J. Korean Math. Soc. **40** (2003), No. 3, pp. 391–408

ROTATIONALLY INVARIANT COMPLEX MANIFOLDS

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Abstract. In this paper we discuss complex manifolds of dimension $n \geq 2$ that admit effective actions of either U_n or SU_n by biholomorphic transformations.

0. Introduction

Let M be a connected complex manifold of dimension $n \geq 2$. It is natural to attempt to describe manifolds M that admit actions by biholomorphic transformations of the product of unitary groups $U_{n_1} \times \cdots \times U_{n_k}$, where $n_1 + \cdots + n_k = n$ and $n_j \geq 1$ for $j = 1, \dots, k$. The special case of $k = n$ and $n_j = 1$ for all j, corresponds to an action of the torus $\mathbb{T}^n$ on M. Under certain conditions such an action is known to be linearizable, which means that M is biholomorphically equivalent to a Reinhardt domain in $\mathbb{C}^n$ [1].

The other extreme is $k = 1$, in which case M admits an action of the group U_n by biholomorphic transformations. It is natural to think that elements of U_n act on M by a kind of "rotations", and we therefore term such manifolds *rotationally invariant.* We also include in this class manifolds that admit actions of the special unitary group SU_n.

If G is a Lie group and $\text{Aut}\,(M)$ is the group of biholomorphic automorphisms of M equipped with the compact-open topology, an action of G on M by biholomorphic transformations is a real-analytic map

$$\Phi : G \times M \to M,$$

such that for every $g \in G$ we have $\Phi(g, \cdot) \in \text{Aut}\,(M)$, and the induced mapping $\Psi : G \to \text{Aut}\,(M)$, $g \mapsto \Phi(g, \cdot)$ is a homomorphism. If the kernel of Ψ is trivial, the action is called effective. For an effective action, $\text{Aut}\,(M)$ contains a subgroup isomorphic to G (note that Ψ is continuous).

Received September 20, 2002.
2000 Mathematics Subject Classification: 32Q57, 32M17.
Key words and phrases: complex manifolds, group actions.

In [8] we classified all n-dimensional complex manifolds that admit effective actions of U_n. One motivation for our study there was the following characterization of the complex space $\mathbb{C}^n$ obtained as a result of the classification. Let M be a connected complex manifold of dimension n and assume that the groups $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\mathbb{C}^n)$ are isomorphic as topological groups; then M is biholomorphically equivalent to $\mathbb{C}^n$.

Another motivation arises from our earlier work in [7] and [3]. In [7] we, in particular, classified all Kobayashi-hyperbolic manifolds for which $\dim \mathrm{Aut}\,(M) > n^2$. The automorphism group dimension n^2 appears to be critical for obtaining such classifications: it looks entirely impossible to produce a classification for dimensions less that n^2 and quite hard for dimension equal to n^2. In fact, we have been able to produce a classification for $\dim \mathrm{Aut}\,(M) = n^2$ only in the case of hyperbolic Reinhardt domains [3]. Since $\dim U_n = n^2$, if M admits an effective action of U_n, we have $\dim \mathrm{Aut}\,(M) \geq n^2$, and hence a classification of such manifolds might provide interesting examples of manifolds with critical automorphism group dimension n^2. Note that for the purposes of this classification we no longer assumed hyperbolicity of M in [8]. We reproduce our classification from [8] in Section 1 below. The classification includes, in particular, quotients of Hopf manifolds with transitive U_n-actions (Theorem 1.7). There are many effective transitive U_n-actions on such quotients, and we included in Section 1 our previously unpublished description of all such actions (see Propositions 1.3 and 1.5).

Actions of the group SU_n on real manifolds have been studied extensively. One motivation for such studies is the importance of SU_n-actions in physics, especially in the case of small values of n (see, e.g., [11]). SU_n-actions have also been of interest to mathematicians, and various classification results for such actions have been obtained (see, e.g., [5], [6], [12]). There is, however, no complete classification for the case of SU_n-actions by biholomorphic transformations on complex manifolds. The best result towards such a classification was obtained in [16], where all real compact connected orientable manifolds of dimension $2n$ admitting actions of SU_n were determined for $n \geq 5$.

In collaboration with N. Kruzhilin, we have recently obtained a classification of all n-dimensional complex manifolds that admit effective actions of SU_n by biholomorphic transformations. In this paper we only describe all possible dimensions of orbits of such actions and make initial steps towards obtaining the complete classification (see Section 2). The final classification will appear elsewhere. In Section 3 we also discuss examples of manifolds that admit SU_n-actions, but not U_n-actions.

