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Strong Laws of Large Numbers for Weighted Sums of Random Variables

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확률변수의 가중합에 대한 대수의 강법칙

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Strong laws of large numbers are established for the weighted sums of i.i.d. random variables which have higher order moment condition. One of the results of Sung (2001) is extended.

높은 차수의 적률을 갖는 i.i.d. 확률 변수의 가중합에 대한 강대수 법칙을 유도한다. 또한 Sung (2001)의 결과 중 하나를 확장한다.

Key words : Strong laws of large numbers, weight sums, i.i.d. random variables, almost sure convergence

1. Introduction

Let $\{X, X_n, n\geq 1\}$ be a sequence of i.i.d. random variables and $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ be an array of constants. The almost sure (a.s.) limiting behavior of weighted sums

$\sum_{i=1}^{n} a_{ni}X_i$ was studied by many authors (see, Bai and Cheng (2000), Choi and Sung (1987), Cuzick (1995), Sung (2001), Wu (1999)). Recently, Sung (2001) proved the strong law of large numbers

$$\sum_{i=1}^{n} a_{ni}X_i/b_n \to 0 \ a.s. \tag{1}$$

when $\{X, X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $EX=0$ and

$$E[\exp(h|X|^{\gamma})] < \infty \text{ for some } h>0 \text{ and some } 0<\gamma\leq 1, \quad v \tag{2}$$

and $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ is an array of constants satisfying

$$A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty, \quad A^{\alpha}_{\alpha,n} = \sum_{i=1}^{n} |a_{ni}|^{\alpha}/n \tag{3}$$

for some $1<\alpha\leq 2$, where $b_n = n^{1/\alpha}(\log n)^{1/\gamma+\beta}$ $(\beta>0)$.

In this paper, we extend the result of Sung (2001).

2. Main Result

We state and prove our main result.

Theorem 1. Let $0<\gamma\leq 1$. Let $\{X, X_n, n\geq 1\}$ be i.i.d. random variables satisfying $EX=0$ and (2). Let $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ be an array of constants such that

$$\max_{1\leq i\leq n} |a_{ni}| = o\left(\frac{1}{(\log n)^{1/\gamma}}\right) \tag{4}$$

and

$$\sum_{i=1}^{n} a^2_{ni} = o\left(\frac{1}{\log n}\right). \tag{5}$$

Then $\sum_{i=1}^{n} a_{ni}X_i \to 0 \ a.s.$

Proof. By (5), we have

$$E\left(\sum_{i=1}^{n} a_{ni}X_i\right)^2 = \sum_{i=1}^{n} a^2_{ni}EX^2 = o\left(\frac{1}{\log n}\right).$$

It follows that

$$\sum_{i=1}^{n} a_{ni}X_i \to 0 \ \textit{in probability.}$$

Hence, by Theorem 3.2.1 in Stout (1974), it suffices to prove that

$$\sum_{i=1}^{n} a_{ni}X_i^s \to 0 \ a.s.,$$

where (X_n^s) is a symmetric version of (X_n). So we need only to prove the result for (X_n) symmetric.

Define $X_{ni}' = X_i I(|X_i| \le (h^{-1}\log n)^{1/\gamma})$ and $X_{ni}'' = X_i - X_{ni}$ for $1 \le i \le n$ and $n \ge 1$. Note that $E[e^{h|X|^\gamma}] < \infty$ is equivalent to $\sum_{n=1}^{\infty} P(|X_n| > (h^{-1}\log n)^{1/\gamma}) < \infty$.

