, H, K and M are independent of $x(\cdot)$.

: (A_2) For the function $x(\cdot)$ in (A_1), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma,\sigma+a)$.

: (A_3) The space $\mathcal{B}$ is complete.

For brevity let us take $\|\cdot\|_{\mathcal{B}}=\|\cdot\|$. We need the following fixed point theorem.

Schafer's Theorem[13]: Let S be a convex subset of a normed linear space E and $0\in S$. Let $F:S\to S$ be a completely continuous operator and let

$$\zeta(F)=\Big\{x\in S:\lambda Fx=x \text{ for some } \lambda\in(0,1)\Big\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

If $x(t)$ is a mild solution of system (1) then it can be written as [5,7]

$$\begin{aligned} x(t) &= T(t)[\phi(0) - g(0,\phi) + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)ds \\ &\quad + \int_0^t T(t-s)\left[(Bu)(s) + f\Big(s, x_s, \int_0^s q(s,\tau,x_\tau)d\tau\Big)\right]ds, \ t \in J, \\ x_0 &= \phi. \end{aligned} \tag{2}$$

We say that the system (1) is said to be controllable on the interval J if for every initial function $\phi \in \mathcal{B}$ and $x_1 \in X$, there exist a control $u \in L^2(J,U)$ such that the mild solution of (1) satisfies $x(b) = x_1$.

Assume the following hypotheses:

: (A_4) A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$, $t > 0$ on X and there exist a constants $M \leq 1$ and $M_1 > 0$ such that

$$|T(t)| \leq M \text{ and } |AT(t)| \leq M_1.$$

: (A_5) The linear operator $W : L^2(J,U) \to X$, defined by

$$Wu = \int_0^b T(b-s)Bu(s)ds,$$

has an inverse operator $\widetilde{W}^{-1}$, which takes values in $L^2(J,U)/\ker W$ and there exist positive constants M_2, M_3 such that $|B| \leq M_2$ and $|\widetilde{W}^{-1}| \leq M_3$.

: (A_6) (i) The function g is completely continuous and such that the operator $G : \mathcal{B} \to \mathcal{B}$ defined by $(G\phi)(t) = g(t,\phi)$ is compact.

: (ii) There exists c_1 and c_2 such that $\widetilde{K}c_1 < 1$ and $|g(t,\phi)| \leq c_1\|\phi\| + c_2, t \in J$, $\phi \in \mathcal{B}$, where $\widetilde{K} = \max\{K(t) : t \in J\}$.

: (A_7) (i) For each $t \in J$, the function $f(t,\cdot,\cdot) : \mathcal{B} \times X \to X$ is continuous and for each $(x,y) \in \mathcal{B} \times X$, the function $f(\cdot,x,y) : J \to X$ is strongly measurable.

: (ii) For each $(t,s) \in J \times J$, the function $q(t,s,\cdot) : \mathcal{B} \to X$ is continuous and for each $x \in \mathcal{B}$, the function $q(\cdot,\cdot,x) : J \times J \to X$ is strongly measurable.

: (iii) For every positive integer k, there exists $\alpha_k(\cdot) \in L^1[(0,b)]$ such that

$$\sup\Big\{|f(t,x,y)| : \|x\|, |y| \leq k\Big\} \leq \alpha_k(t) \text{ for } t \in J \text{ a.e.},$$

: (iv) There exist an integrable function $m : J \to [0,\infty)$ and a constant $\alpha > 0$ such that

$$|q(t,s,x)| \leq \alpha m(s)\Omega_0(\|x\|), \ t \in J, \ x \in \mathcal{B}$$

where $\Omega_0 : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

: (v) There exists an integrable function $p : J \to [0, \infty)$ such that
$$|f(t, x, y)| \leq p(t)\Omega(\|x\| + |y|), \; t \in J, \; x \in \mathcal{B}, \; y \in X$$
where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

