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SUPERCONVERGENCE OF FINITE ELEMENT METHODS FOR LINEAR QUASI-PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider finite element methods applied to a class of quasi parabolic integro-differential equations in R^d. Global strong superconvergence, which only requires that partitions are quasi-uniform, is investigated for the error between the approximate solution and the Sobolev-Volterra projection of the exact solution. Two order superconvergence results are demonstrated in $W^{1,p}(\Omega)$ and $L_p(\Omega)$, for $2 \leq p < \infty$.

1. Introduction

Assume that Ω is a bounded domain in R^d with piecewise smooth boundary $\partial\Omega$. Consider the following initial boundary value problem of linear quasi-parabolic integro-differential equation

$$\begin{cases} u_t = \nabla\cdot\{a(t)\nabla u_t + b(t)\nabla u + \int_0^t c(t,\tau)\nabla u(\tau)d\tau\} + f(x,t), & \text{in } \Omega \times (0,\mathrm{T}], \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathrm{J},\ \mathrm{J} = [0,\mathrm{T}], \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{1.1}$$

Where $a(t) = a(x,t)$, $b(t) = b(x,t)$, $c(t,\tau) = c(t,\tau,x)$, f, u_0 are known functions and bounded together with their derivatives up to certain orders as far as the ensuring analysis requires, and $a(x,t)$ satisfies

$$0 < a_0 \leq a(x,t) \leq a_1,\ (x,t) \in \Omega \times [0,T]\ , \tag{1.2}$$

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for positive constants a_0, a_1. The existence and uniquess of the solution of (1.1) have been considered and analyzed by Cui[1] .

Let $L_p(\Omega)$ and $W^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,2}(\Omega)$, for any integer $m \geq 0$ and $1 \leq p \leq \infty$, denote the usual Lebesgue and Sobolev spaces respectively. L_2 and L_p norms are denoted by $\|\cdot\|$ and $\|\cdot\|_{0,p}$, Sobolev norms by $\|\cdot\|_{m,p}$ and $\|\cdot\|_m$. For any integer $s \geq 0$ and $t \in J$, we define

$$\|u(t)\|_{s,m,p} = \sum_{j=0}^{s} \|D_t^j u(t)\|_{m,p} + \int_0^t \sum_{j=0}^{s} \|D_t^j u(\tau)\|_{m,p} d\tau \ . \tag{1.3}$$

Let $(\cdot,\cdot)$ denote the inner product in $L_2(\Omega)$ or $L_2(\Omega)^2$. In addition, we also use the notation $p' = \frac{p}{p-1}$ to denote the conjugate index of p for $2 \leq p < \infty$. We shall denote by C a constant independent of h, not necessarily the same at different occurences.

The mathematical methods applied to (1.1) have been considered and analyzed by several authors [1], [2]. In Cui [1], optimal order error estimates are obtained in L_2. Because in the right side of (1.1) , there are not only ∇u_t but also ∇u and the integration of t for ∇u, it seems very complex. If we employ the traditional projection of finite element, such as Ritz projection , Sobolev projection and Ritz-Sobolev projection, we can't reflect its eccentric characters. So it's very difficult to get the superconvergence estimate. The paper develops a new projection of finite element (called Sobolev-Volterra projection) and by employing a special method for initial value selection, to study superconvergence of the error between the approximation solution U and the Sobolev-Volterra projection $V_h u$ of the exact solution u of (1.1). Two order superconvergence results in L_p and $W^{1,p}$ for $2 \leq p < \infty$ are demonstrated for the general quasi-uniform partition. The superconvergence estimates for this class of problem have been studied by [3]-[8].

The rest of the paper is organized as follows. In section 2 we lay out the Galerkin approximation formula of the problem, and give the Sobolev-Volterra projection and some necessary preparations. Several Lemmas that are needed for the main results are proved in Section 3. Section 4 derives the main conclusions of the paper, two order superconvergence in $L_p(\Omega)$ and $W^{1,p}(\Omega)$ is given in Theorems 4.1 and 4.3, respectively.

2. Finite-Element Formulations And Sobolev-Volterra Projection

In this section we will consider the semidiscrete Galerkin method with respect to space for (1.1) and introduce the Sobolev-Volterra projection.

Let $\{S_h\}_{0<h\leq 1}$ be a family of finite-dimentional subspaces of $H_0^1(\Omega)$ satisfying the following approximation property: for some $r \geq 2$, $1 \leq s \leq r$, $1 \leq p \leq \infty$,

$$\inf_{\chi \in S_h} \{\|v-\chi\|_{0,p} + h\|v-\chi\|_{1,p}\} \leq Ch^s \|v\|_{s,p} \ , \ v \in W^{s,p}(\Omega) \bigcap H_0^1(\Omega). \tag{2.1}$$

We also assume that the standard inverse properties holds in subspaces S_h.

We first introduce the Ritz projection oprator $R_h = R_h(t) : H_0^1(\Omega) \longrightarrow S_h$, for $t \in J$, defined by

$$(a(t)\nabla(R_h w - w), \nabla\chi) = 0, \quad \chi \in S_h. \tag{2.2}$$

The weak form of (1.1) is to find $u \in H_0^1(\Omega),\ t \in J,$ such that

$$\begin{cases} (a)\ (u_t, v) + (a(t)\nabla u_t + b(t)\nabla u + \int_0^t c(t,\tau)\nabla u(\tau)d\tau, \nabla v) = (f(t), v), & v \in H_0^1(\Omega), \\ (b)\ u(x,0) = u_0(x), & x \in \Omega\ . \end{cases} \tag{2.3}$$

We introduce the Sobolev-Volterra projection operator $V_h = V_h(t) : H_0^1(\Omega) \to S_h$, for $t \in J$, satisfies

$$(a(t)\nabla(V_h u_t - u_t) + b(t)\nabla(V_h u - u) + \int_0^t c(t,\tau)\nabla(V_h u - u)d\tau, \nabla\chi) = 0\ , \quad \chi \in S_h. \tag{2.4}$$

Now we define the semidiscrete finite element approximation to the solution u of (1.1) is to find a map $U(t) : J \to S_h$, such that

$$\begin{cases} (a)\ (U_t, \chi) + (a(t)\nabla U_t + b(t)\nabla U + \int_0^t c(t,\tau)\nabla U(\tau)d\tau, \nabla\chi) = (f(t), \chi), & \chi \in S_h,\ t \in J \\ (b)\ U(0) = V_h u(0), & x \in \Omega. \end{cases} \tag{2.5}$$

Noting that Sobolev-Volterra projection is Ritz-Sobolev projection when $t = 0$, Ritz-Sobolev projection has been studied in Li [8].

