CHARACTERIZATION OF THE HILBERT BALL BY ITS AUTOMORPHISMS

KANG-TAE KIM AND DAOWEI MA

ABSTRACT. We show in this paper that every domain in a separable Hilbert space, say $\mathcal{H}$, which has a C^2 smooth strongly pseudoconvex boundary point at which an automorphism orbit accumulates is biholomorphic to the unit ball of $\mathcal{H}$. This is the complete generalization of the Wong-Rosay theorem to a separable Hilbert space of infinite dimension. Our work here is an improvement from the preceding work of Kim/Krantz [10] and subsequent improvement of Byun/Gaussier/Kim [3] in the infinite dimensions.

1. Introduction

The primary goal of this article is to establish the following theorem, which gives a full generalization, to a separable Hilbert space of infinite dimension, of the Wong-Rosay Theorem of finite dimension.

THEOREM 1.1. *If a domain Ω in a separable Hilbert space $\mathcal{H}$ admits a C^2 strongly pseudoconvex boundary point at which a holomorphic automorphism orbit accumulates, then Ω is biholomorphic to the open unit ball in $\mathcal{H}$.*

Since there are many subtleties in setting up the necessary terminology in the infinite dimensions, we shall present the precise definitions in the next section.

There have been several important contributions by several authors concerning this line of research. Chronologically speaking, Wong [18] proved in 1977 the above theorem in $\mathbb{C}^n$ with the assumption that the domain Ω is bounded and strongly pseudoconvex at every boundary

point. Then, Rosay [16] improved it in 1979, using holomorphic peak functions, that the theorem holds if the domain is bounded and an automorphism orbit accumulation point is strongly pseudoconvex. Much later in 1995, Efimov [5] removed the boundedness assumption from Ω. That argument is now well set up using Sibony's analysis of plurisubharmonic peak functions. See [2], [7] and [3], for details. For the infinite dimension, Kim and Krantz [10] in 2000 proved the above theorem with an extra assumption that Ω is bounded and convex. They needed convexity since they were relying upon a weak-normal family argument which they developed. Then, developing an infinite dimensional version of Sibony's analysis on plurisubharmonic peak functions, Byun, Gaussier and Kim ([3] in 2002) removed the boundedness assumption from the theorem of Kim and Krantz. In this article, we remove the convexity assumption from the theorem of Byun-Gaussier-Kim, thus arriving at the optimal version of the theorem of this type. The crux of the proof uses a new method, which concerns a principle of strong convergence for certain holomorphic mappings of the infinite dimensional Hilbert space. This new convergence argument seems worth exploring further, with a separate interest. Finally, it is worth noting that a manifold version of Wong-Rosay theorem have been studied also. See [13] for instance. Now the most general version is known, and is due to Gaussier, Kim and Krantz ([8]). We also present the Hilbert manifold version in this article.

The rest of the paper is organized as follows. Since the proof uses the ideas developed by Kim and Krantz [10] and then the localization methods introduced in [3], we shall introduce the outline of their methods shortly after the notation and basic terminology are introduced. Then we shall present our methods leading to the strong convergence of the scaling sequence in the separable Hilbert space and to the proof of the main theorem.

2. Terminology

We introduce in this section the concepts of smoothness of mappings of infinite dimensional spaces and the strong pseudoconvexity. Further details in great generality can be found in the books of Mujica [14] and Dineen [4], for instance.

Let $\mathbb{E}$ and $\mathbb{F}$ be Banach spaces and let Ω be an open subset of $\mathbb{E}$. Let $u : \Omega \to \mathbb{F}$ be a C^∞ smooth mapping. Then for each point $p \in \Omega$ and vectors $v_1, \ldots, v_k \in \mathbb{E}$, we may define inductively the derivative $d^k u$ of

order k as follows:

$$du(p; v_1) = \lim_{\mathbb{R} \ni r \to 0} \frac{1}{r}(u(p + rv_1) - u(p))$$

$$d^2 u(p; v_1, v_2) = \lim_{\mathbb{R} \ni r \to 0} \frac{1}{r}(du(p + rv_2; v_1) - du(p; v_1))$$

$$\vdots$$

$$d^k u(p; v_1, \ldots, v_k) = \lim_{\mathbb{R} \ni r \to 0} \frac{1}{r}(d^{k-1}u(p + rv_k; v_1, \ldots, v_{k-1})$$
$$- d^{k-1}u(p; v_1, \ldots, v_{k-1})).$$

Notice that these derivatives are symmetric multi-linear over $\mathbb{R}$. If one so prefer, these formulae can be used to define C^k smoothness, requiring in that case that the corresponding derivatives are continuous multi-linear tensors.