: (vi)
$$\int_0^b \widetilde{m}(s)ds < \int_o^\infty \frac{ds}{s + \Omega(s) + \Omega_0(s)},$$

where

$$c = \frac{1}{1 - \widetilde{K}c_1}\Big\{\widetilde{M}\|\phi\| + \widetilde{K}[M(|\phi(0)| + c_1\|\phi\| + c_2) + c_2 + M_1 c_2 b + MNb]\Big\},$$

$$\widetilde{M} = \max\Big\{M(t) : t \in J\Big\}, \widetilde{m}(t) = \max\left\{\frac{\widetilde{K}M_1 c_1}{1 - \widetilde{K}c_1}, \frac{\widetilde{K}Mp(t)}{1 - \widetilde{K}c_1}, \alpha m(t)\right\}, \; t \in J,$$

$$N = M_2 M_3 \Bigg\{ |x_1| + M(|\phi(0)| + c_1\|\phi\| + c_2) + c_1\|x_b\| + c_2$$
$$+ M_1 \int_0^b (c_1\|x_\tau\| + c_2)ds + M \int_0^b p(\tau)\Omega\Big(\|x_\tau\| + \alpha \int_0^\tau m(\theta)\Omega_0(\|x_\theta\|)d\theta\Big)d\tau \Bigg\}.$$

3. MAIN RESULT

Consider the mapping Φ defined by

$$\Phi x(t) = T(t)[\phi(0) - g(0, \phi)] + g(t, x_t) + \int_0^t AT(t-s)g(s, x_s)ds$$
$$+ \int_0^t T(t-s)\left[(Bu)(s) + f\Big(s, x_s, \int_0^s q(s, \tau, x_\tau)d\tau\Big)\right]ds, \; t \in J, \tag{3}$$

and define the control function $u(t)$ as

$$u(t) = \widetilde{W}^{-1}\Bigg\{ x_1 - T(b)[\phi(0) - g(0, \phi)] - g(b, x_b) - \int_0^b AT(b-s)g(s, x_s)ds$$
$$- \int_0^b T(b-s)f\Big(s, x_s, \int_0^\tau q(\tau, \theta, x_\theta)d\theta\Big)ds\Bigg\}(t), \; t \in J. \tag{4}$$

We shall show that the operator Φ has a fixed point, which is then a solution of system (1). Obviously, $(\Phi x)(b) = x_1$, which means that the control u steers the system from the initial function ϕ to x_1 in time b, provided that the nonlinear operator Φ has a fixed point. In order to apply the Schafer theorem we consider the equation

$$x(t) = \lambda \Phi x(t), \quad 0 < \lambda < 1. \tag{5}$$

Lemma 3.1: For system (2), there is a priori bounds $K > 0$ such that $\|x_t\| \leq K$, $t \in J$, where K depends only on b and on the functions $m(\cdot)$ and $\Omega(\cdot)$.

Proof: Now

$$
\begin{aligned}
x(t) &= \lambda T(t)[\phi(0)-g(0,\phi)]+\lambda g(t,x_t)+\lambda\int_0^t AT(t-s)g(s,x_s)ds \\
&\quad +\lambda\int_0^t T(t-\eta)B\widetilde{W}^{-1}\Big\{x_1-T(b)[\phi(0)-g(0,\phi)]-g(b,x_b) \\
&\quad -\int_0^t AT(b-t)g(t,x_t)dt-\int_0^b T(b-\tau)f\Big(\tau,x_\tau,\int_0^\tau q(\tau,\theta,x_\theta)d\theta\Big)d\tau\Big\}(\eta)d\eta \\
&\quad +\lambda\int_0^t T(t-s)f\Big(s,x_s,\int_0^s q(s,\tau,x_\tau)d\tau\Big)ds,\ t\in J, \qquad (6)\\
x_0 &= \phi.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
|x(t)| &\le M[|\phi(0)|+c_1\|\phi\|+c_2]+c_1\|x_t\|+c_2+M_1\int_0^t (c_1\|x_s\|+c_2)ds \\
&\quad +MNb+M\int_0^t p(s)\Omega\Big(\|x_s\|+\int_0^s \alpha m(\tau)\Omega_0(\|x_\tau\|)d\tau\Big)ds,\quad t\in J,
\end{aligned}
$$

from which and Axiom (A_1)(iii) it follows

$$
\begin{aligned}
\|x_t\| &\le K(t)\sup\{|x(s)|:0\le s\le t\}+M(t)\|\phi\| \\
&\le \widetilde{K}\sup\{|x(s)|:0\le s\le t\}+\widetilde{M}\|\phi\| \\
&\le \widetilde{M}\|\phi\|+\widetilde{K}\Big\{M[|\phi(0)|+c_1\|\phi\|+c_2]+c_2+M_1c_2b+MNb\Big\} \\
&\quad +\widetilde{K}c_1\sup_{0\le s\le t}\|x_s\|+\widetilde{K}M_1c_1\int_0^t\|x_\tau\|d\tau \\
&\quad +\widetilde{K}M\int_0^t p(s)\Omega\Big(\|x_s\|+\int_0^s \alpha m(\tau)\Omega_0(\|x_\tau\|)d\tau\Big)ds,\quad t\in J.
\end{aligned}
$$

