J. KSIAM Vol.8, No.2, 75-86, 2004

Dynamic Analysis of a Chain of Rigid Rods

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Abstract

In this study, a recursive algorithm for generating the equations of motion of a chain of rigid rods is presented. The methods rests upon the idea of replacing the rigid body by a dynamically equivalent constrained system of particles. The concepts of linear and angular momentums are used to generate the rigid body equations of motion without either introducing any rotational coordinates or the corresponding transformation matrices. For open-chain, the equations of motion are generated recursively along the serial chains. For closed-chain, the system is transformed to open-chain by cutting suitable kinematic joints with the addition of cut-joints kinematic constraints. An exmaple of a closed-chain of rigid rods is chosen to demonstrate the generality and simplicity of the proposed method.

1. Introduction

Many formulations have been used to carry out the dynamic analysis of planar mechanisms. Some formulations (Orlandea et al. 1977 and Nikravesh 1988) use a large set of dependent coordinates. The location of each rigid body in the system is described in terms of a set of absolute coordinates; translational and rotational coordinates. The constraint equations are imposed to represent the kinematic joints that connect the rigid bodies. This formulation has the advantage that the constraint equations are easily introduced, however, it has the disadvantage of a large number of coordinates defined. Other formulations (Sheth et al. 1972) describe the configuration of the system in terms of relative coordinates. The location of each body is defined with respect to the adjacent body by means of an angle or a distance depending on the type of the kinematic pair joining the two bodies. Although this formulation yields the constraints as a minimal set of algebraic equations, it has the disadvantage that it does not directly determine the positions of the bodies and points of interest.

Other methods for generating the equations of motion use a two-step transformation.

Keywords : Dynamics, open-chain, closed-chain, rigid bodies, equations of motion, recursive equations, simulation.

They group the advantages of the simplicity, generality and efficiency. One method (Kim et al. 1986) uses initially the absolute coordinate formulation. Then, the equations of motion are expressed in terms of the relative joint variables. Another method (Garcia et al. 1986, Attia 1993, Nikravesh et al. 1994, Attia 1999, and Attia et al. 2001) uses initially a dynamically equivalent constrained system of particles to replace the rigid bodies. The equations of motion are derived using Newton's second law and the Lagrange multiplier technique which results in a large number of differential-algebraic equations. The simplicity and the absence of any rotational coordinates from the final form of the equations of motion are considered the main advantages of this formulation. Therefore, the equations of motion that were expressed in matrix form in terms of the Cartesian coordinates of the particles are transformed to a reduced set in terms of the relative joint variables.

In the present paper, a recursive algorithm for generating the equations of motion of an open or closed-chain of rigid rods is presented. The method rests upon the idea of replacing the rigid body by its dynamically equivalent constrained system of particles discussed in (Garcia et al. 1986 and Attia 1993) with essential modifications and improvements. The concepts of the linear and angular momentums of the rigid body are used to formulate the rigid body dynamical equations. However, they are expressed in terms of the rectangular Cartesian coordinates of the equivalent system of particles. This groups the advantages of the automatic elimination of the unknown internal constraint forces as in Newton-Euler formulation while expresses the general motion of the rigid body in terms of a set of Cartesian coordinates without introducing any rotational coordinates or the corresponding transformation matrices. This process results in a reduced system of differential-algebraic equations and also eliminates the necessity of distributing the external forces and moments over the particles. For open-chain, the equations of motion are generated recursively along the serial chains instead of the matrix formulation derived in Attia 1993. Geometric constraints that fix the distances between the particles are introduced while some kinematic constraints due to common types of joints and the associated constraint forces are automatically eliminated by properly selecting the locations of the particles. For a closed-chain, the system is transformed to open-chain by cutting suitable kinematic joints and introducing the cut-joint kinematic constraints. The dynamic analysis of a closed-chain of rigid rods is carried out to demonstrate the generality and simplicity of the suggested method.

