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# $C$-EXISTENCE FAMILY AND EXPONENTIAL FORMULA

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Abstract. In this paper, we show that an exponentially bounded mild $C$-existence family can be represented by the exponential formula.

## 1. Introduction

Consider the following abstract Cauchy Problem (ACP)

$$u'(t) = Au(t), \quad t \geq 0, \quad u(0) = x,$$

where $A$ is a linear operator in a Banach space $X$.

Existence families of bounded linear operators on $X$ has been introduced as a generalization of the strongly continuous semigroup (see [1, 2]). In this paper, we establish the exponential representation of an exponentially bounded $C$-existence family for $A$.

Throughout this paper $X$ will be a Banach space and the space of all bounded operators from $X$ into itself will be denoted by $B(X)$. For an operator $A$, we will write $D(A)$ for the domain of $A$ and $R(A)$ for the range of $A$. $[D(A)]$ is the normed space $D(A)$ with $||x||_{[D(A)]} = ||x|| + ||Ax||$, $x \in D(A)$. By a solution of (ACP) we mean a function $u(t) \in C([0,\ \infty),\ [D(A)]) \cap C^1([0,\ \infty),\ X))$ satisfying (ACP). By a mild solution of (ACP) we mean a function $u(t) \in C([0,\ \infty),\ X)$ such that

$$\int_0^t u(s)ds \in D(A) \quad \text{and} \quad \frac{d}{dt}(\int_0^t u(s)ds) = A(\int_0^t u(s)ds) + x$$

for all $t \geq 0$.

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Let $C$ be an operator in $B(X)$. The strongly continuous family of operators $\{S(t) : t \geq 0\} \subset B(X)$ is called a mild $C$-existence family for $A$ if for each $x \in X$ and $t \geq 0$

(i) $\int_0^t S(s)xds \in D(A)$

(ii) $A\left(\int_0^t S(s)xds\right) = S(t)x - Cx.$

Let $\{S(t) : t \geq 0\}$ be a mild $C$-existence family for $A$. Then $u(t) = S(t)x$ is a mild solution of (ACP) with $u(0) = Cx$ and $u(t) = x + \int_0^t S(s)yds$ is a solution of (ACP) for all $x$ with $Ax = Cy$.

## 2. Exponential Representation

We start with the following lemma given in [3].

*Lemma 2.1* Let $h_1,\ h_2 \in C([0,\ \infty),\ X)$ satisfying $||h_1(t)||,\ ||h_2(t)|| \leq Ce^{\omega t}$, $(t \geq 0)$ for some $C,\ \omega > 0$. Suppose $A$ is a closed operator in $X$ such that for $\lambda > \omega$

$$\int_0^\infty e^{-\lambda t} h_1(t)dt \in D(A) \text{ and } A\left(\int_0^\infty e^{-\lambda t} h_1(t)dt\right) = \int_0^\infty e^{-\lambda t} h_2(t)dt.$$

Then $h_1(t) \in D(A)$ and $Ah_1(t) = h_2(t)$.

*Theorem 2.2* Let $M,\ \omega \geq 0$ and let $\{S(t) : t \geq 0\}$ be a strongly continuous family of operators in $B(X)$ satisfying $||S(t)|| \leq Me^{\omega t}$ for $t \geq 0$. Suppose $A$ is a closed operator in $X$ and $\lambda - A$ is injective for $\lambda > \omega$. Then the following statements are equivalent:

(1) $\{S(t) : t \geq 0\}$ is a mild $C$-existence family for $A$.

(2) $R(C) \subset R((\lambda - A)^n)$ and

$$(\lambda - A)^{-n}Cx = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}S(t)xdt$$

for $\lambda > \omega$, $x \in X$ and a positive integer $n$.

*Proof.* Suppose (2) is satisfied. Then

$$\begin{aligned} A(\lambda - A)^{-1}Cx &= A\left(\int_0^\infty e^{-\lambda t}S(t)xdt\right) \\ &= \lambda A\left(\int_0^\infty e^{-\lambda t}\left(\int_0^t S(s)xds\right)dt\right). \end{aligned}$$

and

$$\begin{aligned}\lambda^{-1}A(\lambda-A)^{-1}Cx &= (\lambda-A)^{-1}Cx-\lambda^{-1}Cx\\ &= \int_0^\infty e^{-\lambda t}(S(t)x-Cx)dt.\end{aligned}$$

Thus we have

$$A\left(\int_0^\infty e^{-\lambda t}\left(\int_0^t S(s)xds\right)dt\right)=\int_0^\infty e^{-\lambda t}(S(t)x-Cx)dt.$$

By Lemma 2.1, $\int_0^t S(s)xds\in D(A)$ and $A(\int_0^t S(s)xds)=S(t)x-Cx$. Therefore $\{S(t):t\geq 0\}$ is a mild $C$-existence family for $A$.