Let $\mu(t)=\sup\{\|x_s\|:0\le s\le t\}$, then the function $\mu(t)$ is continuous and nondecreasing in from Axiom (A_2), and we have

$$
\begin{aligned}
\mu(t) &\le \widetilde{M}\|\phi\|+\widetilde{K}\Big\{M[|\phi(0)|+c_1\|\phi\|+c_2]+c_2+M_1c_2b+MNb\Big\} \\
&\quad +\widetilde{K}c_1\mu(t)+\widetilde{K}M_1c_1\int_0^t\mu(\tau)d\tau+\widetilde{K}M\int_0^t p(s)\Omega\Big(\mu(s) \\
&\quad +\int_0^s \alpha m(\tau)\Omega_0(\mu(\tau))d\tau\Big)ds,\quad t\in J,
\end{aligned}
$$

from which it follows

$$\begin{aligned}\mu(t) &\le c+\frac{\widetilde{K}M_1c_1}{1-\widetilde{K}c_1}\int_0^t \mu(\tau)d\tau+\frac{\widetilde{K}M}{1-\widetilde{K}c_1}\int_0^t p(s)\Omega\Big(\mu(s)\\ &\quad+\int_0^s \alpha m(\tau)\Omega_0(\mu(\tau))d\tau\Big)ds,\ t\in J.\end{aligned}$$

Denoting the right-hand side of the above inequality as $\gamma(t)$, we have $\gamma(0)=c,\ \mu(t)\le \gamma(t),\ t\in J$ and

$$\begin{aligned}\gamma'(t) &= \frac{\widetilde{K}M_1c_1}{1-\widetilde{K}c_1}\mu(t)+\frac{\widetilde{K}M}{1-\widetilde{K}c_1}p(t)\Omega\Big(\mu(t)+\alpha\int_0^t m(\tau)\Omega_0(\mu(\tau))d\tau\Big)\\ &\le \frac{\widetilde{K}M_1c_1}{1-\widetilde{K}c_1}\gamma(t)+\frac{\widetilde{K}M}{1-\widetilde{K}c_1}p(t)\Omega\Big(\gamma(t)+\alpha\int_0^t m(\tau)\Omega_0(\gamma(\tau))d\tau\Big)\\ &\le \frac{1}{1-\widetilde{K}c_1}\widetilde{K}M_1c_1\left[\gamma(t)+\frac{M}{M_1c_1}p(t)\Omega\Big(\gamma(t)+\alpha\int_0^t m(\tau)\Omega_0(\gamma(\tau))d\tau\Big)\right],\ t\in J\end{aligned}$$

Let $w(t)=\gamma(t)+\alpha\int_0^t m(\tau)\Omega(\gamma(\tau))d\tau$. Then $w(0)=\gamma(0), \gamma(t)\le w(t)$ and

$$\begin{aligned}w'(t) &= \gamma'(t)+\alpha m(t)\Omega_0(\gamma(t))\\ &\le \frac{\widetilde{K}M_1c_1}{1-\widetilde{K}c_1}\left[\gamma(t)+\frac{M}{M_1c_1}p(t)\Omega\Big(\gamma(t)+\alpha\int_0^t m(\tau)\Omega_0(\gamma(\tau))d\tau\Big)\right]+\alpha m(t)\Omega_0(\gamma(t))\\ &\le \frac{\widetilde{K}M_1c_1}{1-\widetilde{K}c_1}w(t)+\frac{\widetilde{K}M}{1-\widetilde{K}c_1}p(t)\Omega(w(t))+\alpha m(t)\Omega_0(w(t))\\ &\le \widetilde{m}(t)\Big\{w(t)+\Omega_0(w(t))+\Omega(w(t))\Big\}\end{aligned}$$

which implies

$$\int_{w(0)}^{w(t)}\frac{ds}{s+\Omega(s)+\Omega_0(s)}\le\int_0^t\widetilde{m}(s)ds<\int_c^\infty\frac{ds}{s+\Omega(s)+\Omega_0(s)},\quad t\in J.$$

This inequality implies that there is a constant K such that $\gamma(t)\le K,\ t\in J$ and hence, $\|x_t\|\le\mu(t)\le\gamma(t)\le K$, $t\in J$, where K only depends on b and on the functions $m(\cdot)$ and $\Omega(\cdot)$.