Then the complex differentials can be defined accordingly:

$$\partial u(p; v) = \frac{1}{2}(du(q; v) - i\, du(q; iv))$$

$$\bar{\partial} u(p; v) = \frac{1}{2}(du(q; v) + i\, du(q; iv)).$$

We are now able to introduce the concept of holomorphic maps and the remaining terminology thereof. First, by a *holomorphic mapping* we mean a C^1 smooth map that is annihilated by the $\bar{\partial}$ operator. See [14] for equivalent definitions.

A *domain* in this article is an open connected subset of a Banach space. An *automorphism* of a domain Ω is a bijective holomorphic mapping of Ω with its inverse holomorphic. The *automorphism group* Aut (Ω) of a domain Ω is the group of all automorphisms of Ω. An *automorphism orbit* is a set of the form Aut $(\Omega)q = \{\varphi(q) \mid \varphi \in \text{Aut}\,(\Omega)\}$, where $q \in \Omega$. Thus, a boundary point p of Ω is said to be an *orbit accumulation point*, if p is an accumulation point of an automorphism orbit, i.e., if there is a sequence $\{\varphi_j\} \subset \text{Aut}\,(\Omega)$ and a point $q \in \Omega$ such that $\lim_{j \to \infty} \|\varphi_j(q) - p\| = 0$.

We now introduce the concept of strong pseudoconvexity. Let Ω be a domain in a Banach space $\mathbb{E}$. We say that Ω is strongly pseudoconvex at a boundary point $p \in \partial\Omega$ if there is an open neighborhood U of p and a C^2 smooth local defining function $\rho : U \to \mathbb{R}$ satisfying the following properties:

(i) $\Omega \cap U = \{z \in U \mid \rho(z) < 0\}$,
(ii) $\partial\Omega \cap U = \{z \in U \mid \rho(z) = 0\}$,

(iii) $d\rho(q; \cdot)$ is a non-zero functional for every $q \in \partial\Omega \cap U$,

(iv) there exists a constant $C > 0$ such that $\partial\bar{\partial}\rho(p; v, v) \geq C\|v\|^2$ for every $v \in \mathbb{E}$ satisfying $\partial\rho(p; v) = 0$.

As in the finite dimensional case, we call $\partial\bar{\partial}\rho$ the *Levi form* of ρ.

3. Localization and pluri-subharmonic peak functions

Let Ω be a domain in a Banach space $\mathbb{E}$, and let $p \in \partial\Omega$. A continuous function $\psi : \Omega \to \mathbb{R}$ is said to be *pluri-subharmonic* if it is subharmonic along every complex affine line in Ω. A *pluri-subharmonic peak function* at p of Ω is a pluri-subharmonic function $\psi_p : U \to \mathbb{R}$ defined on an open neighborhood U of the closure $\overline{\Omega}$ of Ω satisfying the following two conditions:

(1) $h(p) = 0$, and $h(z) < 0$ for every $z \in \overline{\Omega} \setminus \{p\}$.
(2) The sets $V_m := \{z \in \overline{\Omega} \mid h(z) > -1/m\}$, where $m = 1, 2, \ldots$, form a neighborhood basis at p in $\overline{\Omega}$.

Unlike the finite dimensional cases, the second condition is essential for the definition in the infinite dimensions. On the other hand, notice that every strongly pseudoconvex boundary point admits a pluri-subharmonic peak function for Ω.

Following the work of Sibony [17], several investigations from the articles of Efimov [5], Berteloot [2], Gaussier [7] and Byun-Gaussier-Kim [3] have been made. We exploit some of them which pertain to the localization and hyperbolicity.

THEOREM 3.1. (cf. p.588, [3]) *Let Ω be a domain in a Banach space $\mathbb{E}$ with a C^2 smooth strongly pseudoconvex boundary point at which an automorphism orbit accumulates. Then, Ω is Kobayashi hyperbolic.*

THEOREM 3.2. (cf. p.588, [3]) *Let Ω be a domain in a Banach space $\mathbb{E}$ with a C^2 smooth strongly pseudoconvex boundary point p which admits a sequence $\varphi_j \in \mathrm{Aut}\,(\Omega)$ of automorphisms and a point $q \in \Omega$ such that $\lim_{j \to \infty} \varphi_j(q) = p$. Then, for every Kobayashi distance ball $B_\Omega^K(x; r)$ of radius r centered at $x \in \Omega$ and for every open neighborhood U of p there exists $N > 0$ such that $\varphi_j(B_\Omega^K(x; r)) \subset U$ for every $j > N$.*

We choose not to include any details of the proofs, in order to avoid an excessive repetition with the references cited above. However, we consider it appropriate to point out that the localization method using pluri-subharmonic peak functions initiated by Sibony seems indeed

more effective than the traditional localization arguments relying upon holomorphic peak functions and normal family arguments.

4. Scaling maps and weak normal family

The contents of this section are mostly from the article of Kim and Krantz [10]. The scaling method introduced here has its finite dimensional origin in the work of Pinchuk [15], Frankel [6], Kim [9] and others. Some other details are in [3].