Before proceeding, we remark that one can attempt to obtain classifications analogous to ours in more general settings, for example, for groups U_n and SU_n acting on k-dimensional complex manifolds with $k \neq n$. In fact, it can be shown that effective actions of either U_n or SU_n do not exist on manifolds of dimension $k < n$. Thus, our classifications are obtained for the smallest possible dimension of manifolds for which there are effective actions. Another generalization is possible if one considers not necessarily effective actions, e.g., actions with non-trivial discrete kernel. In Section 3 we give some examples of manifolds admitting such SU_n-actions. Generalizing our proofs to obtain a complete classification in this situation is not straightforward.

1. U_n-actions

In this section we reproduce our classification from [8]. We start with a description of all possible dimensions of orbits.

PROPOSITION 1.1. *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of U_n by biholomorphic transformations. Let $p \in M$ and let $O(p)$ be the U_n-orbit of p. Then $O(p)$ is either*

(i) *a single point, or*
(ii) *the whole of M, or*
(iii) *a real compact hypersurface in M, or*
(iv) *a complex compact hypersurface in M.*

REMARK 1.2. For actions with fixed points (Case (i) of Proposition 1.1), the complete classification was obtained in [9] (see Folgerung 1.10 there). Namely, if M admits an effective action of U_n with a fixed point, then M is biholomorphically equivalent to either

(i) the unit ball $B^n \subset \mathbb{C}^n$, or
(ii) $\mathbb{C}^n$, or
(iii) $\mathbb{CP}^n$.

The biholomorphic equivalence f can be chosen to be an isomorphism of U_n-spaces, more precisely,

$$f(gq) = \gamma(g)f(q),$$

where either $\gamma(g) = g$ or $\gamma(g) = \overline{g}$ for all $g \in U_n$ and $q \in M$ (here B^n, $\mathbb{C}^n$ and $\mathbb{CP}^n$ are considered with the standard actions of U_n).

Suppose now that M is homogeneous under the U_n-action (Case (ii) of Proposition 1.1). We start with an example of such a manifold.

Let $d \in \mathbb{C} \setminus \{0\}$, $|d| \neq 1$, let M_d^n be the Hopf manifold constructed by identifying $z \in \mathbb{C}^n \setminus \{0\}$ with $d \cdot z$, and let $[z]$ be the equivalence class of z. Choose a complex number λ such that $e^{\frac{2\pi(\lambda - i)}{nK}} = d$ for some $K \in \mathbb{Z} \setminus \{0\}$. We define an action of U_n on M_d^n as follows. Let $A \in U_n$. We can represent A in the form $A = e^{it} \cdot B$, where $t \in \mathbb{R}$ and $B \in SU_n$. Then we set

$$A[z] := [e^{\lambda t} \cdot Bz]. \tag{1.1}$$

Of course, we must verify that this action is well-defined. Indeed, the same element $A \in U_n$ can be also represented in the form $A = e^{i(t+\frac{2\pi k}{n}+2\pi l)} \cdot (e^{-\frac{2\pi i k}{n}} B)$, $0 \leq k \leq n-1$, $l \in \mathbb{Z}$. Then formula (1.1) yields

$$A[z] = [e^{\lambda(t+\frac{2\pi k}{n}+2\pi l)} \cdot e^{-\frac{2\pi i k}{n}} Bz] = [d^{kK+nKl} e^{\lambda t} \cdot Bz] = [e^{\lambda t} \cdot Bz].$$

It is also clear that (1.1) does not depend on the choice of representative in the class $[z]$.

The action in question is obviously transitive. It is also effective. For let $e^{it} \cdot B[z] = [z]$ for some $t \in \mathbb{R}$, $B \in SU_n$, and all $z \in \mathbb{C}^n \setminus \{0\}$. Then, for some $k \in \mathbb{Z}$, $B = e^{\frac{2\pi i k}{n}} \cdot \mathrm{id}$, and some $s \in \mathbb{Z}$ the following holds

$$e^{\lambda t} \cdot e^{\frac{2\pi i k}{n}} = d^s.$$

Using the definition of λ we obtain

$$t = \frac{2\pi s}{nK}, \; e^{\frac{2\pi i k}{n}} = e^{-\frac{2\pi i s}{nK}}.$$

Hence $e^{it} \cdot B = \mathrm{id}$, and thus the action is effective.

Another example is provided by quotients of Hopf manifolds $M_d^n/\mathbb{Z}_m$ obtained from M_d^n by identifying $[z]$ and $[e^{\frac{2\pi i}{m}} z]$, $m \in \mathbb{N}$. Let $\{[z]\} \in M_d^n/\mathbb{Z}_m$ be the equivalence class of $[z]$. We define an action of U_n on $M_d^n/\mathbb{Z}_m$ by the formula $g\{[z]\} := \{g[z]\}$ for $g \in U_n$. This action is clearly transitive; it is also effective if, e.g., $(n, m) = 1$ and $(K, m) = 1$.