Let $\eta = V_h u - u$, then (2.4) becomes

$$(a(t)\nabla\eta_t + b(t)\nabla\eta + \int_0^t c(t,\tau)\nabla\eta(\tau)d\tau, \nabla\chi) = 0, \quad \chi \in S_h. \tag{2.6}$$

3. Lemmas

It is easy to prove the problem (2.6) is equivalent to the following equation

$$(a(t)\nabla\eta, \nabla\chi) + (\int_0^t [(b(\tau)\nabla\eta(\tau) - a_t(\tau)\nabla\eta(\tau) + \int_0^\tau c(\tau,s)\nabla\eta(s)ds]d\tau, \nabla\chi) = 0, \quad \chi \in S_h \tag{3.1}$$

From [9], We derive the following lemma.

Lemma 3.1. Let $2 \le p < \infty$, then there exists a constant C such that for $V_h u$ defined by (2.4)

$$\|V_h u - u\|_{0,p} + h\|V_h u - u\|_{1,p} \le Ch^r \|u\|_{0,r,p}\ . \tag{3.2}$$

Proof. We first estimate $\|\eta\|_{1,p}$. Let η_x be any component of $\nabla\eta$, then

$$\|\eta_x\|_{0,p} = sup\{|(\eta_x,\varphi)|, \varphi \in L_{p'}(\Omega), \|\varphi\|_{0,p'} = 1\}.$$

Introducing the dual auxiliary problem, for any $\varphi \in L_{p'}(\Omega)$, $\|\varphi\|_{0,p'} = 1$, let $\psi \in W^{1,p'}(\Omega)$, be the solution of

$$(a(t)\nabla\psi, \nabla v) = -(v, \varphi_x), v \in H_0^1(\Omega), \quad t \in J, \tag{3.3}$$

Thus we have

$$\|\psi\|_{1,p'} \le C\|\varphi\|_{0,p'} \le C, \quad t \in J. \tag{3.4}$$

Then by Green's formula, (3.3)

$$\begin{aligned}
(\eta_x,\varphi) &= -(\eta,\varphi_x) = (a(t)\nabla\eta, \nabla\psi)\\
&= (a(t)\nabla\eta, \nabla(\psi - R_h\psi)) + (a(t)\nabla\eta, \nabla R_h\psi)\\
&= (a(t)\nabla(V_hu - R_hu + R_hu - u), \nabla(\psi - R_h\psi)) + (a(t)\nabla\eta, \nabla R_h\psi)\\
&= (a(t)\nabla(R_hu - u), \nabla(\psi - R_h\psi)) + (a(t)\nabla\eta, \nabla R_h\psi)\\
&= I_1 + I_2.
\end{aligned}$$

By *Hölder* inequality, (3.4) and (3.1)

$$\begin{aligned}
|\, I_1 \,| &\le C\|R_hu - u\|_{1,p}\ \|\psi - R_h\psi\|_{1,p'}\\
&\le Ch^{r-1}\|u\|_{r,p}\ \|\psi\|_{1,p'}\\
&\le Ch^{r-1}\|u\|_{r,p}\ .
\end{aligned}$$

And

$$\begin{aligned}
|\, I_2 \,| &= |\,(\int_0^t [(a_t(\tau) - b(\tau))\nabla\eta(\tau) - \int_0^\tau c(\tau,s)\nabla\eta(s)ds]d\tau, \nabla R_h\psi)\,|\\
&\le C\int_0^t [\|\eta\|_{1,p} + \int_0^\tau \|\eta\|_{1,p}ds]d\tau\ \|R_h\psi\|_{1,p'}\\
&\le C\int_0^t \|\eta\|_{1,p}d\tau\ \|\psi\|_{1,p'}\\
&\le C\int_0^t \|\eta\|_{1,p}d\tau.
\end{aligned}$$

Then

$$|\,(\eta_x,\varphi)\,| \le |\, I_1 \,| + |\, I_2 \,| \le Ch^{r-1}\|u\|_{r,p} + C\int_0^t \|\eta\|_{1,p}d\tau\ .$$

Thus

$$\|\eta_x\|_{0,p} = \sup_{\substack{\varphi\in L_{p'}(\Omega)\\ \|\varphi\|_{0,p'}=1}} |\,(\eta_x,\varphi)\,| \le Ch^{r-1}\|u\|_{r,p} + C\int_0^t \|\eta\|_{1,p}d\tau,$$