4.1. Pinchuk's scaling sequence

Here, we introduce Pinchuk's scaling sequence. We begin with some notation. We choose an orthonormal basis $e_1, e_2, \ldots$ for a separable Hilbert space $\mathcal{H}$. Then for each $z \in \mathcal{H}$, we write

$$z = \sum_{m=1}^{\infty} z_m e_m,$$

and

$$z' = \sum_{m=2}^{\infty} z_m e_m.$$

Let Ω be a domain in a separable Hilbert space $\mathcal{H}$ with a C^2 strongly pseudoconvex boundary point p. Then, there exist an open neighborhood U of p in $\mathcal{H}$ and an injective holomorphic mapping $G : U \to G(U) \subset \mathcal{H}$ such that the following hold:

(A) $G(p) = 0$,

(B) the domain $\Omega_U := G(\Omega \cap U) = \{z \in G(U) \mid \operatorname{Re} z_1 > \psi(\operatorname{Im} z_1, z')\}$ is strictly convex. Moreover, the function ψ is strongly convex and vanishes precisely to the second order at the origin.

Now, as in the hypothesis of Theorem 1.1, we work with the assumption that there exist $q \in \Omega$ and a sequence $\varphi_j \in \operatorname{Aut}(\Omega)$ such that $\lim_{j \to \infty} \|\varphi_j(q) - p\| = 0$. We may choose a subsequence if necessary so that we have $\varphi_j(q) \in U$ for every $j = 1, 2, \ldots$. Let $q_j = G(\varphi_j(q))$. Choose now for each j the point $p_j \in \partial \Omega_U$ such that $p_j - q_j = r_j e_1$ for some $r_j > 0$. Note that $p'_j = q'_j$. Let us denote by $p_{j1} = \langle p_j, e_1 \rangle$. Then we consider a complex affine linear isomorphism $H_j : \mathcal{H} \to \mathcal{H} : z \mapsto w$ defined by

$$\begin{aligned} w_1 &= e^{\theta_j}(z_1 - p_{j1}) + T_j(z' - p') \\ w' &= z' - p', \end{aligned}$$

where the bounded linear functional $T_j : (e_1)^\perp \to \mathbb{C}$ is chosen for each j to satisfy the following properties:

(C1) $H_j(\Omega_U)$ is supported by the real hyperplane defined by $\operatorname{Re} w_1 = 0$ at the origin of $\mathcal{H}$,

(C2) $0 \in \partial T_j(\Omega_U)$,

(C3) $T_j(q'_j) = e^{\theta_j} r_j$.

Note also that e^{θ_j} can be chosen so that it converges to 1 as $j \to \infty$. See [10] for an explicit choice for these maps and the values for θ_j.

Now we define the Pinchuk scaling sequence. Define the linear map $L_j : \mathcal{H} \to \mathcal{H}$ by

$$L_j(w) = \frac{w_1}{r_j} e_1 + \frac{1}{\sqrt{r_j}} w'.$$

Then the Pinchuk scaling sequence is defined by the composition

$$\omega_j = L_j \circ H_j \circ G \circ \varphi_j.$$

This map is not well-defined on Ω. However, the localization theorems in Section 3 imply that there exists an increasing and exhausting sequence of Kobayashi distance open balls $B_\Omega^K(q, R_k)$ $(k = 1, 2, \ldots; R_1 < R_2 < \cdots)$ so that ω_j is a well-defined map of $B_\Omega^K(q, R_k)$ whenever $j \geq k$.

4.2. Weak normal family theorems

In the finite dimensions, the Pinchuk scaling sequence ω_j defines a normal family whose subsequential limits are holomorphic embeddings of Ω into the ambient Euclidean space. However, it is not the case in the infinite dimensions. In this section, we introduce the concept of weak convergence of holomorphic mappings that produces holomorphic limits. This is again from [10] and [3].

THEOREM 4.1. (Theorem 4.4 of [3]) *Let $\mathbb{E}$ be a separable Banach space, and let $\mathbb{F}$ a reflexive Banach space. Let Ω_1 and Ω_2 be domains in $\mathbb{E}$ and $\mathbb{F}$, respectively. Assume further that Ω_2 is bounded. Then, for every sequence $h_j : \Omega_1 \to \Omega_2$ of holomorphic mappings, there exist a subsequence h_{j_k} and a holomorphic mapping $\widehat{h} : \Omega_1 \to \mathbb{F}$ such that, for each $x \in \Omega_1$, the sequence $h_{j_k}(x)$ converges weakly to $\widehat{h}(x)$.*

Unlike the finite dimensions, this theorem is not so effective. The weak limit $\widehat{h}$ is holomorphic, but not in general injective, even if the maps h_j are injective. Also, it is not even guaranteed at this point that $\widehat{h}(\Omega_1)$ is contained in Ω_2. (Although, we do have that $\widehat{h}(\Omega_1)$ is contained

in the closed convex hull of Ω_2 due to the reflexivity of $\mathbb{F}$.) Nonetheless, this is about the best one can obtain from the general theory.

