Kangweon-Kyungki Math. Jour. **11** (2003), No. 1, pp. 45–49

K_0-PROXIMITY INDUCED BY UNIFORMITY

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Abstract. We introduce the k_0-proximity space as a generalization of the Efremovič -proximity space. We try to show that k_0-proximity structure lies between topological structures and uniform structure in the sense that all topological invariants are k_0-proximity invariants and all k_0-proximity invariants are uniform invariants.

1. Introduction

The proximity relation δ was introduced in 1950 by Efremovič and he showed that the proximity relation δ induces a topology $\tau(\delta)$ in X and that the induced topology is completely regular in [1].

He also showed that every completely regular space (X, τ) admits a compatible proximity δ on X such that $\tau(\delta) = \tau$. He axiomatically characterized the proximity relation, A is near B, which is denoted by $A\delta B$, for subsets A and B of any set X. Efremovič axioms of proximity relation δ are as follows;

E1. $A\delta B$ implies $B\delta A$.

E2. $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$.

E3. $A\delta B$ implies $A \neq \phi$, $B \neq \phi$.

E4. $A\not\delta B$ implies there exists a subset E such that $A\not\delta E$ and $(X - E)\not\delta B$.

E5. $A \cap B \neq \phi$ implies $A\delta B$.

A binary relation δ satisfying axioms E1-E5 on the power set of X is called a (Efremocič) proximity on X. If δ also satisfies the following;

E6. $x\delta y$ implies $x = y$ then δ is called the separated proximity relation.

Received February 6, 2003.
2000 Mathematics Subject Classification: 54A20, 54B10, 54B15.
Key words and phrases: metric space, proximity space, product space, quotient space.

Definition 1.1. Let δ be a binary relation between a set X and its power set $P(X)$ such that

$K_0$1. $x\delta\{y\}$ implies $y\delta\{x\}$.

$K_0$2. $x\delta(A \cup B)$ if and only if $x\delta A$ or $x\delta B$.

$K_0$3. $x\not\delta\phi$ for all $x \in X$.

$K_0$4. $x \in A$ implies $x\delta A$.

$K_0$5. For each subset $E \subset X$, if there is a point $x \in X$ such that either $x\delta A$, $x\delta E$ or $x\delta B$, $x\delta(X - E)$, then we have $y\delta A$ and $y\delta B$ for some $y \in X$. The binary relation δ is called the K_0-proximity on X iff δ satisfies the axioms $K_0 1 - K_0 5$. The pair (X, δ) is called a K_0-proximity space.

$K_0$6. If $x\delta\{y\}$ implies $x = y$, then δ is called the separated K_0-proximity relation.

Lemma 1.2. *In a K_0-proximity space (X, δ) let δ_1 be a binary relation on $P(X)$ defined as follows;*

If we define $A\delta_1 B$ if and only if there is a point $x \in X$ such that $x\delta A$,$x\delta B$, then δ_1 is an Efremovič proximity.

It is well known that a family $\mathcal{L}$ of subsets of a non-empty set X is an ultrafilter if and only if the following condition are satisfied:

(i) If A and B belong to $\mathcal{L}$, then $A \cap B \neq \phi$.

(ii) If $A \cap C \neq \phi$ for every $C \in \mathcal{L}$, then $A \in \mathcal{L}$.

(iii) If $(A \cup B) \in \mathcal{L}$, then $A \in \mathcal{L}$ or $B \in \mathcal{L}$.

Now we consider the family of sets in an K_0-proximity space satisfying condition similar to (i), (ii), (iii), with nearness replacing non-empty intersection and we are led to the following definition:

Definition 1.3. A family σ of subsets of an K_0-proximity space (X, δ) is called a cluster iff the following condition are satisfied;

(1) If A and B belong to σ, then there is a point $x \in X$ such that $x\delta A$ and $x\delta B$.

(2) If for every $C \in \sigma$, there is a point $x \in X$ such that $x\delta A$, $x\delta C$, then $A \in \sigma$.

(3) If $(A \cup B) \in \sigma$, then $A \in \sigma$ or $B \in \sigma$.

2. Main Results

We shall study questions concerning the relationship between uniform structures and K_0-proximity structures.

A uniform structure on X was first defined by Weil in terms of subsets of $X \times X$.