We have

$$\|\eta\|_{1,p} \leq Ch^{r-1}\|u\|_{r,p} + C\int_0^t \|\eta\|_{1,p}d\tau,$$

By Gronwall's Lemma, we have

$$\|\eta\|_{1,p} \leq Ch^{r-1}\|u\|_{r,p} \ . \tag{3.5}$$

Next we turn to estimate $\|\eta\|_{0,p}$. Introducing the other dual auxiliary problem, for any $\phi \in L_{p'}(\Omega)$, $\|\phi\|_{0,p'} = 1$, let $\Phi \in W^{2,p'}(\Omega)$ be the solution of

$$(a(t)\nabla\Phi, \nabla v) = (\phi, v), \ v \in H_0^1(\Omega), \tag{3.6}$$

Thus we have

$$\|\Phi\|_{2,p'} \leq C\|\phi\|_{0,p'} \leq C \ . \tag{3.7}$$

Then $(\eta, \phi) = (a(t)\nabla\eta, \nabla\Phi) = (a(t)\nabla\eta, \nabla(\Phi - R_h\Phi)) + (a(t)\nabla\eta, \nabla R_h\Phi) = I_1 + I_2$.
By *Hölder* inequality , (3.5) and (3.7)

$$| I_1 | \ \leq \ C\|\eta\|_{1,p} \ \|\Phi - R_h\Phi\|_{1,p'} \leq Ch^r\|u\|_{r,p} \ \|\Phi\|_{2,p'} \leq Ch^r\|u\|_{r,p} \ .$$

Hence by using (3.1),

$$\begin{aligned} | I_2 | &= | (a(t)\nabla\eta, \nabla R_h\Phi) | \\ &= | (\int_0^t [(b(\tau) - a_t(\tau))\nabla\eta(\tau) + \int_0^\tau c(\tau, s)\nabla\eta(s)ds]d\tau, \nabla R_h\Phi) | \\ &\leq | (\int_0^t [(b(\tau) - a_t(\tau))\nabla\eta(\tau) + \int_0^\tau c(\tau, s)\nabla\eta(s)ds]d\tau, \nabla(R_h\Phi - \Phi)) | \\ &\quad + | (\int_0^t [(b(\tau) - a_t(\tau))\nabla\eta(\tau) + \int_0^\tau c(\tau, s)\nabla\eta(s)ds]d\tau, \nabla\Phi) | \\ &= J_1 + J_2. \end{aligned}$$

Following from *Hölder* inequality and (3.5)

$$\begin{aligned} J_1 &\leq C\int_0^t \|\eta\|_{1,p}d\tau \ \|R_h\Phi - \Phi\|_{1,p'} \\ &\leq Ch^r\int_0^t \|u\|_{r,p}d\tau \ \|\Phi\|_{2,p'} \\ &\leq Ch^r\int_0^t \|u\|_{r,p}d\tau. \end{aligned}$$

By using integration by parts, we have

$$J_2 \leq C\int_0^t \|\eta\|_{0,p}d\tau \ \|\Phi\|_{2,p'} \leq C\int_0^t \|\eta\|_{0,p}d\tau.$$

Thus

$$| (\eta, \phi) | \leq Ch^r(\|u\|_{r,p} + \int_0^t \|u\|_{r,p}d\tau) + C\int_0^t \|\eta\|_{0,p}d\tau.$$

It follows from Gronwall's Lemma

$$\|\eta\|_{0,p} = \sup_{\substack{\phi\in L_{p'}(\Omega)\\ \|\phi\|_{0,p'}=1}} |(\eta,\phi)| \le Ch^r(\|u\|_{r,p} + \int_0^t \|u\|_{r,p}d\tau)$$

The proof is completed.

We have the following estimate for Sobolev-Volterra projection.

Lemma3.2. Let $0 \le l \le r-2$, $p > 1$, $k = 0, 1, 2$, and $t \in J$, we have

$$|(D_t^k\eta,\phi)| \le Ch^{r+l}\ \|u(t)\|_{k,r,p}\ \|\phi\|_{l,p'}\ ,\ \phi \in W^{l,p'}(\Omega). \tag{3.8}$$

Proof. We only prove Lemma3.2 when $k = 0$, i.e.

$$|(\eta,\phi)| \le Ch^{r+l}\ \|u(t)\|_{0,r,p}\ \|\phi\|_{l,p'},\quad \phi \in W^{l,p'}(\Omega). \tag{3.9}$$

Introducing the dual auxiliary problem, for any $\phi \in W^{l,p'}(\Omega)$, let $\Phi \in W^{l+2,p'}(\Omega)$ be the solution of

$$(a(t)\nabla\Phi, \nabla v) = (v,\phi),\quad v \in H_0^1(\Omega), \tag{3.10}$$

Then we have

$$\|\Phi\|_{l+2,p'} \le C\|\phi\|_{l,p'}\ . \tag{3.11}$$

Let Φ^h be the Galerkin approximation of Φ, then by (3.10), we have

$$\begin{aligned}(\eta,\phi) &= (a(t)\nabla\eta,\nabla\Phi) = (a(t)\nabla\eta,\nabla(\Phi-\Phi^h)) + (a(t)\nabla\eta,\nabla\Phi^h)\\ &= (a(t)\nabla(V_hu - R_hu + R_hu - u),\nabla(\Phi-\Phi^h)) + (a(t)\nabla\eta,\nabla\Phi^h)\\ &= (a(t)\nabla(R_hu-u),\nabla(\Phi-\Phi^h)) + (a(t)\nabla\eta,\nabla\Phi^h)\\ &= I_1 + I_2.\end{aligned}$$

Then by (1.2) and *Hölder* inequality

$$|I_1| \le C\|R_hu-u\|_{1,p}\ \|\Phi-\Phi^h\|_{1,p'} \le Ch^{r+l}\ \|u\|_{r,p}\ \|\Phi\|_{l+2,p'},$$

And by (3.1)

$$\begin{aligned}|I_2| &= |(a(t)\nabla\eta,\nabla\Phi^h)|\\ &= |(\int_0^t [(b(\tau)\nabla\eta(\tau) - a_t(\tau)\nabla\eta(\tau) + \int_0^\tau c(\tau,s)\nabla\eta(s)ds]d\tau, \nabla\Phi^h)|\\ &= |(\int_0^t [(b(\tau)\nabla\eta(\tau) - a_t(\tau)\nabla\eta(\tau) + \int_0^\tau c(\tau,s)\nabla\eta(s)ds]d\tau, \nabla(\Phi^h-\Phi))|\\ &\quad + |(\int_0^t [(b(\tau)\nabla\eta(\tau) - a_t(\tau)\nabla\eta(\tau) + \int_0^\tau c(\tau,s)\nabla\eta(s)ds]d\tau, \nabla\Phi)|\\ &= J_1 + J_2.\end{aligned}$$