Before proceeding further, we point out that one can modify the Pinchuk scaling sequence ω_j so that the range becomes bounded. In [10], a method is described in detail to modify ω_j by composing with an explicit linear fractional transformation Ψ so that the map $\Psi \circ \omega_j$ has its image in $(1 + \epsilon_j)\mathbb{B}$ for each j, where $\mathbb{B}$ is the unit ball in the Hilbert space. Moreover, the sequence of positive numbers ϵ_j converges monotonically to zero.

From here on, we shall denote the map $\Psi \circ \omega_j$ by τ_j for $j = 1, 2, \ldots$.

4.3. Calibration of derivatives and Kim-Krantz scaling sequence

In [10], a new method of modifying the scaling sequence has been introduced. The goal is to have a strong convergence of the derivatives $d\tau_j(q; \cdot)$, as $j \to \infty$. In order to do this, Kim and Krantz have used Hilbert isometries to calibrate each differential $d\tau_j(q; \cdot)$ as follows: first they consider the new basis $d\tau_j(q; e_m)$ $(m = 1, 2, \ldots)$ for the separable Hilbert space $\mathcal{H}$. Then they apply the Gram-Schmidt process to these vectors such as

$$f_{jm} := d\tau_j(q; e_m) - \sum_{k=1}^{m-1} \frac{\langle d\tau_j(q; e_k), f_{jk} \rangle}{\langle f_{jk}, f_{jk} \rangle} f_{jk}.$$

It turns out that the vectors f_{jm} have their norms uniformly bounded away from zero by a constant independent of j and m. Then they define the Hilbert space isometries $S_j : \mathcal{H} \to \mathcal{H}$ arising from the condition $S_j(f_{jm}/\|f_{jm}\|) = e_m$ for every $j, m = 1, 2, \ldots$. Finally, the Kim/Krantz scaling sequence

$$\sigma_j := S_j \circ \tau_j : B_\Omega^K(q; R_j) \to (1 + \epsilon_j)\mathbb{B}$$

is introduced.

If we use the notation $\Sigma_n = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$, we can observe at this point immediately that $d\sigma_j(q; \Sigma_n) = \Sigma_n$ for every positive integer n. Moreover the sequences $\|d\sigma_j(q; \cdot)\|$ and $\|d\sigma_j(q; \cdot)^{-1}\|$ are both uniformly bounded. (See Section 7 of [10] for details.)

In summary, one obtains the following:

PROPOSITION 4.2. *Let Ω be a domain in a separable Hilbert space $\mathcal{H}$ with a C^2 smooth, strongly pseudoconvex boundary point p at which an automorphism orbit accumulates. Then, there exist a point $q \in \Omega$, a decreasing sequence ϵ_j of positive numbers tending to zero, an increasing*

sequence R_j tending to infinity, and a sequence of holomorphic mappings $\sigma_j : B_\Omega^K(q; R_j) \to (1 + \epsilon_j)\mathbb{B}$ such that

  (i) σ_j converges weakly to a holomorphic mapping σ at every point of Ω;

 (ii) $\sigma_j(q) = 0$ and $\sigma(q) = 0$;

(iii) $d\sigma_j(q)$ and $d\sigma(q)$ are calibrated in the sense that they map the flag subspace $\Sigma_n = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$ into Σ_n for each positive integer n; and

(iv) $d\sigma_j(q)$ converges to $d\sigma(q)$ on every Σ_n.

Notice that the arguments up to this point are sufficient to prove the main theorem of [10]. The main theorem of [3] is also in the same line but uses more modifications for the convergence of the sequence of σ_j^{-1} since Ω may be convex but still unbounded. We would like to remark that our proof, as one can see in the subsequent section, goes around such difficulties establishing directly the two facts: (1) σ is injective, and (2) $\sigma(\Omega) = \mathbb{B}$.

5. Strong convergence of the scaling sequence

5.1. Techniques for strong convergence arguments

We now demonstrate a new method of strong normal families in the infinite dimensional Hilbert space. We begin with an estimate on the Kobayashi metric and distance. From here on, d_M and k_M will denote the Kobayashi distance and metric of the complex manifold M, respectively. Let $u : [0, 1) \to [0, \infty)$ be defined by $u(t) = (1/2)\ln[(1+t)/(1-t)]$, so that $u(t) = d_\Delta(0, t)$ and $u^{-1}(s) = \tanh s$.