If $U \subset X \times X$ then $U^{-1} = \{(x, y) : (y, x) \in U\}$. Whenever $U = U^{-1}$ U is called symmetric. For subsets U, V of $X \times X$, $U \circ V = \{(x, z) :$ there exists a $y \in X$ such that $(x, y) \in V$ and $(y, z) \in U\}$.

Let $\Delta = \{(x, x) : x \in X\}$. If $A \subset X$, then $U[A] = \{y : (x, y) \in U$ for some $x \in A\}$

For $x \in X$, $U[x] = U[\{x\}]$

Definition 2.1. A uniform structure (or uniformity) $\mathcal{U}$ on a set X is a collection of subsets (called entourages) of $X \times X$ satisfying the following conditions:

(1) Every entourage contains the diagonal Δ.
(2) If $U \in \mathcal{U}$ and $V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.
(3) Given $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V \circ V \subset U$.
(4) If $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.
(5) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$.

The pair $(X,\mathcal{U})$ is called a uniform space.

A subfamily β of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$ iff each entourage in $\mathcal{U}$ contains a member of β.

A family φ is a subbase for $\mathcal{U}$ iff the family of finite intersections of members of φ is a base for $\mathcal{U}$.

It can be shown that for each $x \in X$, $\{U[x] : U \in \mathcal{U}\}$ is a neighbourhood filter. Thus $\mathcal{U}$ generates a topology $\mathcal{T} = \mathcal{T}(\mathcal{U})$ on X.

As is well known, this topology is always completely regular.

If $\mathcal{U}$ satisfies the additional condition

(6) $\bigcap\limits_{U \in \mathcal{U}} U = \Delta$,

Then $\mathcal{U}$ is called a Hausdorff or separated uniformity.

In this case, $\mathcal{T}(\mathcal{U})$ is Tychonoff. Conversely, every (Tychonoff) completely regular space $(X, \mathcal{T})$ has a compactible(separated) uniformity, i.e. a uniformity $\mathcal{U}$ such that $\mathcal{T} = \mathcal{T}(\mathcal{U})$.

Every uniformity has a base consisting of open(closed) symmetric members, and it is frequently more convenient to work with such a base for $\mathcal{U}$ rather than with $\mathcal{U}$ itself.

Theorem 2.2. *Every uniform space $(X,\mathcal{U})$ has an associated K_0-proximity $\delta = \delta(\mathcal{U})$ defined by that there is a point $x \in X$ such that $x\delta A$, $x\delta B$ iff $(A \times B) \cap U \neq \phi$ for every $U \in \mathcal{U}$.*

Furthermore, $\mathcal{T}(\mathcal{U}) = \mathcal{T}(\delta)$. If $\mathcal{U}$ is separated, then so is $\delta(\mathcal{U})$.

Proof. All the axioms for a K_0-proximity, except perhaps (K_0-5) are easily verified.

To verify (K_0-5), suppose for each $x \in X$, $x \not\delta A$ or $x \not\delta B$. Then there exists entourage U and entourage V such that $(\{x\} \times A) \cap U = \phi$ or $(\{x\} \times B) \cap V = \phi$.

By definition 2.1-(2), there exists an entourage W such that $W = U \cap V$. Then by definition 2.1-(3) there exists an entourage Z such that $Z \circ Z \subset W$.

Let $E = Z^{-1}[B]$. Then $[(\{x\} \times A) \cap Z = \phi$ or $(\{x\} \times E) \cap Z = \phi]$ and $[(\{x\} \times B) \cap Z = \phi$ or $(\{x\} \times (X-E)) \cap Z = \phi]$

i.e. there exists $E \subset X$ such that for each $x \in X$, ($x \not\delta A$ or $x \not\delta E$) and ($x \not\delta B$ or $x \not\delta X - E$).

To show that $\mathcal{T}(\delta) = \mathcal{T}(\mathcal{U})$, we observe that x is in the $\mathcal{T}(\mathcal{U})$- closure of A iff $x \in U[A]$ for every entourage U iff $(x \times A) \cap U \neq \phi$ for every entourage iff $x\delta A$, i.e. x is in the $\mathcal{T}(\delta)$-closure of A. Finally, suppose that $\mathcal{U}$ is separated. If $x\delta y$, then $(x,y) \cap U \neq \phi$ for every entourage U. This implies $(x,y) \cap \Delta \neq \phi$, so that $x = y$. Thus δ is separated. □

δ could equivalently be defined by for some $x \in X$ $x\delta A$ and $x\delta B$ iff $U[A] \cap U[B] \neq \phi$ for every $U \in \mathcal{U}$.