By *Hölder* inequality and (3.2)

$$\begin{aligned} J_1 &\leq C\{\int_0^t [\|\eta\|_{1,p} + \int_0^\tau \|\eta\|_{1,p} ds] d\tau\} \ \|\Phi^h - \Phi\|_{1,p'} \\ &\leq C \int_0^t \|\eta\|_{1,p} d\tau \ \|\Phi^h - \Phi\|_{1,p'} \\ &\leq Ch^{r+l} \int_0^t \|u\|_{0,r,p} d\tau \ \|\Phi\|_{l+2,p'}. \end{aligned}$$

Following from Green's formula

$$\begin{aligned} J_2 &\leq \ | \int_0^t ((b(\tau) - a_t(\tau))\nabla\eta(\tau), \nabla\Phi) d\tau \ | + | \int_0^t \int_0^\tau (c(\tau,s)\nabla\eta(s), \nabla\Phi) ds d\tau \ | \\ &= \ | \int_0^t (\eta, \nabla\cdot((b(\tau) - a_t(\tau))\nabla\Phi)) d\tau \ | + | \int_0^t \int_0^\tau (\eta(s), \nabla\cdot(c(\tau,s)\nabla\Phi)) ds d\tau \ | \\ &= \ | \int_0^t \frac{(\eta, \nabla\cdot((b(\tau) - a_t(\tau))\nabla\Phi))}{\|\nabla\cdot((b(\tau) - a_t(\tau))\nabla\Phi)\|_{l,p'}} \|\nabla\cdot((b(\tau) - a_t(\tau))\nabla\Phi)\|_{l,p'} d\tau \ | \\ &\quad + | \int_0^t \int_0^\tau \frac{(\eta(s), \nabla\cdot(c(\tau,s)\nabla\Phi))}{\|\nabla\cdot(c(\tau,s)\nabla\Phi)\|_{l,p'}} \|\nabla\cdot(c(\tau,s)\nabla\Phi)\|_{l,p'} ds d\tau \ | \\ &\leq C \int_0^t \sup_{\psi\in W^{l,p'}(\Omega)} \frac{|(\eta,\psi)|}{\|\psi\|_{l,p'}} \|\Phi\|_{l+2,p'} d\tau + C \int_0^t \sup_{\psi\in W^{l,p'}(\Omega)} \frac{|(\eta,\psi)|}{\|\psi\|_{l,p'}} \|\Phi\|_{l+2,p'} d\tau \\ &\leq C \int_0^t \sup_{\psi\in W^{l,p'}(\Omega)} \frac{|(\eta,\psi)|}{\|\psi\|_{l,p'}} d\tau \ \|\Phi\|_{l+2,p'}. \end{aligned}$$

Therefore,by (3.11), combing $I_1,\ J_1,\ J_2$,

$$| (\eta,\phi) | \leq Ch^{r+l}[\|u\|_{r,p} + \int_0^t \|u\|_{0,r,p} d\tau] \ \|\phi\|_{l,p'} + C \int_0^t \sup_{\psi\in W^{l,p'}(\Omega)} \frac{|(\eta,\psi)|}{\|\psi\|_{l,p'}} d\tau \ \|\phi\|_{l,p'}$$

Then

$$\frac{| (\eta,\phi) |}{\|\phi\|_{l,p'}} \leq Ch^{r+l} \|u\|_{0,r,p} + C \int_0^t \sup_{\psi\in W^{l,p'}(\Omega)} \frac{|(\eta,\psi)|}{\|\psi\|_{l,p'}} d\tau$$

$$\sup_{\psi\in W^{l,p'}(\Omega)} \frac{|(\eta,\psi)|}{\|\psi\|_{l,p'}} \leq Ch^{r+l} \ \|u\|_{0,r,p} + C \int_0^t \sup_{\psi\in W^{l,p'}(\Omega)} \frac{|(\eta,\psi)|}{\|\psi\|_{l,p'}} d\tau$$

By Gronwall's Lemma, we have

$$\sup_{\psi\in W^{l,p'}(\Omega)} \frac{|(\eta,\psi)|}{\|\psi\|_{l,p'}} \leq Ch^{r+l} \|u\|_{0,r,p}$$

Hence

$$\frac{|(\eta,\phi)|}{\|\phi\|_{l,p'}} \leq \sup_{\psi\in W^{l,p'}(\Omega)} \frac{|(\eta,\psi)|}{\|\psi\|_{l,p'}} \leq Ch^{r+l} \ \|u\|_{0,r,p}$$

Then

$$|(\eta,\phi)| \;\le\; \sup_{\psi\in W^{l,p'}(\Omega)} \frac{|(\eta,\psi)|}{\|\psi\|_{l,p'}}\,\|\phi\|_{l,p'} \le Ch^{r+l}\|u\|_{0,r,p}\;\|\phi\|_{l,p'}$$

The proof is completed.

The rest of the paper, u and U will stand for the solution of problems (1.1) and (2.5), respectively. Writing the error $U-u=(U-V_hu)+(V_hu-u)=\xi+\eta$, we then derive superconvergence estimates of the initial value error $\xi_t(0)$.