Hence, by the Borel-Cantelli lemma, $\sum_{i=1}^{n} |X_{ni}''|$ is bounded a.s. It follows by (4) that

$$|\sum_{i=1}^{n} a_{ni}X_{ni}''| \le \max_{1\le i\le n}|a_{ni}| \sum_{i=1}^{n} |X_{ni}''| = o(1)\frac{\sum_{i=1}^{n}|X_{ni}''|}{(\log n)^{1/\gamma}} \to 0$$

a.s. as $n \to \infty$. Hence, to prove the theorem, it remains to prove that

$$\sum_{i=1}^{n} a_{ni}X_{ni}' \to 0 \ a.s. \tag{6}$$

From an inequality $e^x \le 1 + x + \frac{x^2}{2}e^{|x|}$ for all $x \in R$, we have

$$E[e^{ta_{ni}X_{ni}'}] \le 1 + E[\frac{1}{2}t^2 a_{ni}^2 |X_{ni}'|^2 e^{t|a_{ni}||X_{ni}'|}]$$

for any $t > 0$. Let $\varepsilon > 0$ be given. By putting $t = 2\log n/\varepsilon$, we obtain

$$\begin{aligned} E[e^{ta_{ni}X_{ni}'}] &\le 1 + (\frac{1}{2})(\frac{2}{\varepsilon})^2 a_{ni}^2 \log^2 n\, E[|X_{ni}'|^2 e^{\frac{2}{\varepsilon}|a_{ni}||X_{ni}'|\log n}] \\ &= 1 + (\frac{1}{2})(\frac{2}{\varepsilon})^2 a_{ni}^2 \log^2 n\, E[e^{\log|X_{ni}'|^2} e^{\frac{2}{\varepsilon}|a_{ni}||X_{ni}'|\log n}]. \end{aligned}$$

Since $|X_{ni}'| = |X_i| I(|X_i| \le (h^{-1}\log n)^{1/\gamma})$, we have that

$$
\begin{aligned}
&E[e^{\log|X_{ni}'|^2} e^{\frac{2}{\varepsilon}|a_{ni}||X_{ni}'|\log n}] \\
&= \sum_{j=2}^{n} E[e^{\log|X|^2 I(|X|\le (h^{-1}\log n)^{1/\gamma})} e^{\frac{2}{\varepsilon}|a_{ni}||X|I(|X|\le (h^{-1}\log n)^{1/\gamma})\log n} \times \\
&\qquad I((h^{-1}\log(j-1))^{1/\gamma} < |X| \le (h^{-1}\log j)^{1/\gamma})] \qquad (7)\\
&\le \sum_{j=2}^{n} e^{2\log(h^{-1}\log j)^{1/\gamma}} e^{\frac{2}{\varepsilon}|a_{ni}|(h^{-1}\log j)^{1/\gamma}\log n} P((h^{-1}\log(j-1))^{1/\gamma} < |X| \le (h^{-1}\log j)^{1/\gamma}).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&2\log(h^{-1}\log j)^{1/\gamma} + \frac{2}{\varepsilon}|a_{ni}|(h^{-1}\log j)^{1/\gamma}\log n \\
&= 2\log(h^{-1}\log j)^{1/\gamma} + \frac{2}{\varepsilon}\frac{o(1)}{(\log n)^{1/\gamma}}(h^{-1}\log j)^{1/\gamma}\log n \\
&= 2\log(h^{-1}\log j)^{1/\gamma} + o(1)\frac{\log n}{(\log n)^{1/\gamma}}(\log j)^{1/\gamma} \\
&\le \log(j-1)
\end{aligned}
$$

for all sufficiently large $j(j_0 < j \le n)$. Thus we have by (7) that

$$
\begin{aligned}
&E[e^{\log|X_{ni}'|^2} e^{\frac{2}{\varepsilon}|a_{ni}||X_{ni}'|\log n}] \\
&\le \sum_{j=2}^{j_0} e^{2\log(h^{-1}\log j)^{1/\gamma}} e^{\frac{2}{\varepsilon}|a_{ni}|(h^{-1}\log j)^{1/\gamma}\log n} P((h^{-1}\log(j-1))^{1/\gamma} < |X| \le (h^{-1}\log j)^{1/\gamma}) \\
&\quad + \sum_{j=j_0+1}^{n} e^{\log(j-1)} P((h^{-1}\log(j-1))^{1/\gamma} < |X| \le (h^{-1}\log j)^{1/\gamma}) \\
&\le C + E[e^{h|X|^\gamma}]
\end{aligned}
$$