One can consider more general actions by choosing λ such that $e^{\frac{2\pi(\lambda - i)}{n}} = d^K$, but not all such actions are effective. It is in fact possible to give a complete description of all effective transitive U_n-actions on quotients of Hopf manifolds as shown in the following two propositions.

PROPOSITION 1.3. *If $M_d^n/\mathbb{Z}_m$ admits an effective action of U_n by biholomorphic transformations, then $(n, m) = 1$. Further, every effective* *transitive action of U_n on $M_d^n/\mathbb{Z}_m$ by biholomorphic transformations has either the form*

$$A\{[z]\} = \left\{\left[e^{i\left(1+(r+\frac{\rho}{m})n\right)t} d^{\frac{nqt}{2\pi}} CBC^{-1}z\right]\right\}, \tag{1.2}$$

or the form

$$A\{[z]\} = \left\{\left[e^{i\left(-1+(r+\frac{\rho}{m})n\right)t} d^{\frac{nqt}{2\pi}} C\overline{B}C^{-1}z\right]\right\}, \tag{1.3}$$

where $A \in U_n$ is represented as $A = e^{it} \cdot B$ with $t \in \mathbb{R}$ and $B \in SU_n$, $A\{[z]\}$ denotes the action of A on $\{[z]\} \in M_d^n/\mathbb{Z}_m$, $r, \rho, q \in \mathbb{Z}$, $q \neq 0$, $C \in GL_n(\mathbb{C})$.

Proof. As we noted in the proof of Theorem 4.5 in [8], $\mathrm{Aut}(M_d^n/\mathbb{Z}_m)$ is naturally isomorphic to $Q_{d,m}^n := (GL_n(\mathbb{C})/H)/\mathbb{Z}_m$, where $H := \{d^k \cdot \mathrm{id},\ k \in \mathbb{Z}\}$ and $\mathbb{Z}_m$ is identified with with the subgroup of $GL_n(\mathbb{C})/H$ that consists of elements of the form $e^{\frac{2\pi i l}{m}}H$, $l \in \mathbb{Z}$.

We will now find maximal compact subgroups of $Q_{d,m}^n$. First, consider the subgroup $G \subset GL_n(\mathbb{C})$, $G := \{d^t \cdot \mathrm{id}, t \in \mathbb{R}\}$. Then the subgroup $K \subset GL_n(\mathbb{C})/H$, $K := G/H$, is isomorphic to S^1. We also denote by K the natural embedding of K in $Q_{d,m}^n$: $g \mapsto g\mathbb{Z}_m$, $g \in K$. Further, consider the natural embedding of U_n in $GL_n(\mathbb{C})/H$: $g \mapsto gH$, $g \in U_n$. Then $(U_n/\mathbb{Z}_m) \cdot K$ is a maximal compact subgroup of $Q_{d,m}^n$. We note that $(U_n/\mathbb{Z}_m) \cap K = \{e\}$. Any other maximal compact subgroup of $Q_{d,m}^n$ has the form $s_0(U_n/\mathbb{Z}_m)s_0^{-1} \cdot K$, where $s_0 \in Q_{d,m}^n$.

Suppose now that we are given an effective action of U_n on $M_d^n/\mathbb{Z}_m$ by biholomorphic transformations. Clearly, the action induces an embedding $\tau : U_n \to Q_{d,m}^n$. Since $\tau(U_n)$ is a compact subgroup of $Q_{d,m}^n$, we have $\tau(U_n) \subset s_0(U_n/\mathbb{Z}_m)s_0^{-1} \cdot K$ for some $s_0 \in Q_{d,m}^n$. Consider the restriction of τ to SU_n. Since there does not exist a nontrivial homomorphism of SU_n to S^1, we have $\tau(SU_n) \subset s_0(U_n/\mathbb{Z}_m)s_0^{-1}$. Since the action is effective, $\tau(SU_n)$ is isomorphic to SU_n. Clearly, $s_0(U_n/\mathbb{Z}_m)s_0^{-1}$ contains a subgroup isomorphic to SU_n if and only if $(n, m) = 1$, in which case $\tau(SU_n) = s_0 SU_n s_0^{-1}$, where in the right-hand side SU_n is embedded in $Q_{d,m}^n$ in the natural way. Hence there exists an automorphism γ of SU_n and $D \in GL_n(\mathbb{C})$ such that $B\{[z]\} = \{[D\gamma(B)D^{-1}z]\}$ for all $z \in \mathbb{C}^n \setminus \{0\}$ and $B \in SU_n$.