Lemma 3.3. For $2\le p\le\infty$, we have

$$\begin{cases} (a)\ \|\xi_t(0)\|_{1,p}\le Ch^{r+1}[\|u_0\|_{r,p}+\|u_t(0)\|_{r,p}]\ , & \mathrm{r}>2, \\ (b)\ \|\xi_t(0)\|_{1,p}\le Ch^{2}\ [\|u_0\|_{2,p}+\|u_t(0)\|_{2,p}]\ , & \mathrm{r}=2, \\ (c)\ \|\xi_t(0)\|_{0,p}\le Ch^{r+2}\ [\|u_0\|_{r,p}+\|u_t(0)\|_{r,p}]\ , & \mathrm{r}>3. \end{cases} \tag{3.12}$$

Proof. We first combine (2.3), (2.5) and (2.6) to obtain the error equation

$$(\xi_t,\chi)+(a(t)\nabla\xi_t+b(t)\nabla\xi+\int_0^t c(t,\tau)\nabla\xi(\tau)d\tau,\nabla\chi)=-(\eta_t,\chi),\ \chi\in S_h,\ t\in J. \tag{3.13}$$

To bound $\|\xi_t(0)\|_{1,p}$, we need an auxiliary problem. For any $\phi\in W^{1,p'}(\Omega)$ with $\|\phi\|_{0,p'}=1$. Let $\Phi\in H_0^1(\Omega)$ be the solution of

$$(a(t)\nabla v,\nabla\Phi)+(v,\Phi)=-(v,\phi_x),\ v\in H_0^1(\Omega),\ t\in J, \tag{3.14}$$

where ϕ_x is some component of $\nabla\phi$. Thus we have

$$\|\Phi\|_{1,p'}\le C_p(t)\|\phi\|_{0,p'}\le C_p(t),\ t\in J, \tag{3.15}.$$

According to our assumptions, $C_p(t)$ is uniformly bounded on t. Let Φ_h be the Galerkin approximation of Φ in S_h. Setting $t=0$ in (3.13) and by (2.5), noting $\xi(0)=0$, we see that

$$(\xi_t(0),\chi)+(a(0)\nabla\xi_t(0),\nabla\chi)=-(\eta_t(0),\chi). \tag{3.16}$$

Then by Green's formula, (3.14), (3.16), (3.8) and (3.15) , we know that

$$\begin{aligned} (\xi_{tx}(0),\phi) &= -(\xi_t(0),\phi_x) \\ &= (a(0)\nabla\xi_t(0),\nabla\Phi)+(\xi_t(0),\Phi) \\ &= (a(0)\nabla\xi_t(0),\nabla\Phi^h)+(\xi_t(0),\Phi^h) \\ &= -(\eta_t(0),\Phi^h) \\ &\le Ch^{r+1}\|u(0)\|_{1,r,p}\;\|\Phi^h\|_{1,p'} \\ &\le Ch^{r+1}\|u(0)\|_{1,r,p}\;\|\Phi\|_{1,p'} \\ &\le Ch^{r+1}\|u(0)\|_{1,r,p}\ . \end{aligned}$$

Thus

$$\|\xi_{tx}(0)\|_{0,p} = \sup_{\substack{\phi\in L_{p'}(\Omega)\\ \|\phi\|_{0,p'}=1}} |\ (\xi_{tx}(0),\phi)\ | \leq Ch^{r+1}\|u(0)\|_{1,r,p}\ . \tag{3.17}$$

By summing on $\|\xi_{tx}(0)\|_{0,p}$,we have

$$\|\xi_t(0)\|_{1,p} \leq Ch^{r+1}\|u(0)\|_{1,r,p} \leq Ch^{r+1}[\|u_0\|_{r,p} + \|u_t(0)\|_{r,p}], \quad r > 2.$$

Arguing as before, we can also derive (3.12b).

Finally we turn to the proof of (3.12c). For this end, we introduce the other auxiliary problem. For any $\psi \in L_{p'}(\Omega)$ with $\|\psi\|_{0,p'} = 1$. Let $\Psi \in W^{2,p'}(\Omega)$ be the solution of

$$(a(t)\nabla v, \nabla\Psi) + (v,\Psi) = (v,\psi), \ v \in H_0^1(\Omega), \ t \in J. \tag{3.18}$$

Then we have

$$\|\Psi\|_{2,p'} \leq C_p(t)\|\psi\|_{0,p'} \leq C_p(t), \ t \in J. \tag{3.19}$$

Here $C_p(t)$ is uniformly bounded on t. Let Ψ^h be the Galerkin approximation of Ψ. Using argument similar to those used to get (3.12a), by (3.18), (3.16), (3.8) and (3.19), we have

$$\begin{aligned}
(\xi_t(0),\psi) &= (a(0)\nabla\xi_t(0),\nabla\Psi) + (\xi_t(0),\Psi)\\
&= (a(0)\nabla\xi_t(0),\nabla\Psi^h) + (\xi_t(0),\Psi^h)\\
&= -(\eta_t(0),\Psi^h)\\
&= (\eta_t(0),\Psi-\Psi^h) - (\eta_t(0),\Psi)\\
&\leq |\ (\eta_t(0),\Psi-\Psi^h)\ | + |\ (\eta_t(0),\Psi)\ |\\
&\leq Ch^{r+1}\|u(0)\|_{1,r,p}\ \|\Psi-\Psi^h\|_{1,p'} + Ch^{r+2}\|u(0)\|_{1,r,p}\ \|\Psi\|_{2,p'}\\
&\leq Ch^{r+2}[\|u(0)\|_{r,p} + \|u_t(0)\|_{r,p}].
\end{aligned}$$

Thus,

$$\|\xi_t(0)\|_{0,p} = \sup_{\substack{\psi\in L_{p'}(\Omega)\\ \|\psi\|_{0,p'}=1}} |\ (\xi_t(0),\psi)\ | \leq Ch^{r+2}[\|u(0)\|_{r,p} + \|u_t(0)\|_{r,p}].$$

The proof is completed.

4. Superconvergence In $W^{1,p}(\Omega)$ And $L_p(\Omega)$

Our object of this section is to demonstrate the main results of the paper. We shall begin by proving two order global superconvergence estimate of ξ and ξ_t in $W^{1,p}(\Omega)$ for $2 \leq p < \infty$.