THEOREM 2.3. *Let $(X,\mathcal{U})$ be a uniform space and let $\delta = \delta(\mathcal{U})$. Then $A \ll B$ if and only if there is an entourage U such that $U[A] \subset B$.*

Proof. $A \ll B$ iff for each $x \in X$ $x \not\delta A$ or $x \not\delta (X-B)$ iff $(A \times (X - B)) \cap U = \phi$ for some $U \in \mathcal{U}$. But the last statement is equivalent to $U[A] \subset B$. □

As is well known, let $X \in \xi$, $Y \in \mathcal{U}$, a function $f : X \to Y$ is uniformly continuous iff for each $E \in \mathcal{U}$, there is some $D \in \xi$ such that $(x,y) \in D$ implies $(f(x), f(y)) \in E$.

If f is one-one, onto and both f and f^{-1} are uniformly continuous, we call f a uniform isomorphism and say X and Y are uniformly isomorphic.

THEOREM 2.4. *If $f : (X,\mathcal{U}_1) \to (Y,\mathcal{U}_2)$ is uniformly continuous, then $f : (X,\delta_1) \to (Y,\delta_2)$ is K_0-proximally continuous where $\delta_i = \delta(\mathcal{U}_i)$ for $i = 1,2$.*

Proof. Suppose on the contrary that for some $x \in X$ $x\delta_1 A$ and $x\delta_1 B$, but $f(x) \not\delta_2 f(A)$ or $f(x) \not\delta_2 f(A)$ for each $x \in X$.

Then there exists a $U_2 \in \mathcal{U}_2$ such that $(f(A) \times f(B)) \cap U_2 = \phi$. Since f is uniformly continuous, there exists a $U_1 \in \mathcal{U}_1$ such that $(x,y) \in$

U_1 implies $(f(x), f(y)) \in U_2$. But for some $x \in X$ $x\delta_1 A$ and $x\delta_1 B$, so that $(A \times B) \cap U_1 \neq \phi$ which implies $(f(A) \times f(B)) \cap U_2 \neq \phi$, a contradiction. □

The converse of the above theorem is not true. Consider the identity mapping $i : (X, \mathcal{U}_2) \to (X, \mathcal{U}_1)$ where X, $\mathcal{U}_1$ and $\mathcal{U}_2$ are defined as $\mathcal{U}_1$ is the usual metric uniformity and $\mathcal{U}_2$ is the subspace uniformity on X induced by the uniformity of its Smirnov compactification corresponding to the usual metric proximity and X is the real line. Then i is a K_0-proximity mapping from (X, δ) onto itself, but it is not uniformly continuous.

References

[1] V. A. Efremovič, *Infinitesimal spaces*, Dokl. Akad. Nauk SSSR, **76** (1951), 341–343.

[2] V. A. Efremovič, *The geometry of proximity I*, Mat. Sb. **31**(73) (1952), 189–200.

[3] Y. M. Smirov,*On proximity spaces*, Mat. Sb. **31**(73) (1952), 543–574.

[4] Y. M. Smirov, *On proximity spaces*, Mat. Sb. **31**(73), 543–574(in Russian); English translation in Am. math. Soc Transl. Ser. 2, **38** (1972), 5–35; MR 14, 1107.

[5] S. Leader, *On Products of proximity spaces*, Math. Ann, **154** (1964), 185–194.

[6] C. Y. Kim, *On the R-Proximities*, Yonsei Nochong, Vol **13** (1976), 1–5.

[7] Tadir Huasin, *Topology and Maps*, Mcmaseter University Hamilton, 1977.

[8] S. H. Han, *The cartesian products on extended Jiang subgroup*, J, of Kangweon-Kyungki Math. **2** (1994), 73–77.

[9] S. H. Han, *Product space and quotient space in K_0-proximity spaces*, J, of Kangweon-Kyungki Math. **1** (2002), 59–66.

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