Now, we introduce the space $\mathcal{B}_b$ of all function $z:(-\infty,b]\to X$ such that $z_0\in\mathcal{B}$ and the restriction $z:[0,b]\to X$ is continuous. Let $\|\cdot\|_b$ be a semi norm in $\mathcal{B}_b$ defined by

$$\|z\|_b=\|z_0\|+\sup\Big\{|z(s)|:0\le s\le b\Big\},\quad z\in\mathcal{B}_b.$$

For $\hat{\phi}\in\mathcal{B}$, we define $\hat{\phi}$ by

$$\hat{\phi}(t)=\phi(t),-\infty<t\le 0;\ \hat{\phi}(t)=T(t)\phi(0),\quad 0\le t\le b,$$

then $\hat{\phi} \in \mathcal{B}_b$. Let $x(t) = y(t) + \hat{\phi}(t), -\infty < t \leq b$, then it is clear that x satisfies (2) if and only if y satisfies

$$\begin{aligned}
y(t) &= -T(t)g(0,\phi) + g(t, y_t + \hat{\phi}_t) + \int_0^t AT(t-s)g(s, y_s + \hat{\phi}_s)ds \\
&\quad + \int_0^t T(t-\eta)B\widetilde{W}^{-1}\Big\{ x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b, y_b + \hat{\phi}_b) \\
&\quad - \int_0^b AT(b-s)g(s, y_s + \hat{\phi}_s)ds \\
&\quad - \int_0^b T(b-s)f\Big(s, y_s + \hat{\phi}_s, \int_0^\tau q(\tau, \theta, y_\theta + \hat{\phi}_\theta)d\theta\Big)ds\Big\}(\eta)d\eta \\
&\quad + \int_0^t T(t-s)f\Big(s, y_s + \hat{\phi}_s, \int_0^s q(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau\Big)ds, \quad t \in J, \\
y_0 &= 0,
\end{aligned}$$

Let $\mathcal{B}_b^0 = \{z \in \mathcal{B}_b : z_0 = 0\}, B_k = \{z \in \mathcal{B}_b^0 : |z(t)| \leq \frac{k}{K}, t \in J\}$ for some $k \geq 1$. Clearly, $\mathcal{B}_b^0$ is convex and closed, and B_k is uniformly bounded. Define $\Psi : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ by

$$\begin{aligned}
(\Psi y)(t) &= 0, -\infty \leq t \leq 0 \\
(\Psi y)(t) &= -T(t)g(0,\phi) + g(t, y_t + \hat{\phi}_t) + \int_0^t AT(t-s)g(s, y_s + \hat{\phi}_s)ds \\
&\quad + \int_0^t T(t-\eta)B\widetilde{W}^{-1}\Big\{ x_1 - T(b)[\phi(0) - g(0,\phi)] - g(b, y_b + \hat{\phi}_b) \\
&\quad - \int_0^b AT(b-s)g(s, y_s + \hat{\phi}_s)ds \\
&\quad - \int_0^b T(b-s)f\Big(s, y_s + \hat{\phi}_s, \int_0^\tau q(\tau, \theta, y_\theta + \hat{\phi}_\theta)d\theta\Big)ds\Big\}(\eta)d\eta \\
&\quad + \int_0^t T(t-s)f\Big(s, y_s + \hat{\phi}_s, \int_0^s q(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau\Big)ds, \quad t \in J, \qquad (7)
\end{aligned}$$

Lemma 3.2: The operator $\Psi : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ is compact and continuous.

Proof: First we show that Ψ maps B_k into an equicontinuous family. To this end, for any $y \in B_k$, let $0 < t_1 < t_2 \leq b$, then