Theorem4.1. For $r > 2$ and $2 \le p < \infty$, we have, for $t \in J$,

$$\begin{cases} (a)\ \|\xi\|_{1,p} \le Ch^{r+1} \int_0^t \|u\|_{1,r,p} d\tau \ , \\ (b)\ \|\xi_t\|_{1,p} \le Ch^{r+1}[\|u(0)\|_{1,r,p} + \int_0^t \|u\|_{2,r,p} d\tau] \ . \end{cases} \tag{4.1}$$

Proof. We first estimate $\|\xi\|_{1,p}$. Let Φ, ϕ satisfy (3.14),(3.15), then an analogy of the proof for (3.12a) implies, by Green's formula, (3.14), (3.13) ,(3.8) and (3.15),

$$\begin{aligned}
(\xi_x, \phi) &= \int_0^t (\xi_{tx}, \phi) d\tau = -\int_0^t (\xi_t, \phi_x) d\tau \\
&= \int_0^t [(a(\tau)\nabla\xi_t, \nabla\Phi) + (\xi_t, \Phi)] d\tau \\
&= \int_0^t [(a(\tau)\nabla\xi_t, \nabla\Phi^h) + (\xi_t, \Phi^h)] d\tau \\
&= -\int_0^t [(\eta_t(\tau), \Phi^h) + (b(\tau)\nabla\xi + \int_0^\tau c(\tau, s)\nabla\xi(s) ds, \nabla\Phi^h)] d\tau \\
&\le C\{\int_0^t [h^{r+1}\|u(\tau)\|_{1,r,p}\ \|\Phi^h\|_{1,p'} + \|\xi\|_{1,p}\ \|\Phi^h\|_{1,p'} + \int_0^\tau \|\xi\|_{1,p} ds\ \|\Phi^h\|_{1,p'}] d\tau\} \\
&\le Ch^{r+1} \int_0^t \|u(\tau)\|_{1,r,p} d\tau\ \|\Phi\|_{1,p'} + C\int_0^t \|\xi\|_{1,p} d\tau\ \|\Phi\|_{1,p'} \\
&\le Ch^{r+1} \int_0^t \|u(\tau)\|_{1,r,p} d\tau + C\int_0^t \|\xi\|_{1,p} d\tau,
\end{aligned}$$

Then (4.1a) follows from Gronwall's Lemma.

To bound $\|\xi_t\|_{1,p}$, we differentiate (3.13) with respect to t to get

$$\begin{aligned}
&(\xi_{tt}, \chi) + (a(t)\nabla\xi_{tt}, \nabla\chi) + ((a_t(t) + b(t))\nabla\xi_t, \nabla\chi) + ((b_t(t) + c(t,t))\nabla\xi(t), \nabla\chi) \\
&+(\int_0^t c_t(t,\tau)\nabla\xi(\tau) d\tau, \nabla\chi) = -(\eta_{tt}, \chi), \quad \chi \in S_h.
\end{aligned} \tag{4.2}$$

By applying Green's formula, (3.14), (4.2), *Hölder* inequality ,Lemma 3.2, (3.15), (3.12a) and (4.1a), we have

$$
\begin{aligned}
(\xi_{tx},\phi) &= (\xi_{tx}(0),\phi)+\int_0^t(\xi_{ttx},\phi)d\tau\\
&= (\xi_{tx}(0),\phi)-\int_0^t(\xi_{tt},\phi_x)d\tau\\
&= (\xi_{tx}(0),\phi)+\int_0^t[(a(\tau)\nabla\xi_{tt},\nabla\Phi^h)+(\xi_{tt},\Phi^h)]d\tau\\
&= (\xi_{tx}(0),\phi)-\int_0^t[(\eta_{tt},\Phi^h)+(a_t(\tau)+b(\tau))\nabla\xi_t,\nabla\Phi^h)\\
&\quad +(b_t(\tau)+c(\tau,\tau))\nabla\xi(\tau),\nabla\Phi^h)+(\int_0^\tau c_\tau(\tau,s)\nabla\xi(s)ds,\nabla\Phi^h)]d\tau\\
&\le \|\xi_{tx}(0)\|_{0,p}\ \|\phi\|_{0,p'}+C[\int_0^t\{h^{r+1}\|u\|_{2,r,p}\ \|\Phi^h\|_{1,p'}\\
&\quad +\|\xi_t\|_{1,p}\ \|\Phi^h\|_{1,p'}+\|\xi\|_{1,p}\ \|\Phi^h\|_{1,p'}+\int_0^\tau\|\xi\|_{1,p}d\tau\ \|\Phi^h\|_{1,p'}\}d\tau]\\
&\le \|\xi_t(0)\|_{1,p}+C[\int_0^t\{h^{r+1}\|u\|_{2,r,p}+\|\xi_t\|_{1,p}+\|\xi\|_{1,p}+\int_0^\tau\|\xi\|_{1,p}ds\}d\tau]\\
&\le Ch^{r+1}\|u(0)\|_{1,r,p}+Ch^{r+1}\int_0^t\|u\|_{2,r,p}d\tau+C\int_0^t\|\xi\|_{1,p}d\tau+C\int_0^t\|\xi_t\|_{1,p}d\tau\\
&\le Ch^{r+1}\|u(0)\|_{1,r,p}+Ch^{r+1}\int_0^t\|u\|_{2,r,p}d\tau+C\int_0^t\|\xi_t\|_{1,p}d\tau.
\end{aligned}
$$

The desired estimate (4.1b) is now concluded by Gronwall's Lemma.
The proof is completed.

For the case of $r=2$, we also have the following superconvergence.