2. The Dynamic Model

2.1 Construction of the Equivalent System of Particles

A system of three particles is chosen to replace the rigid rod in spatial motion as shown in Fig. 1. Particles 1 and 3 are located at both ends of the rod while particle 2 is located at the middle of the rod. The rigid rod and its dynamically equivalent system of particles should have the same mass, the same position of the centre of mass and the same moment of inertia about an axis perpendicular to the rod. For a rigid rod of length l and mass m, these conditions read as

$$m = m_1 + m_2 + m_3, ml_c = (m_2/2 + m_3)l, I_o = (m_2/4 + m_3)l^2,$$

and can be solved to determine the unknown masses of the particles in the form,

$$m_2 = \frac{4}{l^2}(mll_c - I_o), m_3 = \frac{2}{l^2}(I_o - \frac{m}{2}ll_c), m_1 = m - m_2 - m_3,$$

where l_c is the location of the centre of mass of the rod and I_o is the moment of inertia about an axis perpendicular to the rod and passing through its left end. If the rigid rod is connected to other rods in an open-chain by spherical joints, then particles 1 and 3 can be conveniently located at the centers of these joints which reduces the total number of particles replacing the whole system and leads to the automatic elimination of the constraint forces associated with the joints.

2.2 Equations of Motion of a Single Rigid Rod in Plane Motion

Consider a rigid rod which is acted upon by external forces and force couples. The rigid rod is replaced by an equivalent system of three particles as shown in Fig. 1. The distances between the three particles are invariant as a result of the internal constraint forces existing between them. The vector sum of these unknown internal forces and also the vector sum of their moments about any point each vanishes by the law of action and reaction. Then, the linear momentum equation for the whole system of particles yields,

$$R = \sum_{i=1}^{3} m_i \ddot{r}_i \tag{1}$$

where **R** is the vector sum of the external forces acting on the rigid rod and $\ddot{r}_i$ is the acceleration vector of particle i with respect to the global coordinate frame. Also, the angular momentum equation for the whole system of particles with respect to particle 1 takes the form

$$G_1 = \sum_{i=2}^{3} r_{i,1} \wedge m_i \ddot{r}_i = \sum_{i=2}^{3} \tilde{r}_{i,1} \wedge m_i \ddot{r}_i = m_2 \tilde{r}_{2,1} \ddot{r}_2 + m_3 \tilde{r}_{3,1} \ddot{r}_3 \tag{2}$$

where G_1 represents the vector sum of the moments of the external forces and force couples acting on the rod with respect to the location of particle 1 and $r_{i,1}$ is the relative position vector from particle 1 to i. It should be mentioned that, in the spatial case, since a rod has no moment of inertia about its axis, the three components of the vector G_1 are not independent but only two equations of the vector relation can be used. The constraints between the three particles are given as

$$r_{3,1}^T r_{3,1} - l^2 = 0 \tag{3a}$$

$$r_2 - (r_1 + r_3)/2 = 0 \tag{3b}$$

Differentiating Eqs. (3) with respect to time leads to the velocity constraints

$$r_{3,1}^T \dot{r}_{3,1} = 0 \tag{4a}$$

$$\dot{r}_2 - (\dot{r}_1 + \dot{r}_3)/2 = 0 \tag{4b}$$

and further differentiation yields the acceleration constraints

$$r_{3,1}^T (\ddot{r}_3 - \ddot{r}_1) = -\dot{r}_{3,1}^T \dot{r}_{3,1} \tag{5a}$$

$$\ddot{r}_2 - (\ddot{r}_1 + \ddot{r}_3)/2 = 0 \tag{5b}$$

The equations of motion (1), (2) and (5) represent a linear system of 6 scalar algebraic equations for the planar motion and 9 scalar algebraic equations for the spatial case, that can be solved to determine the unknown acceleration vectors $\ddot{r}_i, i = 1,\ldots,3$, of the particles at any instant of time. Substituting Eq. (5b) into Eqs. (1) and (2) results in

$$R = \left(m_1 + \frac{m_2}{2}\right)\ddot{r}_1 + \left(m_3 + \frac{m_2}{2}\right)\ddot{r}_3 \tag{6}$$

$$G_1 = \frac{m_2}{2}\tilde{r}_{2,1}\ddot{r}_1 + \left(m_3\tilde{r}_{3,1} + \frac{m_2}{2}\tilde{r}_{2,1}\right)\ddot{r}_3 \tag{7}$$

Equations (6), (7) and (5a) represent, for the planar case, four differential-algebraic equations of motion for the planar rod expressed in the four Cartesian coordinates of the endpoints 1 and 3.