LEMMA 5.1. *Let $\Omega \subset H$ be a Kobayashi hyperbolic domain. For $q, x \in \Omega$, denote by $a = d_\Omega(x, q)$. If Ω' is a subdomain of Ω such that $\Omega' \supset \{y \in \Omega : d_\Omega(y, q) < b\}$, where $b > a$, then $d_{\Omega'}(x, q) \leq a/\tanh(b-a)$, $k_{\Omega'}(x, v) \leq k_\Omega(x, v)/\tanh(b - a)$ for $v \in H$.*

Proof. We first prove the second inequality. Let $s = \tanh(b - a)$ and $\epsilon > 0$. Then, there exists a holomorphic map $f : \Delta \to \Omega$ such that $f(0) = x$ and $f'(0) = v/(k_\Omega(x, v) + \epsilon)$. If $\zeta \in \Delta(0, s)$, then

$$\begin{aligned} d_\Omega(q, f(\zeta)) &\leq d_\Omega(q, x) + d_\Omega(x, f(\zeta)) \\ &= a + d_\Omega(f(0), f(\zeta)) \\ &\leq a + d_\Delta(0, \zeta) \\ &< a + (b - a) = b. \end{aligned}$$

So $f(\Delta(0, s)) \subset \Omega'$. Define $g : \Delta \to \Omega'$ by $g(\zeta) = f(s\zeta)$. Then we have $g(0) = x$ and $g'(0) = sf'(0) = sv/(k_\Omega(x, v) + \epsilon)$. Thus, it holds that $k_{\Omega'}(x, v) \leq (k_\Omega(x, v) + \epsilon)/s$. Since ϵ can be arbitrarily small, we obtain that $k_{\Omega'}(x, v) \leq k_\Omega(x, v)/s$.

We now prove the first inequality. Let $\epsilon \in (0, b - a)$. There exists a C^1 curve $z : [0, 1] \to \Omega$ such that $z(0) = q$, $z(1) = x$, and $\int_0^1 k_\Omega(z(t), z'(t)) \, dt < a + \epsilon$. It follows that $d_\Omega(q, z(t)) < a + \epsilon < b$ and $z(t) \in \Omega'$ for each $t \in [0, 1]$. By the inequality that we proved in the preceding paragraph, $k_{\Omega'}(z(t), z'(t)) \leq k_\Omega(z(t), z'(t))/\tanh(b - a - \epsilon)$. Therefore,

$$
\begin{aligned}
d_{\Omega'}(x, q) &\leq \int_0^1 k_{\Omega'}(z(t), z'(t)) \, dt \\
&\leq \frac{1}{\tanh(b - a - \epsilon)} \int_0^1 k_\Omega(z(t), z'(t)) \, dt.
\end{aligned}
$$

So we see that $d_{\Omega'}(x, q) < (a + \epsilon)/\tanh(b - a - \epsilon)$. Allowing ϵ tend to 0, we obtain the desired conclusion. $\square$

The next lemma follows immediately by a standard normal family argument.

LEMMA 5.2. *For each positive number $\epsilon < 1$ there exists a constant $\delta > 0$ such that for each holomorphic function $f : \Delta \to \Delta$ with $f(0) = 0$ and $f'(0) > 1 - \delta$ it holds that*

$$|f(z) - z| < \epsilon, \qquad \text{whenever } |z| \leq 1 - \epsilon.$$

We now present a crucial lemma, which establishes the strong uniform convergence on exhausting open subsets for the Kim/Krantz scaling sequence introduced in Section 4.3. We begin with some notation. For bounded linear operators S and T of the Hilbert space $\mathcal{H}$, we use the standard notation $S \leq T$ (or $T \geq S$, equivalently) which means that $\langle (T - S)x, x \rangle \geq 0$ for each $x \in \mathcal{H}$. We also use two more standard notation: I for the identity map of $\mathcal{H}$, and the notation $\mathbb{B}$ for the open unit ball in $\mathcal{H}$.

LEMMA 5.3. *Let $\{a_j\}$ be a sequence of positive numbers with $a_j \to 0$ and let $g_j : \mathbb{B} \to \mathbb{B}$ be a sequence of holomorphic mappings such that $g_j(0) = 0$ and $dg_j(0) \geq (1 - a_j)I$. Then the sequence g_j converges to I uniformly on each $r\mathbb{B}$ with $0 < r < 1$.*

Proof. Fix $0 < r < 1$ and $0 < \epsilon < 1/8$. Let ζ be a unit vector in the Hilbert space $\mathcal{H}$. Define a function $f_j : \Delta \to \Delta$ by $f_j(z) = \langle g_j(z\zeta), \zeta \rangle$.