$$
\begin{aligned}
&|(\Psi y)(t_1) - (\Psi y)(t_2)| \\
&\leq \ |T(t_1) - T(t_2)||g(0,\phi)| + |g(t_1, y_{t_1} + \hat{\phi}_{t_1}) - g(t_2, y_{t_2} + \hat{\phi}_{t_2})| \\
&\quad + |\int_0^{t_1} A[T(t_1 - s) - T(t_2 - s)]g(s, y_s + \hat{\phi}_s)ds| \\
&\quad + |\int_{t_1}^{t_2} AT(t_2 - s)g(s, y_s + \hat{\phi}_s)ds| \\
&\quad + |\int_0^{t_1} [T(t_1 - \eta) - T(t_2 - \eta)]B\widetilde{W}^{-1}\Big\{ x_1 - T(b)[\phi(0) - g(0,\phi)] \\
&\quad - g(b, y_b + \hat{\phi}_b) - \int_0^b AT(b - s)g(s, y_s + \hat{\phi}_s)ds \\
&\quad - \int_0^b T(b - s)f\Big(s, y_s + \hat{\phi}_s, \int_0^\tau q(\tau, \theta, y_\theta + \hat{\phi}_\theta)d\theta\Big)ds\Big\}(\eta)d\eta| \\
&\quad + |\int_{t_1}^{t_2} T(t_2 - \eta)]B\widetilde{W}^{-1}\Big\{ x_1 - T(b)[\phi(0) - g(0,\phi)] \\
&\quad - g(b, y_b + \hat{\phi}_b) - \int_0^b AT(b - s)g(s, y_s + \hat{\phi}_s)ds \\
&\quad - \int_0^b T(b - s)f\Big(s, y_s + \hat{\phi}_s, \int_0^\tau q(\tau, \theta, y_\theta + \hat{\phi}_\theta)d\theta\Big)ds\Big\}(\eta)d\eta| \\
&\quad + |\int_0^{t_1} [T(t_1 - s) - T(t_2 - s)]f\Big(s, y_s + \hat{\phi}_s, \int_0^s q(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau\Big)ds| \\
&\quad + |\int_{t_1}^{t_2} T(t_2 - s)|f\Big(s, y_s + \hat{\phi}_s, \int_0^s q(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau\Big)ds| \\
&\leq \ |T(t_1) - T(t_2)||g(0,\phi)| + |g(t_1, y_{t_1} + \hat{\phi}_{t_1}) - g(t_2, y_{t_2} + \hat{\phi}_{t_2})| \\
&\quad + \int_0^{t_1} |A[T(t_1 - s) - T(t_2 - s)]|(c_1\|y_s + \hat{\phi}_s\| + c_2)ds \\
&\quad + \int_{t_1}^{t_2} |AT(t_2 - s)|(c_1\|y_s + \hat{\phi}_s\| + c_2)ds \\
&\quad + \int_0^{t_1} |T(t_1 - \eta) - T(t_2 - \eta)|M_2M_3\Big\{ |x_1| + M[|\phi(0) - g(0,\phi)|] \\
&\quad + c_1\|y_b + \hat{\phi}_b\| + c_2 + M_1\int_0^b (c_1\|y_s + \hat{\phi}_s\| + c_2)ds + M\int_0^b\int_0^\tau \alpha_{k'}(\theta)d\theta d\tau\Big\}d\eta
\end{aligned}
$$

$$+\int_{t_1}^{t_2}|T(t_2-\eta)|M_2M_3\Big\{|x_1|+M[|\phi(0)-g(0,\phi)|]$$

$$+c_1\|y_b+\hat{\phi}_b\|+c_2+M_1\int_0^b(c_1\|y_s+\hat{\phi}_s\|+c_2)ds+M\int_0^b\int_0^\tau\alpha_{k'}(\theta)d\theta d\tau\Big\}d\eta$$

$$(8)\qquad +\int_0^{t_1}|T(t_1-s)-T(t_2-s)|\int_0^s\alpha_{k'}(\tau)d\tau ds+\int_{t_1}^{t_2}|T(t_2-s)|\int_0^s\alpha_{k'}(\tau)d\tau ds$$

where $k'=k^*+\alpha\sup_{s\in[0,b]}m(s)\Omega_0(k^*)$, $k^*=k+M\widetilde{K}|\phi(0)|+\widetilde{M}\|\phi\|$ (note that $\|y_s+\hat{\phi}_s\|\le\|y_s\|+\|\hat{\phi}_s\|\le\widetilde{K}\sup_{0\le r\le s}|y(\tau)|+\widetilde{M}\|y_0\|+\widetilde{K}\sup_{0\le t\le s}|T(\tau)\phi(0)|+\widetilde{M}\|\phi\|\le k+M\widetilde{K}|\phi(0)|+\widetilde{M}\|\phi\|$).