Theorem4.2. For $r=2$ and $2\le p<\infty$, we have, for $t\in J$

$$
\begin{cases}
(a)\ \|\xi\|_{1,p}\le Ch^2\int_0^t\|u\|_{1,2,p}d\tau\ ,\\[2ex]
(b)\ \|\xi_t\|_{1,p}\le Ch^2[\|u(0)\|_{1,2,p}+\int_0^t\|u\|_{2,2,p}d\tau]\ .
\end{cases}
\tag{4.3}
$$

We finally derive two order superconvergence of ξ and ξ_t in $L_p(\Omega)$, for $2 \leq p < \infty$.
Theorem4.3. For $r > 3,\ 2 \leq p < \infty,\ t \in J$, we have

$$\begin{cases} (a)\ \|\xi\|_{0,p} \leq Ch^{r+2} \int_0^t \|u\|_{1,r,p} d\tau, \\ (b)\ \|\xi_t\|_{0,p} \leq Ch^{r+2}[\|u(0)\|_{1,r,p} + \int_0^t \|u\|_{2,r,p} d\tau]\ . \end{cases} \tag{4.4}$$

Proof. Let $\psi,\ \Psi$ satisfy (3.18), (3.19), and Ψ^h is the Galerkin approximation of Ψ. In the same way as the proof for (3.12c), it follows from (3.18) and (3.13) that

$$\begin{aligned}(\xi, \psi) &= \int_0^t (\xi_t, \psi) d\tau \\ &= \int_0^t [(a(\tau)\nabla\xi_t, \nabla\Psi^h) + (\xi_t, \Psi^h)] d\tau \\ &= -\int_0^t [(\eta_t, \Psi^h) + (b(\tau)\nabla\xi + \int_0^\tau c(\tau, s)\nabla\xi(s) ds, \nabla\Psi^h)] d\tau \\ &= \int_0^t (\eta_t, \Psi - \Psi^h) d\tau - \int_0^t (\eta_t, \Psi) d\tau + \int_0^t (b(\tau)\nabla\xi \\ &\quad + \int_0^\tau c(\tau, s)\nabla\xi(s) ds, \nabla(\Psi - \Psi^h) d\tau \\ &\quad - \int_0^t (b(\tau)\nabla\xi + \int_0^\tau c(\tau, s)\nabla\xi(s) ds, \nabla\Psi) d\tau \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Next we estimate $I_1 - I_4$ respectively.
By Lemma3.2, we have

$$\begin{aligned}|\, I_1 \,| + |\, I_2 \,| &= |\int_0^t (\eta_t, \Psi - \Psi^h) d\tau \,| + |\int_0^t (\eta_t, \Psi) d\tau \,| \\ &\leq Ch^{r+1} \int_0^t \|u\|_{1,r,p}\ \|\Psi - \Psi^h\|_{1,p'} d\tau + Ch^{r+2} \int_0^t \|u\|_{1,r,p} d\tau\ \|\Psi\|_{2,p'} \\ &\leq Ch^{r+2} \int_0^t \|u\|_{1,r,p} d\tau\ \|\Psi\|_{2,p'}.\end{aligned} \tag{4.5}$$

By *Hölder* Inequality and (4.1a)

$$\begin{aligned}
| I_3 | &= | \int_0^t (b(\tau)\nabla\xi + \int_0^\tau c(\tau,s)\nabla\xi(s)ds, \nabla(\Psi - \Psi^h))d\tau | \\
&\leq C \int_0^t (\|\xi\|_{1,p} + \int_0^\tau \|\xi\|_{1,p}ds)\ \|\Psi - \Psi^h\|_{1,p'}d\tau \\
&\leq C \int_0^t \|\xi\|_{1,p}d\tau\ \|\Psi - \Psi^h\|_{1,p'} \\
&\leq Ch^{r+2} \int_0^t \|u\|_{1,r,p}d\tau\ \|\Psi\|_{2,p'} \qquad (4.6)
\end{aligned}$$

In order to estimate I_4, let $B(\tau,\xi,\Psi) = (b(\tau)\nabla\xi + \int_0^\tau c(\tau,s)\nabla\xi(s)ds, \nabla\Psi)$, B^* is the adjoint operator of B, then by Green's formula, we have

$$\begin{aligned}
| I_4 | &= | \int_0^t (b(\tau)\nabla\xi + \int_0^\tau c(\tau,s)\nabla\xi(s)ds, \nabla\Psi)d\tau | \\
&= | \int_0^t (\xi, B^*(\tau)\Psi)d\tau | \\
&\leq C \int_0^t \|\xi\|_{0,p}d\tau\ \|\Psi\|_{2,p'}. \qquad (4.7)
\end{aligned}$$

Combine (4.5)-(4.7) and by (3.19), we have

$$| (\xi,\psi) | \leq Ch^{r+2} \int_0^t \|u\|_{1,r,p}d\tau + C \int_0^t \|\xi\|_{0,p}d\tau.$$

Then (4.4a) follows from Gronwall's Lemma.

We next turn to the proof of (4.4b). An analogue of the proof for(4.1b), let ψ, Ψ