Every automorphism of SU_n has either the form

$$B \mapsto h_0 B h_0^{-1}, \tag{1.4}$$

or the form

$$B \mapsto h_0 \overline{B} h_0^{-1}, \tag{1.5}$$

for some $h_0 \in SU_n$ (see, e.g., [17]). If γ has the form (1.4), then there exists $C \in GL_n(\mathbb{C})$ such that $B\{[z]\} = \{[CBC^{-1}z]\}$ for all $z \in \mathbb{C}^n \setminus \{0\}$ and $B \in SU_n$. If γ has the form (1.5), then there exists $C \in GL_n(\mathbb{C})$ such that $B\{[z]\} = \{[C\overline{B}C^{-1}z]\}$ for all $z \in \mathbb{C}^n \setminus \{0\}$ and $B \in SU_n$.

Consider first the case corresponding to (1.4). Consider the restriction of τ to the center Z of U_n. Clearly, there exist homomorphisms $\tau_1 : Z \to Z/\mathbb{Z}_m$ and $\tau_2 : Z \to K$ such that $\tau(g) = \tau_1(g) \cdot \tau_2(g)$ for all $g \in Z$. Obviously, there exists $\sigma \in \mathbb{R}$ such that $\tau_1(e^{it} \cdot \mathrm{id}) = ((e^{i\sigma t} \cdot \mathrm{id})H)\mathbb{Z}_m$. Further, there exists $\mu \in \mathbb{R}$ such that $\tau_2(e^{it} \cdot \mathrm{id}) = (d^{\mu t}H)\mathbb{Z}_m$. Since $\tau_2(e^{it} \cdot \mathrm{id}) = \tau_2(e^{i(t+2\pi)} \cdot \mathrm{id})$, μ has to be of the form $\mu = \frac{R}{2\pi}$ for some $R \in \mathbb{Z}$. Further, since τ_2 is trivial on the center of SU_n, we get $R = nq$, $q \in \mathbb{Z}$. Since the action is transitive, $q \neq 0$.

Let $A \in U_n$. Represent it in the form $A = e^{it} \cdot B$, where $t \in \mathbb{R}$, $B \in SU_n$. Then for every $z \in \mathbb{C}^n \setminus \{0\}$ we have

$$\begin{aligned} A\{[z]\} &= (e^{it}B)\{[z]\} = e^{it}(B\{[z]\}) \\ &= e^{it}\{[CBC^{-1}z]\} = \left\{\left[e^{i\sigma t} d^{\frac{nqt}{2\pi}} CBV^{-1}z\right]\right\}. \end{aligned} \tag{1.6}$$

Representing A as $A = e^{i(t+\frac{2\pi k}{n}+2\pi l)} \cdot (e^{-\frac{2\pi ik}{n}} \cdot B)$ with $k, l \in \mathbb{Z}$, we obtain from (1.6) that σ has to be of the form $\sigma = 1+(r+\frac{\rho}{m})n$ for some $r, \rho \in \mathbb{Z}$. This gives (1.2).

Similarly, the case corresponding to (1.5) leads to (1.3).

The proposition is proved. □

Remark 1.4. For $n = 2$ every action of the form (1.3) is in fact an action of the form (1.2).

We will now find necessary and sufficient conditions for actions of each of the form (1.2) and (1.3) to be effective. The answer is given by the following proposition.

Proposition 1.5. *We have:*

(i) *Action* (1.2) *is effective if and only if there do not exist* $L, l \in \mathbb{Z}$ *such that the following conditions are satisfied*:

(a) $-\frac{L}{q}(1 + (r + \frac{\rho}{m})n) + \frac{nl}{m} \in \mathbb{Z}$;

(b) $-\frac{L}{q}(r + \frac{\rho}{m}) + \frac{l}{m} \notin \mathbb{Z}$.

(ii) *Action* (1.3) *is effective if and only if there do not exist* $L, l \in \mathbb{Z}$ *such that* (b) *and the following condition are satisfied*:

(c) $-\frac{L}{q}(-1 + (r + \frac{\rho}{m})n) + \frac{nl}{m} \in \mathbb{Z}$.