for some constant $C > 0$. It follows that

$$
\begin{aligned}
E[e^{ta_{ni}X_{ni}'}] &\le 1 + (\tfrac{1}{2})(\tfrac{2}{\varepsilon})^2 a_{ni}^2 \log^2 n \ (C + E[e^{h|X|^\gamma}]) \\
&\le \exp\{(\tfrac{1}{2})(\tfrac{2}{\varepsilon})^2 a_{ni}^2 \log^2 n \ (C + E[e^{h|X|^\gamma}])\}.
\end{aligned}
$$

Hence, by (5)

$$
\begin{aligned}
P(\sum_{i=1}^{n} a_{ni}X_{ni}' > \varepsilon) &= P(e^{t\sum_{i=1}^{n} a_{ni}X_{ni}'} > e^{t\varepsilon}) \\
&\le e^{-t\varepsilon} E[e^{t\sum_{i=1}^{n} a_{ni}X_{ni}'}] \\
&= e^{-t\varepsilon} \prod_{i=1}^{n} E[e^{ta_{ni}X_{ni}'}] \\
&= e^{-t\varepsilon} \prod_{i=1}^{n} \exp\{(\tfrac{1}{2})(\tfrac{2}{\varepsilon})^2 a_{ni}^2 \log^2 n \ (C + E[e^{h|X|^\gamma}])\} \\
&= e^{-t\varepsilon} \exp\{(\tfrac{1}{2})(\tfrac{2}{\varepsilon})^2 \log^2 n \ (C + E[e^{h|X|^\gamma}]) \sum_{i=1}^{n} a_{ni}^2\} \\
&= e^{-2\log n} \exp\{o(1)\log n\} \\
&= \exp\{(-2 + o(1))\log n\},
\end{aligned}
$$

which implies that

$$\sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} a_{ni} X_{ni}' > \varepsilon\right) < \infty.$$

By the Borel-Cantelli lemma, we have

$$limsup_{n\to\infty} \sum_{i=1}^{n} a_{ni} X_{ni}' \leq 0 \quad a.s.$$

By replacing X_{ni}' by $-X_{ni}'$ from the above statement, we obtain

$$liminf_{n\to\infty} \sum_{i=1}^{n} a_{ni} X_{ni}' \geq 0 \quad a.s.$$

Hence (6) is satisfied.

The following corollary is proved by Sung (2001).

Corollary 2. Let $0 < \gamma \leq 1$. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables satisfying $EX = 0$ and (2). Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying (3) for some $1 < \alpha \leq 2$. Then for $b_n = n^{1/\alpha}(\log n)^{1/\gamma+\beta}$ $(\beta > 0)$

$$\sum_{i=1}^{n} a_{ni} X_i / b_n \to 0 \quad a.s.$$

Proof. Note that

$$\left|\frac{a_{ni}}{b_n}\right| \leq \frac{\left(\sum_{i=1}^{n} |a_{ni}|^{\alpha}\right)^{1/\alpha}}{b_n} = \frac{(nA_{\alpha,n}^{\alpha})^{1/\alpha}}{n^{1/\alpha}(\log n)^{1/\gamma+\beta}} = o\left(\frac{1}{(\log n)^{1/\gamma}}\right),$$

and

$$\sum_{i=1}^{n} \left|\frac{a_{ni}}{b_n}\right|^2 \leq \frac{\left(\sum_{i=1}^{n} |a_{ni}|^{\alpha}\right)^{2/\alpha}}{n^{2/\alpha}((\log n)^{1/\gamma+\beta})^2} = \frac{(nA_{\alpha,n}^{\alpha})^{2/\alpha}}{n^{2/\alpha}((\log n)^{1/\gamma+\beta})^2} = o\left(\frac{1}{\log n}\right)$$

since $2(1/\gamma + \beta) > 1$. Thus it follows by Theorem 1 that $\sum_{i=1}^{n} a_{ni} X_i / b_n \to 0 \quad a.s.$

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