For the spatial case, in order to determine the independent components of the vector equation (7), we first rotate the x axis to coincide with the rod. This can be achieved by rotating the global coordinate frame about the z axis by an angle Θ and then about the resulting y axis by an angle -φ. The unit vectors along the resulting y and z axes are represented as

$$\bar{u} = [-\sin\theta, \cos\theta, 0]^T, \hat{u} = [-\cos\theta\sin\varphi, -\sin\theta\sin\varphi, \cos\varphi]^T$$

where $\tan\theta = y_3 / x_3$, $\tan\varphi = z_3 / \sqrt{x_3^2 + y_3^2}$, Θ=π/2 for $x_3 = 0$, φ=π/2 for $x_3 = y_3 = 0$ and $r_{3,1} = (x_3, y_3, z_3)$ are the Cartesian coordinates of particle 3 with respect to the global frame centered at point 1.

By projecting the vectors in Eq. (7) along $\bar{u}$ and $\hat{u}$ respectively, we obtain, two independent moment equations in the form,

$$G_1^T \bar{u} = \left[\frac{m_2}{2} \tilde{r}_{2,1} \ddot{r}_1 + \left(m_3 \tilde{r}_{3,1} + \frac{m_2}{2} \tilde{r}_{2,1} \right) \ddot{r}_3 \right]^T \bar{u} \tag{1}$$

$$G_1^T \hat{u} = \left[\frac{m_2}{2} \tilde{r}_{2,1} \ddot{r}_1 + \left(m_3 \tilde{r}_{3,1} + \frac{m_2}{2} \tilde{r}_{2,1} \right) \ddot{r}_3 \right]^T \hat{u} \tag{2}$$

Equations (6), (8) and (5a) represent, for the spatial case, six differential-algebraic equations of motion for the spatial rod expressed in the six Cartesian coordinates of the endpoints 1 and 3.

2.3 Equations of Motion of a Serial Chain of Rigid Rods

Figure 2 shows a serial chain of N rigid rods with the equivalent system of (2N+1) particles where connected particles are unified from both bodies. For the last rod "N" in the chain, the equations of motion are derived in a similar way as Eqs. (6) and (7) (for the planar case) or (8) (for the spatial case) of a single rigid rod. The angular momentum equation takes, for the planar case, the form

$$G_{N,2N-1} = \frac{m_{2N}}{2} \tilde{r}_{2N,2N-1} \ddot{r}_{2N-1} + \left(m_{2N+1} \tilde{r}_{2N+1,2N-1} + \frac{m_{2N}}{2} \tilde{r}_{2N,2N-1} \right) \ddot{r}_{2N+1} \tag{9}$$

and for the spatial case, the form

$$GT_{N,2N-1}\bar{u}_N = \left[\frac{m_{2N}}{2}\tilde{r}_{2N,2N-1}\ddot{r}_{2N-1} + \left(m_{2N+1}\tilde{r}_{2N+1,2N-1} + \frac{m_{2N}}{2}\tilde{r}_{2N,2N-1}\right)\ddot{r}_{2N+1}\right]^T \bar{u}_N \tag{10a}$$

$$GT_{N,2N-1}\hat{u}_N = \left[\frac{m_{2N}}{2}\tilde{r}_{2N,2N-1}\ddot{r}_{2N-1} + \left(m_{2N+1}\tilde{r}_{2N+1,2N-1} + \frac{m_{2N}}{2}\tilde{r}_{2N,2N-1}\right)\ddot{r}_{2N+1}\right]^T \hat{u}_N \tag{10b}$$

where $GT_{N,2N-1}$ is the vector sum of the moments of the external forces and force couples acting on rod N with respect to the location of particle 2N-1. The distance constraint is given as

$$r_{2N+1,2N-1}^T(\ddot{r}_{2N+1} - \ddot{r}_{2N-1}) = -\dot{r}_{2N+1,2N-1}^T\dot{r}_{2N+1,2N-1} \tag{11}$$