Proof. We start from considering action (1.2). It is not effective if and only if for some nontrivial $A = e^{it} \cdot B$, $t \in \mathbb{R}$, $B \in SU_n$, one has $A\{[z]\} = \{[z]\}$ for all $z \in \mathbb{C}^n \setminus \{0\}$. It is easy to show that this identity is equivalent to the existence of $L, l \in \mathbb{Z}$ such that

$$e^{i\left(1+(r+\frac{\rho}{m})n\right)t} d^{\frac{nqt}{2\pi}} Bw = d^L e^{\frac{2\pi i l}{m}} w, \tag{1.7}$$

for all $w \in \mathbb{C}^n \setminus \{0\}$. Identity (1.7) implies that B is a scalar matrix $B = e^{\frac{2\pi i\nu}{n}} \cdot \mathrm{id}$, $\nu \in \mathbb{Z}$, and therefore gives:

$$e^{i\left((1+(r+\frac{\rho}{m})n)t+\frac{2\pi\nu}{n}-\frac{2\pi l}{m}\right)} d^{\frac{nqt}{2\pi}-L} = 1. \tag{1.8}$$

Identity (1.8) is equivalent to

$$\begin{aligned} t &= \tfrac{2\pi L}{nq}, \\ \nu &= nS - \tfrac{L}{q}(1 + (r + \tfrac{\rho}{m})n) + \tfrac{nl}{m}, \end{aligned} \tag{1.9}$$

for some $S \in \mathbb{Z}$. Then A is nontrivial if and only if $-\frac{L}{q}(r + \frac{\rho}{m}) + \frac{l}{m} \notin \mathbb{Z}$ which is condition (b). Condition (a) arises from the second identity in (1.9).

The proof in the case of action (1.3) is similar to the above.

The proposition is proved. □

The formulas from Proposition 1.3 and the conditions from Proposition 1.5 simplify for the case of Hopf manifolds (i.e., when $m = 1$).

COROLLARY 1.6. *Every effective transitive action of U_n on M_d^n, has either the form*

$$A[z] = \left[e^{i(1+rn)t} d^{\frac{nqt}{2\pi}} CBC^{-1} z\right], \tag{1.10}$$

or the form

$$A[z] = \left[e^{i(-1+rn)t} d^{\frac{nqt}{2\pi}} C\overline{B}C^{-1} z\right], \tag{1.11}$$

where $A \in U_n$ is represented as $A = e^{it} \cdot B$ with $t \in \mathbb{R}$ and $B \in SU_n$, $A[z]$ denotes the action of A on $[z] \in M_d^n$, $r, \rho, q \in \mathbb{Z}$, $q \neq 0$, $C \in GL_n(\mathbb{C})$. Action (1.10) is effective if and only if there does not exist $L \in \mathbb{Z}$ such that q divides $L(1+rn)$, but does not divide Lr. Action (1.11) is effective if and only if there does not exist $L \in \mathbb{Z}$ such that q divides $L(-1+rn)$, but does not divide Lr.

As we have seen above, there is no canonical transitive effective U_n-action on a quotient of a Hopf manifold. This explains why the following theorem deals with SU_n-equivariance rather than U_n-equivariance of the equivalence mapping.

THEOREM 1.7. *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective transitive action of U_n by biholomorphic transformations. Then M is biholomorphically equivalent to a Hopf manifold $M_d^n/\mathbb{Z}_m$, where $m \in \mathbb{N}$ and $(n, m) = 1$. The equivalence $f : M \to M_d^n/\mathbb{Z}_m$ can be chosen to satisfy either the relation*

$$f(gq) = gf(q), \tag{1.12}$$

or, for $n \geq 3$, the relation

$$f(gq) = \overline{g}f(q), \tag{1.13}$$

for all $g \in SU_n$ and $q \in M$ (here $M_d^n/\mathbb{Z}_m$ is considered with the standard action of SU_n).

We now turn to the case when all orbits of the U_n-action on M are real hypersurfaces (Case (iii) of Proposition 1.1). All such manifolds are classified in the following

THEOREM 1.8. *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of U_n by biholomorphic transformations. Suppose that all orbits of this action are real hypersurfaces. Then there exists $k \in \mathbb{Z}$ such that, for $m = |nk + 1|$, M is biholomorphically equivalent to either*

(i) *$S_{r,R}^n/\mathbb{Z}_m$, where $S_{r,R}^n := \{z \in \mathbb{C}^n : r < |z| < R\}$, $0 \leq r < R \leq \infty$, is a spherical layer, or*
(ii) *$M_d^n/\mathbb{Z}_m$.*

The biholomorphic equivalence f can be chosen to satisfy either the relation

$$f(gq) = \phi_{n,m}^{-1}(g)f(q), \tag{1.14}$$

or the relation

$$f(gq) = \phi_{n,m}^{-1}(\overline{g})f(q), \tag{1.15}$$

for all $g \in U_n$ and $q \in M$, where $\phi_{n,m}$ is the isomorphism

$$\phi_{n,m} : U_n/\mathbb{Z}_m \to U_n, \qquad \phi_{n,m}(A\mathbb{Z}_m) = (\det A)^k \cdot A, \qquad A \in U_n,$$

and $S_{r,R}^n/\mathbb{Z}_m$ and $M_d^n/\mathbb{Z}_m$ are equipped with the standard actions of $U_n/\mathbb{Z}_m$.