The right-hand side of (8) is independent of $y\in B_k$ and tends to zero as $t_2-t_1\to 0$, since g is completely continuous and the compactness of $T(t)$ for $t>0$ implies the continuity in the uniform operator topology. Thus, Ψ maps B_k into an equicontinuous family of functions.

Next, we show that $\overline{\Psi B_k}$ is compact. Since we have shown that ψB_k is equicontinuous collection, it is sufficient by Arzela-Ascoli's theorem to show that Ψ maps B_k into a precompact set in X. Let $0<t\le b$ be fixed and ϵ a real number satisfying $0<\epsilon<t$. For $y\in B_k$, we define

$$
\begin{aligned}
&(\Psi_\epsilon y)(t)\\
= &- T(t)g(0,\phi)+g(t-\epsilon,y_{t-\epsilon}+\hat{\phi}_{t-\epsilon})+\int_0^{t-\epsilon}AT(t-s)g(s,y_s+\hat{\phi}_s)ds\\
&+ \int_0^{t-\epsilon}T(t-\eta)B\widetilde{W}^{-1}\Big\{x_1-T(b)[\phi(0)-g(0,\phi)]-g(b,y_b+\hat{\phi}_b)\\
&- \int_0^b AT(b-s)g(s,y_s+\hat{\phi}_s)ds\\
&- \int_0^b T(b-s)f\Big(s,y_b+\hat{\phi}_b,\int_0^\tau q(\tau,\theta,y_\theta+\hat{\phi}_\theta)d\theta\Big)ds\Big\}(\eta)d\eta\\
&+ \int_0^{t-\epsilon}T(t-s)f\Big(s,y_s+\hat{\phi}_s,\int_0^s q(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\Big)ds\\
= &- T(t)g(0,\phi)+g(t-\epsilon,y_{t-\epsilon}+\hat{\phi}_{t-\epsilon})\\
&+ T(\epsilon)\int_0^{t-\epsilon}AT(t-s-\epsilon)g(s,y_s+\hat{\phi}_s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \; T(\epsilon)\int_0^{t-\epsilon} T(t-\eta-\epsilon)B\widetilde{W}^{-1}\Big\{x_1 - T(b)[\phi(0)-g(0,\phi)] \\
&- \; g(b,y_b+\hat{\phi}_b) - \int_0^b AT(b-s)g(s,y_s+\hat{\phi}_s)ds \\
&- \; \int_0^b T(b-s)f\Big(s,y_s+\hat{\phi}_s,\int_0^\tau q(\tau,\theta,y_\theta+\hat{\phi}_\theta)d\theta\Big)ds\Big\}(\eta)d\eta \\
&+ \; T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)f\Big(s,y_s+\hat{\phi}_s,\int_0^s q(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\Big)ds.
\end{aligned}
$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(\Psi_\epsilon y)(t) : y \in B_k\}$ is precompact in X for every ϵ , $0 < \epsilon < t$. Moreover, for every $y \in B_k$ we have

$$
\begin{aligned}
&|(\Psi y)(t) - (\Psi_\epsilon y)(t)| \\
&\le \; |g(t,y_t+\hat{\phi}_t) - g(t-\epsilon,y_{t-\epsilon}+\hat{\phi}_{t-\epsilon})| + \int_{t-\epsilon}^t |AT(t-s)g(s,y_s+\hat{\phi}_s)|ds \\
&+ \int_{t-\epsilon}^t |T(t-\eta)|M_2M_3\Big\{|x_1| + M|\phi(0)-g(0,\phi)| + |g(b,y_b+\hat{\phi}_b)| \\
&+ M_1\int_0^b |g(s,y_s+\hat{\phi}_s)|ds + \int_0^b |T(t-s)|\int_0^\tau \alpha_{k'}(\theta)d\theta ds\Big\}d\eta \\
&+ \int_{t-\epsilon}^t |T(t-s)|\alpha_{k'}d\tau ds.
\end{aligned}
$$

Clearly, $|(\Psi y)(t) - (\Psi_\epsilon y)(t)| \to 0$ as $\epsilon \to 0^+$. Therefore, there is a family of precompact sets which are arbitrarily close to the set $\{(\Psi y)(t) : y \in B_k\}$. Hence, the set $\{(\Psi y)(t) : y \in B_k\}$ is precompact in X, which is the desired conclusion.