Then $f_j'(0) = \langle dg_j(0)\zeta, \zeta \rangle \geq 1 - a_j$. By Lemma 5.2, there is a positive integer $k = k(r, \epsilon)$ such that $|f_j(z) - z| < \epsilon$ whenever $j \geq k$ and $|z| \leq r$. Let $h_j(z) := g_j(z\zeta) - f_j(z)\zeta$. Then $\langle h_j(z), \zeta \rangle = 0$. By Schwarz's Lemma, we have $|z|^2 \geq \|g_j(z\zeta)\|^2$. This implies that

$$|z|^2 \geq |f_j(z)|^2 + \|h_j(z)\|^2 \geq (|z| - \epsilon)^2 + \|h_j(z)\|^2.$$

Consequently, we obtain $\|h_j(z)\|^2 < 2\epsilon - \epsilon^2$ and

$$|g_j(z\zeta) - z\zeta|^2 = \|h_j(z)\|^2 + \|f_j(z)\zeta - z\zeta\|^2 \leq (2\epsilon - \epsilon^2) + \epsilon^2 = 2\epsilon$$

for $j \geq k$ and $|z| \leq r$. Therefore, we see immediately that the sequence g_j converges to I uniformly on each $r\mathbb{B}$.   □

We introduce one more technical lemma before we present the proof of our main theorem.

LEMMA 5.4. *Let* $\psi_j : \mathbb{B} \to \mathbb{B}$ *be a sequence of holomorphic mappings such that the sequence* ψ_j *converges to* I *uniformly on* $r\mathbb{B}$ *for every* $0 < r < 1$. *Then for each* $0 < r < 1$ *there exists a* $k \geq 1$ *such that* ψ_j *is injective on* $r\mathbb{B}$ *and* $\psi_j(\mathbb{B}) \supset r\mathbb{B}$ *whenever* $j \geq k$.

Proof. By the Cauchy estimates, we can deduce that $d\psi_j$ converges to I uniformly on each $r\mathbb{B}$. Thus, for fixed constants $r > 0$ and $\epsilon < 1$, it holds that

$$\|\psi_j(x) - \psi_j(y) - (x - y)\| < \epsilon\|x - y\|$$

for any $x, y \in rB$ and any sufficiently large j. Observe that the injectivity of ψ_j on $r\mathbb{B}$ follows from this inequality immediately.

Now, let $0 < r < 1$. Choose $0 < \epsilon < 1/8$ so that $(1 + 2\epsilon)r < 1$. Let j be sufficiently large so that $\psi = \psi_j$ satisfies $\|d\psi - I\| < \epsilon$ on $(1 + 2\epsilon)r\mathbb{B}$. Fix $x \in r\mathbb{B}$. Let $y_0 = x$ and let $y_k = x + y_{k-1} - \psi(y_{k-1})$ for $k = 1, 2, \ldots$. Then we see that

$$\begin{aligned} \|\psi(y_k) - x\| &= \|\psi(y_k) - \psi(y_{k-1}) - (y_k - y_{k-1})\| \\ &\leq \epsilon\|y_k - y_{k-1}\| \\ &= \epsilon\|\psi(y_{k-1}) - x\|. \end{aligned}$$

It follows that $\|y_{k+1} - y_k\| = \|\psi(y_k) - x\| \leq \epsilon^k\|y_1 - y_0\|$. Thus $\|\psi(y_k) - x\| \to 0$ and $\{y_k\}$ is a Cauchy sequence. The completeness of $\mathcal{H}$ implies that y_k converges to a certain y. Moreover, it is obvious now that $\psi(y) = x$. The remaining assertion follows immediately.   □

5.2. Proof of Theorem 1.1

By the hypothesis, we are given a point $q \in \Omega$ such that the $\mathrm{Aut}\,(\Omega)$-orbit of $q \in \Omega$ accumulates at a strongly pseudoconvex boundary point p.

For $j = 2, 3, \dots$ let $b_j = u(1 - 1/j) = (1/2) \ln(2j - 1)$. Recall that the following have been proved in [3] (Also see Proposition 4.2 of this article):

(i) Ω is hyperbolic.

(ii) There exist subdomains $\Omega_j \subset \Omega$, injective holomorphic mappings $\sigma_j : \Omega_j \to \mathcal{H}$ $(j = 2, 3, \dots)$, and a holomorphic mapping $\sigma : \Omega \to \mathcal{H}$, satisfying the following.

   (a) $\bigcup\limits_{j=1}^{\infty} \Omega_j = \Omega$;

   (b) $\Omega_j \supset \{x \in \Omega : d_\Omega(x, q) < b_j\}$;

   (c) $\sigma_j(q) = 0$, $\sigma(q) = 0$;

   (d) $d\sigma_j(q)$ and $d\sigma(q)$ are *calibrated* in the sense that they map the flag subspace $\Sigma_n = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$ into Σ_n for each positive integer n;

   (e) $(1 - 1/j)\mathbb{B} \subset \sigma_j(\Omega_j) \subset (1 + 1/j)\mathbb{B}$ and $\sigma(\Omega) \subset \overline{\mathbb{B}}$;

   (f) σ_j converges weakly to σ at every point of Ω;

   and

   (g) $d\sigma_j(q)$ converges to $d\sigma(q)$ on every Σ_n.

Our present goal is to show that σ is a biholomorphic mapping onto the open unit ball $\mathbb{B}$ of the separable Hilbert space $\mathcal{H}$.