We will now also include complex hypersurface orbits (Case (iv) of Proposition 1.1). First we introduce some notation. Let $\widehat{B^n}$, $\widehat{\mathbb{C}^n}$ and $\widehat{\mathbb{CP}^n}$ denote the blow-ups at the origin of B^n, $\mathbb{C}^n$ and $\mathbb{CP}^n$ respectively. Let $\widetilde{S_{r,\infty}^n}$ be the union of the spherical layer with infinite outer radius

$S^n_{r,\infty}$, $r \geq 0$, and the hyperplane at infinity in $\mathbb{CP}^n$. Further, for $m \in \mathbb{N}$ denote by $\widehat{B^n}/\mathbb{Z}_m$, $\widehat{\mathbb{C}^n}/\mathbb{Z}_m$, $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$ and $\widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$ the quotients of B^n, $\mathbb{C}^n$, $\mathbb{CP}^n$ and $\widetilde{S^n_{r,\infty}}$ by $\mathbb{Z}_m$ respectively. Then we have the following

THEOREM 1.9. *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of U_n by biholomorphic transformations. Suppose that each orbit of this action is either a real or complex hypersurface and at least one orbit is a complex hypersurface. Then there exists $k \in \mathbb{Z}$ such that, for $m = |nk+1|$, M is biholomorphically equivalent to either*

(i) $\widehat{B^n}/\mathbb{Z}_m$, *or*
(ii) $\widehat{\mathbb{C}^n}/\mathbb{Z}_m$, *or*
(iii) $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$, *or*
(iv) $\widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$, $0 \leq r < \infty$.

The biholomorphic equivalence f can be chosen to satisfy either (1.14) *or* (1.15) *for all $g \in U_n$ and $q \in M$.*

Thus, Remark 1.2 and Theorems 1.7, 1.8 and 1.9 give a complete classification of all complex manifolds of dimension $n \geq 2$ that admit effective actions of U_n by biholomorphic transformations.

2. Dimensions of orbits of SU_n-actions

The goal of this section is to determine all possible dimensions of orbits of an effective action of SU_n on M and to obtain some preliminary results towards a complete classification. We start with the following proposition which is analogous to Proposition 1.1.

PROPOSITION 2.1. *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by biholomorphic transformations. Let $p \in M$ and let $O(p)$ be the SU_n-orbit of p. Then $O(p)$ is either*

(i) *a single point, or*
(ii) *the whole of M, or*
(iii) *a compact real hypersurface in M, or*
(iv) *a compact complex hypersurface in M.*

Proof. For $p \in M$ let I_p be the isotropy subgroup of SU_n at p, i.e., $I_p := \{g \in SU_n : gp = p\}$. We denote by Ψ the continuous homomorphism of SU_n into $\mathrm{Aut}(M)$ induced by the action of SU_n on M. Let

$L_p := \{d_p(\Psi(g)) : g \in I_p\}$ be the linear isotropy subgroup, where $d_p f$ is the differential of a map f at p. Clearly, L_p is a compact subgroup of $GL(T_p(M), \mathbb{C})$. Since the action of SU_n is effective, L_p is isomorphic to I_p. The isomorphism is given by the map

$$\alpha : I_p \to L_p, \qquad \alpha(g) := d_p(\Psi(g)). \tag{2.1}$$

Let $V \subset T_p(M)$ be the tangent space to $O(p)$ at p. Clearly, V is L_p-invariant. We assume now that $O(p) \neq M$ (and therefore $V \neq T_p(M)$) and consider the following three cases.

I. $d := \dim_{\mathbb{C}}(V + iV) < n$.

Since L_p is compact, one can choose coordinates in $T_p(M)$ such that $L_p \subset U_n$. Further, the action of L_p on $T_p(M)$ is completely reducible and the subspace $V + iV$ is invariant under this action. Hence L_p can in fact be embedded in $U_d \times U_{n-d} \subset GL(T_p(M), \mathbb{C})$. Since $\dim O(p) \leq 2d$, it follows that

$$n^2 - 1 \leq d^2 + (n-d)^2 + \dim O(p) \leq d^2 + (n-d)^2 + 2d,$$

and therefore either $d = 0$ or $d = n - 1$. If $d = 0$, we obtain (i). If $d = n - 1$, then either $\dim O(p) = 2n - 2$ or $\dim O(p) = 2n - 3$. For $\dim O(p) = 2n - 2$ we have $iV = V$, which yields (iv).