Suppose $\{S(t):t\geq 0\}$ is a mild $C$-existence family for $A$. Since $A$ is closed,

$$\begin{aligned}A\left(\int_0^\infty e^{-\lambda t}S(t)xdt\right) &= A\left(\lambda\int_0^\infty e^{-\lambda t}\left(\int_0^t S(s)xds\right)dt\right)\\ &= \lambda\int_0^\infty e^{-\lambda t}(S(t)x-Cx)dt\\ &= \lambda\int_0^\infty e^{-\lambda t}S(t)xdt-Cx.\end{aligned}$$

So we have $Cx=(\lambda-A)(\int_0^\infty e^{-\lambda t}S(t)xdt)$. Therefore

$$R(C)\subset R(\lambda-A)\ \text{ and }\ (\lambda-A)^{-1}Cx=\int_0^\infty e^{-\lambda t}S(t)xdt.$$

Assume that $(\lambda-A)^n(\int_0^\infty t^{n-1}e^{-\lambda t}S(t)xdt)=(n-1)!Cx$. Then

$$\int_0^\infty t^n e^{-\lambda t}S(t)xdt=\int_0^\infty(\lambda t^n-nt^{n-1})e^{-\lambda t}(\int_0^t S(s)xds)dt.$$

Since $A$ is closed,

$$\begin{aligned}A\left(\int_0^\infty t^n e^{-\lambda t}S(t)xdt\right) &= \int_0^\infty(\lambda t^n-nt^{n-1})e^{-\lambda t}(S(t)x-Cx)dt\\ &= \int_0^\infty \lambda t^n e^{-\lambda t}S(t)xdt-n\int_0^\infty t^{n-1}e^{-\lambda t}S(t)xdt.\end{aligned}$$

Thus

$$(\lambda-A)\int_0^\infty t^n e^{-\lambda t}S(t)xdt=n\int_0^\infty t^{n-1}e^{-\lambda t}S(t)xdt=n!(\lambda-A)^{-n}Cx.$$

Therefore, the result follows.

*Theorem 2.3* Let $M,\ \omega \geq 0$ and let $\{S(t) : t \geq 0\}$ be a mild $C$-existence family for $A$ satisfying $||S(t)|| \leq Me^{\omega t}$ for $t \geq 0$. Suppose $A$ is a closed operator and $\lambda - A$ is injective for $\lambda > \omega$. Then

$$\lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} Cx = S(t)x$$

for all $x \in X$ and the convergence is uniform on bounded $t$-intervals.

*Proof.* Let $0 \leq t \leq T$ and $x \in X$. Then

$$\begin{aligned}\left(\frac{n}{t} - A\right)^{-n} Cx &= \frac{1}{(n-1)!}\int_0^\infty s^{n-1}e^{-\frac{n}{t}s}S(s)xds\\ &= \frac{t^n}{(n-1)!}\int_0^\infty u^{n-1}e^{-nu}S(ut)xdu.\end{aligned}$$

Thus

$$\left(I - \frac{t}{n}A\right)^{-n} Cx = \frac{n^n}{(n-1)!}\int_0^\infty (ue^{-u})^{n-1}e^{-u}S(ut)xdu.$$

Let $\varepsilon > 0$ be given. Choose $0 < a < 1 < b < \infty$ such that

$$||S(ut)x - S(t)x|| < \varepsilon \quad \text{for} \quad a < u < b.$$

Then

$$\begin{aligned}\left(I - \frac{t}{n}A\right)^{-n} Cx - S(t)x &= \frac{n^n}{(n-1)!}\int_0^\infty u^{n-1}e^{-nu}(S(ut)x - S(t)x)du\\ &= \frac{n^n}{(n-1)!}\int_0^a u^{n-1}e^{-nu}(S(ut)x - S(t)x)du\\ &+ \frac{n^n}{(n-1)!}\int_a^b u^{n-1}e^{-nu}(S(ut)x - S(t)x)du\\ &+ \frac{n^n}{(n-1)!}\int_b^\infty u^{n-1}e^{-nu}(S(ut)x - S(t)x)du.\end{aligned}$$

Since $ue^{-u}$ is increasing for $0 < u < 1$ and $e^{-u} \leq 1$,

$$\begin{aligned}&\frac{n^n}{(n-1)!}\left|\left|\int_0^a u^{n-1}e^{-nu}(S(ut)x - S(t)x)du\right|\right|\\ &\qquad \leq \frac{n^n}{(n-1)!}(ae^{-a})^{n-1}\int_0^a ||S(ut)x - S(t)x||du.\end{aligned}$$

Let $b_n = \frac{n^n}{(n-1)!}(ae^{-a})^{n-1}\int_0^a ||S(ut)x - S(t)x||du$. Then $\lim_{n\to\infty} b_{n+1}/b_n = ae^{1-a} < 1$. Thus $\lim_{n\to\infty} b_n = 0$.

Since $ue^{-u}$ is decreasing for $u > 1$ and $e^{-u} < 1$,

$$\frac{n^n}{(n-1)!}\left|\left|\int_b^\infty u^{n-1}e^{-nu}(S(ut)x - S(t)x)du\right|\right|$$
$$\leq \frac{n^n}{(n-1)!}b^{n-1}e^{-(n-1)b}\int_b^\infty ||S(ut)x - S(t)x||du.$$

Let $c_n = \frac{n^n}{(n-1)!}b^{n-1}e^{-(n-1)b}\int_b^\infty ||S(ut)x - S(t)x||du$. Then $\lim_{n\to\infty} c_{n+1}/c_n = be^{-b} < 1$. So $\lim_{n\to\infty} c_n = 0$.

Finally, we have

$$\frac{n^n}{(n-1)!}\left|\left|\int_a^b u^{n-1}e^{-nu}(S(ut)x - S(t)x)du\right|\right|$$
$$\leq \frac{n^n}{(n-1)!}\int_a^b u^{n-1}e^{-nu}\varepsilon du = \frac{n^n}{(n-1)!}\int_0^\infty u^{n-1}e^{-nu}\varepsilon du = \varepsilon.$$

Therefore we obtain the result.

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