First we observe that $\sigma(\Omega) \subset \mathbb{B}$ by the maximum modulus principle. It follows by (b), (e) and Lemma 5.1 that, for every $v \in \mathcal{H}$,

$$(1) \quad (1 - 1/j)k_\Omega(q, v) \leq \|d\sigma_j(q)(v)\| \leq (1 + 1/j)(1 - 1/j)^{-1}k_\Omega(q, v).$$

With (g) and the fact that $d\sigma_j(q)$ are uniformly bounded, we see that $\|d\sigma_j(q)(v)\| \to \|d\sigma(q)(v)\|$ for every $v \in \mathcal{H}$, as $j \to \infty$. It follows now that

$$(2) \qquad\qquad \|d\sigma(q)(v)\| = k_\Omega(q, v).$$

Consider $\sigma_j^{-1} : (1 - 1/j)\mathbb{B} \to \Omega$. By (1) and (2), it follows that

$$\|d(\sigma \circ \sigma_j^{-1})(0)(v)\| \geq (1 - 1/j)(1 + 1/j)^{-1}\|v\|.$$

Let $d(\sigma \circ \sigma_j^{-1})(0) = P_j U_j$ be the polar decomposition of the invertible operator $d(\sigma \circ \sigma_j^{-1})(0)$, where P_j is positive and U_j unitary. Define a map $\tau_j : \mathbb{B} \to \Omega$ by $\tau_j(x) = \sigma_j^{-1}((1 - 1/j)U_j^{-1}x)$. Then $\sigma \circ \tau_j : \mathbb{B} \to \mathbb{B}$, $\sigma \circ \tau_j(0) = 0$. Moreover, the positive operator $d(\sigma \circ \tau_j)(0) = (1 - 1/j)P_j$ satisfies

$$\|d(\sigma \circ \tau_j)(0)(v)\| \geq c_j \|v\|,$$

where $c_j = (1 - 1/j)^2(1 + 1/j)^{-1}$. At this point, we pause for the following observation.

LEMMA 5.5. *If P is a positive operator, and if $\|Pv\| \geq c\|v\|$ for a positive number c, then $P \geq cI$.*

Proof of the Lemma. Recall that a bounded operator is positive if, and only if it is self-adjoint and its spectrum is contained in $[0, +\infty)$. If $0 \leq b < c$, then $|(P - bI)v| \geq (c - b)|v|$. It follows that the operator $P - bI$ is invertible. Hence, $b \notin \sigma(P)$. Thus the spectrum $\sigma(P)$ of P is contained in $[c, +\infty)$. Now, it follows that $\sigma(P - cI) = (\sigma(P) - c) \subset (0, +\infty)$. Since P is self-adjoint, so is $P - cI$. We conclude that $P \geq cI$. So the conclusion of the lemma follows immediately.

Therefore, we can now deduce that each eigenvalue of the operator $d(\sigma \circ \tau_j)(0)$ is at least c_j, hence each eigenvalue of the operator $d(\sigma \circ \tau_j)(0) - c_j I$ is nonnegative. It follows that $d(\sigma \circ \tau_j)(0) \geq c_j I$, and that $c_j \to 1$. By Lemma 5.3, $\sigma \circ \tau_j$ converges to I uniformly on $r\mathbb{B}$ for every $0 < r < 1$.

Fix $0 < r < 1$. By Lemma 5.4, $\sigma \circ \tau_j(\mathbb{B}) \supset r\mathbb{B}$ for sufficiently large j. Hence $\sigma(\Omega) \supset r\mathbb{B}$. Since this is true for each $0 < r < 1$, we see that $\sigma(\Omega) = \mathbb{B}$.

Fix $a > 0$ and consider $Q_a = \{x \in \Omega : d_\Omega(x, q) < a\}$. Let $r = (1 + \tanh(a))/2$ and $t_j = \tanh(a/\tanh(b_j - a))$. If j is sufficiently large, then $b_j > a$ and $t_j(1 + 1/j)(1 - 1/j)^{-1} < r$. By Lemma 5.1, $Q_a \subset \{x \in \Omega_j : d_{\Omega_j}(x, q) < a/\tanh(b_j - a)\}$. This, together with $\sigma_j(\Omega_j) \subset (1 + 1/j)\mathbb{B}$, implies that $\sigma_j(Q_a) \subset t_j(1 + 1/j)\mathbb{B}$. It follows that

$$\tau_j(r\mathbb{B}) \supset \tau_j(t_j(1 + 1/j)(1 - 1/j)^{-1}\mathbb{B}) = \sigma_j^{-1}(t_j(1 + 1/j)\mathbb{B}) \supset Q_a.$$

For sufficiently large j, $\sigma \circ \tau_j$ is injective on $r\mathbb{B}$, hence σ is injective on $\tau_j(r\mathbb{B}) \supset Q_a$. Since σ is injective on Q_a for every $a > 0$, it must be injective on Ω. Therefore, σ is an injective holomorphic mapping from Ω onto $\mathbb{B}$. It follows that Ω is biholomorphic to $\mathbb{B}$. $\qquad \square$

6. Concluding remarks

Recall that the localization theorem is obtained from the existence of pluri-subharmonic peak functions. Since the pluri-subharmonic functions are extremely flexible as far as the extension properties are concerned, the whole argument of this article is valid for the domains in a separable Hilbert manifold. More precisely our main theorem extends to the following.