Suppose now that $\dim O(p) = 2n-3$. In this case $\dim I_p = n^2-2n+2$. Since L_p can be embedded in $U_1 \times U_{n-1}$, it follows that L_p — and hence I_p — are isomorphic to $U_1 \times U_{n-1}$. It is now clear from Lemma 2.1 of [7] that I_p is conjugate in U_n to $U_1 \times U_{n-1}$ (realized in the block-diagonal form in the obvious way). But this is impossible since then I_p is not contained in SU_n. Hence, in fact, $\dim O(p) \neq 2n - 3$.

II. $T_p(M) = V + iV$ and $r := \dim_{\mathbb{C}}(V \cap iV) > 0$.

As above, L_p can be embedded in $U_r \times U_{n-r}$ (clearly, we have $r < n$). Moreover, $V \cap iV \neq V$ and since L_p preserves V, it follows that $\dim L_p < r^2 + (n-r)^2$. We have $\dim O(p) \leq 2n - 1$, and therefore

$$n^2 - 1 < r^2 + (n-r)^2 + \dim O(p) \leq r^2 + (n-r)^2 + 2n - 1,$$

which shows that either $\dim O(p) = 2n - 1$ or $\dim O(p) = 2n - 2$.

The case $\dim O(p) = 2n - 1$ yields (iii).

Assume now that $\dim O(p) = 2n - 2$. Then $\dim I_p = (n-1)^2$ and by Lemma 2.1 of [8], I_p^c, the connected component of the identity in I_p, is conjugate in SU_n to the group H^n of all matrices of the form

$$\begin{pmatrix} 1/\det B & 0 \\ 0 & B \end{pmatrix}, \tag{2.2}$$

where $B \in U_{n-1}$. Therefore, I_p contains the center of SU_n. Let $g \neq \text{id}$ be an element of this center. Then g acts trivially on $O(p)$, i.e., $gq = q$ for all $q \in O(p)$. Therefore, $\alpha(g)(v) = v$ for all $v \in V$, where α is the isomorphism defined in (2.1). Since $T_p(M) = V + iV$ and $\alpha(g)$ is complex-linear on $T_p(M)$, it follows that $\alpha(g) = \text{id}$ and $g = \text{id}$, which is a contradiction. Hence $\dim O(p) \neq 2n - 2$.

III. $T_p(M) = V \oplus iV$.

In this case $\dim V = n$ and L_p can be embedded in the real orthogonal group $O_n(\mathbb{R})$, therefore

$$\dim L_p + \dim O(p) \leq \frac{n(n-1)}{2} + n.$$

Thus, for $n \geq 3$ we have $\dim L_p + \dim O(p) < n^2 - 1$ which is a contradiction. Assume now that $n = 2$. In this case $\dim I_p = 1 = (n-1)^2$, and we arrive at a contradiction by arguing as in **II** above.

The proof of the proposition is complete. □

The case of an action with a fixed point (Case (i) in Proposition 2.1) easily follows from the results in [4] and [2] as shown in the following proposition.

PROPOSITION 2.2. *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by biholomorphic transformations that has a fixed point in M. Then M is biholomorphically equivalent to either*

(i) *the unit ball $B^n \subset \mathbb{C}^n$, or*
(ii) *$\mathbb{C}^n$, or*
(iii) *$\mathbb{CP}^n$.*

The biholomorphic equivalence f can be chosen to satisfy either relation (1.12) *or, if $n \geq 3$, relation* (1.13) *for all $g \in SU_n$ and $q \in M$ (here B^n, $\mathbb{C}^n$ and $\mathbb{CP}^n$ are considered with the standard action of SU_n).*

Proof. Let p be a fixed point of the action of SU_n on M. Then $I_p = SU_n$. Let L_p be as above the linear isotropy subgroup. Clearly, L_p is also isomorphic to SU_n. Since L_p is a compact subgroup of $GL(T_p(M), \mathbb{C})$, one can find coordinates in $T_p(M)$ such that $L_p \subset U_n$. In these coordinates $L_p = SU_n$ (note that SU_n can be embedded in U_n in the unique way). The group SU_n acts transitively on the unit sphere in $T_p(M)$.

Assume first that M is non-compact. Then by [4] the manifold M is biholomorphically equivalent to either B^n or $\mathbb{C}^n$, and a biholomorphism F may be chosen to satisfy $F(gq) = \gamma(g)F(q)$ for all $g \in SU_n$ and $q \in M$, and some automorphism γ of SU_n, where the action of SU_n on $\mathbb{C}^n$ in the right-hand side is standard. Every automorphism of SU_n has either the form (1.4) or the form (1.5) with $h_0 \in SU_n$. Thus, setting $f = \hat{h}_0^{-1} \circ F$, where $\hat{h}_0$ is the automorphism of B^n or $\mathbb{C}^n$ induced by h_0, we obtain either (1.12) or (1.13), respectively.