Addition of one more rod in the chain leads to the inclusion of an angular momentum equation that takes into consideration the contributions of all the ascending rods in the chain together with one distance constraint between the particles belonging to this rod. These two equations are appended to the equations of motion derived for the leading rods in the chain. For rod j, the appended equations of motion take, for the planar case, the form

$$\sum_{i=j}^{N} G_{i,2j-1} = \sum_{i=j}^{N} \left\{\frac{m_{2i}}{2}\tilde{r}_{2i,2j-1}\ddot{r}_{2i-1} + \left(m_{2i+1}\tilde{r}_{2i+1,2j-1} + \frac{m_{2i}}{2}\tilde{r}_{2i,2j-1}\right)\ddot{r}_{2i+1}\right\} \tag{12}$$

and for the spatial case, the form

$$\left[\sum_{i=j}^{N} G_{i,2j-1}\right]^T \bar{u}_j = \left[\sum_{i=j}^{N} \left\{\frac{m_{2i}}{2}\tilde{r}_{2i,2j-1}\ddot{r}_{2i-1} + \left(m_{2i+1}\tilde{r}_{2i+1,2j-1} + \frac{m_{2i}}{2}\tilde{r}_{2i,2j-1}\right)\ddot{r}_{2i+1}\right\}\right]^T \bar{u}_j \tag{13a}$$

$$\left[\sum_{i=j}^{N} G_{i,2j-1}\right]^T \hat{u}_j = \left[\sum_{i=j}^{N} \left\{\frac{m_{2i}}{2}\tilde{r}_{2i,2j-1}\ddot{r}_{2i-1} + \left(m_{2i+1}\tilde{r}_{2i+1,2j-1} + \frac{m_{2i}}{2}\tilde{r}_{2i,2j-1}\right)\ddot{r}_{2i+1}\right\}\right]^T \hat{u}_j \tag{13b}$$

with the constraint equation,

$$r_{2j-1,2j+1}^T\ddot{r}_{2j-1} + r_{2j+1,2j-1}^T\ddot{r}_{2j+1} = -\dot{r}_{2j+1,2j-1}^T\dot{r}_{2j+1,2j-1} \tag{14}$$

If rod "j" is the floating base rod in the chain then, two linear momentum equations, similar to Eq. (6), are required to solve for the unknown acceleration components of particle 1. These linear momentum equations equate the sum of the external forces acting on all the rods in the chain to the time rate of change of the linear momentums of all the equivalent particles that replace the chain and take the form

$$\sum_{i=j}^{N} R_i = \sum_{i=j}^{N}\left\{\left(m_{2i-1} + \frac{m_{2i}}{2}\right)\ddot{r}_{2i-1} + \frac{m_{2i}}{2}\ddot{r}_{2i+1}\right\} + m_{2N+1}\ddot{r}_{2N+1} \tag{15}$$

In general, for a serial chain of N rods, an equivalent system of (2N+1) particles is constructed. By eliminating the coordinates of N particles, we are left with N+1 particles and consequently, 2N+2 unknown acceleration components. To solve for these unknowns, N angular momentum equations can be generated recursively along the serial chain together with N distance constraints between the pair of particles located on each rod. Finally, two linear momentum equations can be used to solve for the unknown acceleration components of particle 1 or for the unknown reaction forces if there is a fixation at point 1. If the chain is closed at its final end, a cut-joint at this end can be used to produce an open chain with the introduction of unknown reaction forces. The cut-joint constraint equations substitute for these unknown reactions.

If rods "j" and "j-1" in a serial chain are connected by a prismatic joint, then particles 2j-3, 2j-2, and 2j-1 are located at body "j-1" while particles 2j, 2j+1, and 2j+2 are assigned to body "j". Particles 2j-3 and 2j-1 on body "j-1" and particles 2j and 2j-2 on body "j" are arbitrarily located along the axis of the prismatic joint. Then the equations of motion are generated recursively along the serial chain as discussed above with the introduction of two kinematical constraints associated with the prismatic joint in the form:

$$(r_{2j-3} - r_{2j-1}) \wedge (r_{2j} - r_{2j+2}) = 0, \tag{16a}$$

$$(r_{2j-3} - r_{2j-1}) \wedge (r_{2j} - r_{2j-1}) = 0, \tag{16b}$$

Similar treatment can be used in dealing with all other kinds of lower or higher pair kinematical joints.