THEOREM 6.1. *Let Ω be a domain in a separable Hilbert manifold X. If Ω admits an automorphism orbit accumulating at a strongly pseudoconvex boundary point, then it is biholomorphic to the unit ball in a separable Hilbert space $\mathcal{H}$.*

Notice that this is the infinite dimensional version of the main theorem of [8].

The Wong-Rosay theorem in $\mathbb{C}^n$ has been generalized to several other domains that are not necessarily strongly pseudoconvex. Better known theorems include [1], [9], [12] and [11]. They characterized the Thullen domains and the polydiscs from Wong type conditions on existence of boundary accumulating orbits. However, the authors do not know at this time of writing as how to generalize these theorems to infinite dimensions. Thus we close this article posing the following question.

PROBLEM 6.2. *Formulate and prove the Wong-Rosay type characterization of the unit ball in the space c_0 of complex sequences converging to zero, or in the space ℓ^∞ of bounded complex sequences.*

References

[1] E. Bedford and S. Pinchuk, *Domains in $\mathbb{C}^{n+1}$ with noncompact automorphism group*, J. Geom. Anal. **1** (1992), 165–191.

[2] F. Berteloot, *Characterization of models in $\mathbb{C}^2$ by their automorphism groups*, Internat. J. Math. **5** (1994), 619–634.

[3] J. Byun, H. Gaussier and K. Kim, *Weak-type normal families of holomorphic mappings in Banach spaces and Characterization of the Hilbert ball by its automorphism group*, J. Geom. Anal. **12** (2002), 581–599.

[4] S. Dineen, *Complex analysis on infinite dimensional spaces*, Springer Monograph Ser., 1999.

[5] A. Efimov, *A generalization of the Wong-Rosay theorem for the unbounded case*, Sb. Math. **186** (1995), 967–976.

[6] S. Frankel, *Complex geometry with convex domains that cover varieties*, Acta Math. **163** (1989), 109–149.

[7] H. Gaussier, *Tautness and complete hyperbolicity of domains in* $\mathbb{C}^n$, Proc. Amer. Math. Soc. **127** (1999), 105–116.

[8] H. Gaussier, K. T. Kim and S. G. Krantz, *A note on the Wong-Rosay theorem in complex manifolds*, Complex Variables **47** (2002), 761–768.

[9] K. T. Kim, *Domains in* $\mathbb{C}^n$ *with a piecewise Levi flat boundary which possess a noncompact automorphism group*, Math. Ann. **292** (1992), 575–586.

[10] K. T. Kim and S. G. Krantz, *Characterization of the Hilbert ball by its automorphism group*, Trans. Amer. Math. **354** (2002), 2797–2838.

[11] K. T. Kim, S. G. Krantz and A. F. Spiro, *Analytic polyhedra in* $\mathbb{C}^2$ *with a noncompact automorphism group*, preprint.

[12] K. T. Kim and A. Pagano, *Normal analytic polyhedra in* $\mathbb{C}^2$ *with a non-compact automorphism group*, J. Geom. Anal. **11** (2001), 283–293.

[13] D. Ma and S. J. Kan, *On rigidity of Grauert tubes over locally symmetric spaces*, J. Reine Angew. Math. **524** (2000), 205–225.

[14] J. Mujica, *Complex analysis in Banach spaces*, North-Holland, 1986.

[15] S. Pinčuk, *Holomorphic inequivalence of certain classes of domains in* $\mathbb{C}^n$, Mat. Sb. **111 (153)** (1981), No. 1, 67–94.

[16] J. P. Rosay, *Une caracterization de la boule parmi les domaines de* $\mathbb{C}^n$ *par son groupe d'automorphismes*, Ann. Inst. Fourier (Grenoble) **XXIX** (1979), 91–97.

[17] N. Sibony, *A class of hyperbolic manifolds*, Recent developments in several complex variables, Ann. Math. Studies **100** (1981), 357–372.

[18] B. Wong, *Characterization of the unit ball in* $\mathbb{C}^n$ *by its automorphism group*, Invent. Math. **41** (1977), 253–257.

Kang-Tae Kim
Department of Mathematics
Pohang University of Science and Technology
Pohang 790-784, Korea
E-mail: kimkt@postech.edu

Daowei Ma
Department of Mathematics
Wichita State University
Wichita, KS 67260-0033 U.S.A.
E-mail: dma@math.twsu.edu