Next we prove that the operator $\Psi : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ is continuous. Let $\{y_n\}_{n\ge 1} \subset \mathcal{B}_b^0$

with $y_n \to y$ in $\mathcal{B}_b^0$. We have

$$
\begin{aligned}
&\|(\Psi y_n)(t) - (\Psi y)(t)\| \\
&\leq \ |g(t, y_{n_t} + \hat{\phi}_t) - g(t, y_t + \hat{\phi}_t)| \\
&\quad + \int_0^t |AT(t-s)||g(s, y_{n_s} + \hat{\phi}_s) - g(s, y_s + \hat{\phi}_s)|ds \\
&\quad + \int_0^t |T(t-\eta)|M_2 M_3 \Bigg\{ M_1 \int_0^b |g(s, y_{n_s} + \hat{\phi}_s) - g(s, y_s + \hat{\phi}_s)|ds \\
&\quad + M \int_0^b \Big|f\Big(s, y_{n_s} + \hat{\phi}_s, \int_0^\tau q(\tau, \theta, y_{n_\theta} + \hat{\phi}_\theta)d\theta\Big) \\
&\quad - f\Big(s, y_s + \hat{\phi}_s, \int_0^\tau q(\tau, \theta, y_\theta + \hat{\phi}_\theta)d\theta\Big)\Big|d\tau \Bigg\}(\eta)d\eta \\
&\quad + \int_0^t |T(t-s)|\Big|f\Big(s, y_{n_s} + \hat{\phi}_s, \int_0^s q(s, \tau, y_{n_\tau} + \hat{\phi}_\tau)d\tau\Big) \\
&\quad - f\Big(s, y_s + \hat{\phi}_s, \int_0^s q(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau\Big)\Big|ds \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Thus Ψ is continuous.

Theorem 3.3: Assume that $(A_1)-(A_7)$ hold. Then the system (1) is controllable on J.

Proof: Let $\zeta(\Psi) = \{y \in \mathcal{B}_b^0 : y = \lambda \Psi y, \lambda \in (0,1)\}$, where the nonlinear operator Ψ is defined by (7). For any $y \in \zeta(\Psi)$, the function $x = y + \hat{\phi}$ is a mild solution of system (1), for which we have proved in Lemma 3.1 that $\|x_t\| \leq K,\ t \in J$, and hence from Axiom (A_1),

$$
\begin{aligned}
\|y\|_b &= \|y_0\| + \sup\{|y(t)| : 0 \leq t \leq b\} \\
&= \sup\{|y(t)| : 0 \leq t \leq b\} \\
&\leq \sup\{|x(t)| : 0 \leq t \leq b\} + \sup\{|\hat{\phi}(t)| : 0 \leq t \leq b\} \\
&\leq \sup\{H\|x_t\| : 0 \leq t \leq b\} + \sup\{|T(t)\phi(0)| : 0 \leq t \leq b\} \\
&\leq HK + M|\phi(0)|.
\end{aligned}
$$

From Lemma 3.2 it follows that Ψ is completely continuous in $\mathcal{B}_b^0$. Therefore, it follows from the Schaefer theorem that the operator Ψ has a fixed point $z \in \mathcal{B}_b^0$. Let $x(t) = z(t) + \hat{\phi}(t),\ t \in (-\infty, b]$. Then x is a fixed point of the operator Φ. Hence, the system (1) is controllable on J.

4. EXAMPLE

Consider the partial neutral integrodifferential equation of the form

$$\frac{\partial}{\partial t}\left[v(t,x) - \int_{-\infty}^{t} b(s-t)v(s,x)ds\right] = \frac{\partial^2 v(t,x)}{\partial x^2} + \mu(t,x)$$

$$(9) \quad + \int_{-\infty}^{t} a(t,x,s-t)F\Big(v(s,x), \int_0^s g(s,\tau,v_\tau)d\tau\Big)ds, 0 \le x \le \pi, t \in [0,b] = J$$

$$\begin{aligned} v(t,0) &= v(t,\pi) = 0, t \geq 0 \\ v(t,x) &= \psi(t,x), \ -\infty < t \leq 0, \ x \in [0,\pi]. \end{aligned}$$

Let $X = L^2[0,\pi]$ endowed with usual norm $|\cdot|$ and let $u(t)(x) = \mu(t,x)$ and be such that $u \in L^2(J,U)$ with $U \subset J$. Define $A : X \to X$ by $Aw' = w''$ and

$$D(A) = \{w \in X : w, w' \text{are absolutely continuous, } w'' \in X, w(0) = w(\pi) = 0\}$$

It is well known that A generates an analytic semigroup $T(t),\ t \geq 0$ on X and (A_4) holds [4,10]. Let $\gamma < 0$ and assume that

: (i) The function $b(\theta) \geq 0$ is continuous in $(-\infty, 0]$, $\int_{-\infty}^{t} b^2(\theta)d\theta < \infty$ and $\widetilde{K}c_1 < 1$ where $\widetilde{K} = e^{-\gamma b}$, $c_1 = \frac{-1}{2\gamma}\sqrt{\int_{-\infty}^{0} b^2(\theta)d\theta}$.