In the case of an open-chain or a closed-chain, it can be transformed to a system of serial chains by cutting suitable joints. Cut-joint constraints and the associated constraint reaction forces are introduced. For the multi-branch system shown in Fig. 3, the system is divided into 4 chains by cutting the connection joints at points 1, 2, 3 and 4. Equivalent particles are conveniently chosen to locate at the positions of the connection joints and in terms of their Cartesian coordinates the cut-joint constraint equations are easily formulated. These kinematical constraints substitute for the unknown constraint reaction forces that appear explicitly in the linear and angular momentum equations. It is also shown in Fig. 3 that some rods are connected with the others in many points. In such a case, though the number of particles that dynamically replace the rigid rod is three which can be used to define two joints, more particles may be added to describe additional joints.

It should be noted that in this formulation, the kinematical constraints due to some common types of kinematical joints (e.g. revolute or spherical joints) can be automatically eliminated by properly locating the equivalent particles. The remaining kinematical constraints along with the geometric constraints are, in general, either linear or quadratic in the Cartesian coordinates of the particles. Therefore, the coefficients of their Jacobian matrix are constants or linear in the rectangular Cartesian coordinates. Where as in the formulation based on the relative coordinates, the constraint equations are derived based on loop closure equations which have the disadvantage that they do not directly determine the positions of the links and points of interest which makes the establishment of the dynamic problem more difficult. Also, the resulting constraint equations are highly nonlinear and contain complex circular functions. The absence of these circular functions in the point coordinate formulation leads to faster convergence and better accuracy. Furthermore, preprocessing the mechanism by the topological graph theory is not necessary as it would be the case with loop constraints.

Also, in comparison with the absolute coordinates formulation, the manual work of the local axes attachment and local coordinates evaluation as well as the use of the rotational variables and the rotation matrices in the absolute coordinate formulation are not required in the point coordinate formulation. This leads to fully computerized analysis and accounts for a reduction in the computational time and memory storage. In addition to that, the constraint equations take much simpler forms as compared with the absolute coordinates. Furthermore, the use of absolute coordinates may cause numerical problems if differences of large values of the absolute coordinates are used, e.g. for the calculation of spring or damper forces or constraint residuals.

The elimination of the rotational coordinates, in the presented formulation, leads to possible savings in computation time when this procedure is compared against the absolute or relative coordinate formulation. It has been determined that numerical computations associated with rotational transformation matrices and their corresponding coordinate transformations between reference frames is time consuming and, therefore, if these computations are avoided more efficient codes may be developed. The elimination of rotational coordinates can also be found very beneficial in design sensitivity analysis of multibody systems. In most procedures for design sensitivity analysis, leading to an optimal design process, the derivatives of certain functions with respect to a set of design parameters are required. Analytical evaluation of these derivatives are much simpler if the rotational coordinates are not present and if we only deal with translational coordinates.

Some practical applications of multibody dynamics require one or more bodies in the system to be described as deformable in order to obtain a more realistic dynamic response (Nikravesh et al. 1994). Deformable bodies are normally modeled by the finite element technique. Assume that the deformable body is connected to a rigid body described by a set of particles. Then, one or more particles of the rigid body can coincide with one or more nodes of the deformable body in order to describe the kinematical joint between the two bodies. This is a much simpler process that when the rigid body is described by a set of translational and rotational coordinates. In general, the point coordinates have additional advantages over the other systems of coordinates since they are the most suitable coordinates for the graphics routines and the animation programs.

2. Dynamic Analysis of The "Straight-Line-Generator" Mechanism

The straight-line-generator mechanism shown in Fig. 4a is chosen as an example of a closed-chain. It is a one degree of freedom mechanism which has three independent closed loops. The Cartesian coordinates (cm) of the end-points of the different links of the mechanism as given as: "O" (0,0), "A" (-2,0), "B" (2,4), "C" (6,0), "D" (2,-4), and "E" (2,0). The mechanism is divided into three independent branches by cutting suitable joints and introducing cut-joint constraints. As shown in Fig. 4b, the first branch consists of 5 bodies OA, AB, BC, CD, and DE connected from ground to ground. This branch is produced by cutting joints at points B and D. The second and third branches each consists of only one body BE and AD respectively. Each rigid body is replaced by an equivalent system of 3 particles. Two particles are conveniently located at the centers of the joints connecting it to the adjacent bodies in the branch, while the Cartesian coordinates of the third particle are expressed in terms of the coordinates of the other two particles with the aid of the two distance constraints. Locating the particles belonging to adjacent bodies together at the connection joints reduces the total number of particles replacing the whole system and leads to an automatic elimination of the kinematical constraints at these joints. Additional particles are added at the joints connecting separated branches due to cut-joints. An overall equivalent system of 10 particles is constructed. The equations of motion are generated recursively along each serial branch as discussed in section 4 while the unknown constraint forces resulting from cutting the joints are introduced. The cut-joint constraints are expressed as,