satisfy (3.18), (3.19), by (3.18) and (4.2), we have

$$\begin{aligned}
(\xi_t,\psi) &= (\xi_t(0),\psi)+\int_0^t(\xi_{tt},\psi)d\tau\\
&= (\xi_t(0),\psi)+\int_0^t[(a(\tau)\nabla\xi_{tt},\nabla\Psi)+(\xi_{tt},\Psi)]d\tau\\
&= (\xi_t(0),\psi)+\int_0^t[a(\tau)\nabla\xi_{tt},\nabla\Psi^h)+(\xi_{tt},\Psi^h)]d\tau\\
&= (\xi_t(0),\psi)-\int_0^t[(\eta_{tt},\Psi^h)+((a_t(\tau)+b(\tau))\nabla\xi_t,\nabla\Psi^h)\\
&\quad+((b_t(\tau)+c(\tau,\tau))\nabla\xi(\tau),\nabla\Psi^h)+(\int_0^\tau c_\tau(\tau,s)\nabla\xi(s)ds,\nabla\Psi^h)]d\tau\\
&= (\xi_t(0),\psi)+\int_0^t(\eta_{tt},\Psi-\Psi^h)d\tau-\int_0^t(\eta_{tt},\Psi)d\tau\\
&\quad+\int_0^t((a_t(\tau)+b(\tau))\nabla\xi_t,\nabla(\Psi-\Psi^h))d\tau\\
&\quad-\int_0^t((a_t(\tau)+b(\tau))\nabla\xi_t,\nabla\Psi)d\tau+\int_0^t((b_t(\tau)+c(\tau,\tau))\nabla\xi(\tau),\nabla(\Psi-\Psi^h))d\tau\\
&\quad-\int_0^t((b_t(\tau)+c(\tau,\tau))\nabla\xi(\tau),\nabla\Psi)d\tau+\int_0^t(\int_0^\tau c_\tau(\tau,s)\nabla\xi(s)ds,\nabla(\Psi-\Psi^h))d\tau\\
&\quad-\int_0^t(\int_0^\tau c_\tau(\tau,s)\nabla\xi(s)ds,\nabla\Psi)d\tau\\
&= I_1+\cdots+I_9. \qquad (4.8)
\end{aligned}$$

Then by *Hölder* Inequality , (3.12c) and Lemma3.2, (4.1b) and (3.19), we have

$$\begin{aligned}
|I_1|+|I_2|+|I_3|+|I_4| &\le Ch^{r+2}[\|u_0\|_{r,p}+\|u_t(0)\|_{r,p}]\\
&\quad+Ch^{r+1}\int_0^t\|u\|_{2,r,p}d\tau\ \|\Psi-\Psi^h\|_{1,p'}\\
&\quad+Ch^{r+2}\int_0^t\|u\|_{2,r,p}d\tau\ \|\Psi\|_{2,p'}\\
&\quad+C\int_0^t\|\xi_t\|_{1,p}d\tau\ \|\Psi-\Psi^h\|_{1,p'}\\
&\le Ch^{r+2}[\|u_0\|_{1,r,p}+\int_0^t\|u\|_{2,r,p}d\tau]\ \|\Psi\|_{2,p'}\ .
\end{aligned}$$

By (4.1a)

$$\begin{aligned} | I_6 | + | I_8 | &\leq C\int_0^t \|\xi\|_{1,p} d\tau \ \|\Psi - \Psi^h\|_{1,p'} + C\int_0^t (\int_0^\tau \|\xi\|_{1,p} ds) d\tau \ \|\Psi - \Psi^h\|_{1,p'} \\ &\leq C\int_0^t \|\xi\|_{1,p} d\tau \ \|\Psi - \Psi^h\|_{1,p'} \\ &\leq Ch^{r+2}\int_0^t \|u\|_{1,r,p} d\tau \ \|\Psi\|_{2,p'} \end{aligned}$$

As in the proof for (4.7), we have, by (4.4a)

$$\begin{aligned} | I_5 | + | I_7 | &\leq C\int_0^t \|\xi_t\|_{0,p} d\tau \ \|\Psi\|_{2,p'} + C\int_0^t \|\xi\|_{0,p} d\tau \ \|\Psi\|_{2,p'} \\ &\leq C[h^{r+2}\int_0^t \|u\|_{1,r,p} d\tau + \int_0^t \|\xi_t\|_{0,p} d\tau] \ \|\Psi\|_{2,p'}. \end{aligned}$$

And

$$| I_9 | \leq C\int_0^t (\int_0^\tau \|\xi\|_{0,p} ds) d\tau \ \|\Psi\|_{2,p'} \leq C\int_0^t \|\xi\|_{0,p} d\tau \ \|\Psi\|_{2,p'}.$$

Combining $I_1 - I_9$ and (4.8), we have

$$| (\xi_t, \psi) | \leq Ch^{r+2}[\|u_0\|_{1,r,p} + \int_0^t \|u\|_{2,r,p} d\tau] \|\Psi\|_{2,p'} + C\int_0^t \|\xi_t\|_{0,p} d\tau \ \|\Psi\|_{2,p'}.$$

The proof of (4.4b) is now concluded by (3.19) and Gronwall's Lemma.
The proof is completed.

References

[1] X.Cui, The numerical analysis for the finite element of the Sobolev-Volterra projection and integro-differential equation, ACTA mathematic applicatae sinica, 2001.

[2] S.B.Cui,The global solution of the nonlinear integro-differential equations. ACTA mathematic applicatae sinica, 1993

[3] D.N.Arnold, J.Douglas Jr and V.Thomee, Superconvergence of a finite element approximation to the solution of Sobolev equation in a single space variable, Math.Comp. , 36(1981), 53-63.

[4] Do.Y.Kwak, S.G.Lee, Q.Li, Superconvergence of a finite element methods for linear integro-differential problems, Internet.J.Math.and Math.sci. , 23(2000), No.3, 595-607.

[5] Q.Li and H.W.Du, L^p error estimates and superconvergence for finite element approximations for nonliear parabolic problems, J.KSSIAM, Vol.4, No.1, 67-77, 2000.

[6] Q.Lin and Q.D.Zhu, Theory of supercongence for the finite element, Changsha,Hunan Scientic Publisher, China, 1989.

[7] Y.H.Cao, Some numerical methods for some involving equations. Master Thesis, Shandong Normal University, Jinan, Shandong, China, 2003.

[8] Q.Li, Two order superconvergence of finite element methods for Sobolev equations. The Korea Society for Computational and Applied Mathematics and Korea SIGCOAM. 2000.

[9] Y.P.Lin, V.Thomee and L.B.Wthlbin, A Ritz-Volterra projection to finite-element spaces and approximations to integro-differential and related equations, SIAM J.Numer.Anal., 27(1990), 608-621.

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