: (ii) The function $a(t,x,\theta) \geq 0$ is continuous in $J \times [0,\pi] \times (-\infty,0]$ and $\int_{-\infty}^{0} a(t,x,\theta)d\theta = m(t,x) < \infty$.

: (iii) The function $g(\cdot)$ is continuous such that $0 \leq g(t,s,x) \leq \Omega_0(|x|)$, where $\Omega_0(\cdot)$ is positive, continuous and nondecreasing in $[0,\infty)$.

: (iv) The function $F(\cdot)$ is continuous such that $0 \leq F(v_1,v_2) \leq \Omega_1(|v_1| + |v_2|)$, where $\Omega_1(\cdot)$ is positive, continuous and nondecreasing in $[0,\infty)$.

: (v) There exists an inverse operator $\widetilde{W}^{-1}$ which takes values in $L^2[J,U]/KerW$ such that $Wu = \int_0^b T(b-s)u(s)ds$.

Take $\mathcal{B} = C_\gamma$ which is defined as

$$C_\gamma = \Big\{\phi \in C((-\infty,0],X) : \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } X\Big\},\ \gamma \in R,$$

and let

$$\|\phi\|_\gamma = \sup\Big\{e^{\gamma\theta}|\phi(\theta)| : -\infty < \theta \leq 0\Big\} \text{ for } \phi\ \in C_\gamma.$$

Then $(C_\gamma, \|\cdot\|_\gamma)$ is a Banach space which satisfies $(A_1)-(A_3)$. For $(t,\phi) \in [0,b]\times C_\gamma$, let $\phi(\theta)(x) = \phi(\theta,x), (\theta,x) \in (-\infty,0]\times[0,\pi]$, $v(t)(x) = v(t,x)$,

$$g(t,\phi)(x) = \int_{-\infty}^{0} b(\tau)\phi(\tau,x)d\tau,$$

$$f(t,\phi,\psi)(x) = \int_{-\infty}^{0} a(t,x,\tau)F\left(\phi(\tau,x), \int_0^\tau \psi(\tau,\theta,v_\theta)d\theta\right)d\tau.$$

Then the system (9) can be reduced to the abstract equation given by

$$\begin{aligned} \frac{d}{dt}\Big[v(t) - g(t,v_t)\Big] &= Av(t) + u(t) + f\left(t, v_t, \int_0^t q(t,s,v_s)ds\right), \ t \in J \\ u_0 &= \phi \in C_\gamma. \end{aligned} \tag{10}$$

Under the hypotheses as above $(A_5), (A_6)(i)$ and $(A_7)(i)(ii)$, hold and

$$|g(t,\phi)| \leq C_1\|\phi\|_\gamma, \quad |q(t,s,\psi)| \leq \alpha m_1(s)\Omega_0(\|\psi\|_\gamma),$$
$$|f(t,\phi,\psi)| \leq p_1(t)\Omega_1(\|\phi\|_\gamma + |\psi|_\gamma),$$

where $m_1(t) = \sqrt{\pi}\max\{m(t,x) : 0 \leq x \leq \pi\}$.

Further, suppose that

$$\int_0^b \hat{m}_1(s)ds \ < \ \int_c^\infty \frac{ds}{s + \Omega_0(s) + \Omega_1(s)},$$

where

$$\begin{aligned} c &= \frac{1}{1 - \widetilde{K}c_1}[\widetilde{M}\|\phi\|_\gamma + M(|\phi(0)| + c_1\|\phi\|_\gamma) + MNb], \\ \hat{m}_1(t) &= \max\left\{\frac{\widetilde{K}M_1c_1}{1 - \widetilde{K}c_1}, \frac{\widetilde{K}Mp(s)}{1 - \widetilde{K}c_1}, \alpha m(t)\right\} \end{aligned}$$

and N depends on ϕ, f and g. Then from Theorem 3.3, the system (9) is controllable on J.

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