$$r_2 - r_9 = 0, r_3 - r_7 = 0, r_5 - r_{10} = 0, r_6 - c_1 = 0, r_8 - c_2 = 0$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are two constant known vectors. A linear system of 28 algebraic equations can be solved at every time step to determine 18 unknown acceleration components of particles 2,...,9 as well as 10 unknown reaction forces at the cut-joints. The motion is started from the rest position under the action of gravity forces where link OA is in the horizontal position shown in Fig. 4a. Figure 5 presents the straight line trajectory of particle 4 in the plane of motion and indicates the oscillations in the vertical coordinate $y4$ due to the action of the constraint forces. The results of the simulation are tested and compared with DAP-2D program which is based on the absolute coordinates (Nikravesh 1988). The comparison shows a complete agreement between the two simulations.

Conclusions

In the present work, the concepts of linear and angular momentums are used to derive the equations of motion of a chain of rigid rods. However, they are expressed in terms of the rectangular Cartesian coordinates of a dynamically equivalent constrained system of particles. This groups the advantages of the automatic elimination of the unknown internal constraint forces and describing the general motion of the rigid body in terms of a set of Cartesian coordinates without either introducing any rotational coordinates or distributing the external forces and force couples over the particles. The method results in a reduced system of differential-algebraic equations with the absence of the inconvenient rotational coordinates. The methodology is extended to a system of rigid rods with all common types of kinematic joints, revolute or prismatic.

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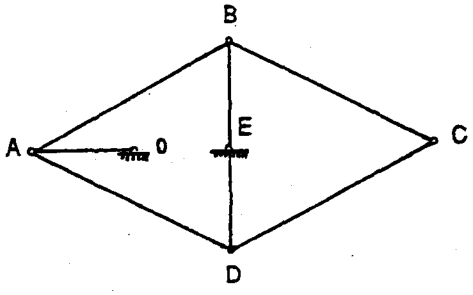

Fig. 4a. The Peaucellier mechanism indicating links and joint types

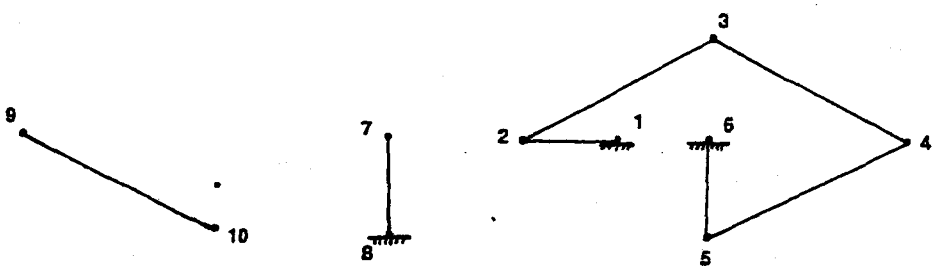

Fig. 4b. The three serial chains replacing the peaucellier mechanism and indicating the equivalent particles

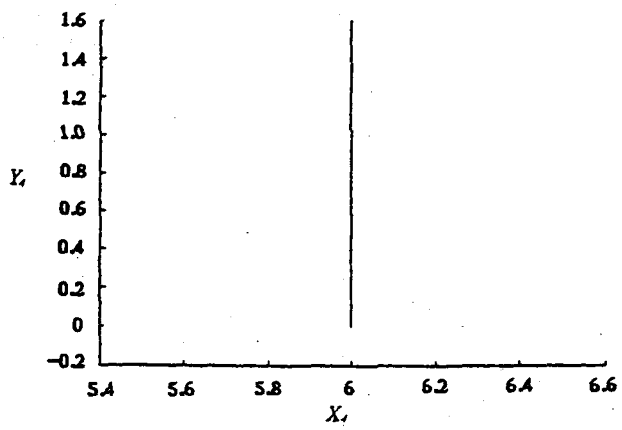

Fig. 5. the trajectory of particle 4

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