Assume now that M is compact. Then by [2] M is biholomorphically equivalent to $\mathbb{CP}^n$. We will now show that a biholomorphism between M and $\mathbb{CP}^n$ can be chosen to satisfy (1.12) or (1.13).

The action of SU_n on M induces an embedding $\tau : SU_n \to \text{Aut}(\mathbb{CP}^n)$, and $\tau(SU_n)$ has a fixed point in $\mathbb{CP}^n$. Therefore, $\tau(SU_n)$ is conjugate in $\text{Aut}(\mathbb{CP}^n)$ to SU_n embedded in $\text{Aut}(\mathbb{CP}^n)$ in the standard way. Hence there exists an automorphism γ of SU_n such that for some $s \in \text{Aut}(\mathbb{CP}^n)$ we have $(s \circ F)(gq) = \gamma(g)(s \circ F)(q)$ for all $g \in SU_n$ and $q \in M$, where the action of SU_n on $\mathbb{CP}^n$ in the right-hand side is standard. We again use that γ has an explicit expression as in (1.4) or (1.5) and setting $f = \hat{h}_0^{-1} \circ s \circ F$ obtain either (1.12) or (1.13), respectively.

The proof is complete. □

We will now show that Case (ii) in Proposition 2.1 in fact does not realize. First we need the following

LEMMA 2.3. *Let G be a connected closed subgroup of SU_n of dimension $n^2 - 2n - 1$, $n \geq 3$. Then either*
(i) *$n = 3$ and G is conjugate in SU_3 to $(U_1 \times U_1 \times U_1) \cap SU_3$ embedded in SU_3 in the standard way, or*
(ii) *$n = 4$ and G is conjugate in SU_4 to $(U_2 \times U_2) \cap SU_4$ embedded in SU_4 in the standard way.*

Proof. Since G is compact, it is completely reducible, i.e., $\mathbb{C}^n$ decomposes into a sum of G-invariant pairwise orthogonal complex subspaces, $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_m$, such that the restriction G_j of G to every V_j is irreducible. Let $n_j := \dim_{\mathbb{C}} V_j$ (hence $n_1 + \cdots + n_m = n$) and let U_{n_j} be the unitary transformation group of V_j. Clearly, $G_j \subset U_{n_j}$, and therefore $\dim G \leq n_1^2 + \cdots + n_m^2$. On the other hand $\dim G = n^2 - 2n - 1$, which shows that $m \leq 2$ for $n \neq 3$. If $n = 3$, then it is also possible that $m = 3$, which means that G is conjugate in SU_3 to $(U_1 \times U_1 \times U_1) \cap SU_3$ embedded in U_3 in the standard way.

Now let $m = 2$. Then either $n = 4$ and G is conjugate in SU_4 to $(U_2 \times U_2) \cap SU_4$ embedded in SU_4 in the standard way, or G is conjugate in SU_n to a subgroup $\hat{G}$ of the group H^n defined in (2.2). The group H^n has dimension $(n-1)^2$ and is isomorphic to U_{n-1} in the obvious way. Hence $\hat{G}$ is isomorphic to a subgroup of U_{n-1} of codimension 2. It was shown in Lemma 2.1 in [7] that U_{n-1} does not have subgroups of codimension 2 unless $n = 3$, in which case $\hat{G}$ is conjugate to the group $(U_1 \times U_1 \times U_1) \cap SU_3$. But this is impossible since for this group $m = 3$.

Let $m = 1$. We proceed as in the proof of Lemma 2.1 in [8]. Let $\mathfrak{g} \subset \mathfrak{su}_n \subset \mathfrak{sl}_n$ be the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} + i\mathfrak{g} \subset \mathfrak{sl}_n$ its complexification. Then $\mathfrak{g}^{\mathbb{C}}$ acts irreducibly on $\mathbb{C}^n$ and by a theorem of É. Cartan is semisimple.

Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be the decomposition of $\mathfrak{g}^{\mathbb{C}}$ into the direct sum of simple ideals. Then the irreducible n-dimensional representation of $\mathfrak{g}^{\mathbb{C}}$ given by the embedding of $\mathfrak{g}^{\mathbb{C}}$ in $\mathfrak{gl}_n$ is the tensor product of some irreducible faithful representations of the $\mathfrak{g}_j$. Let n_j be the dimension of the corresponding representation of $\mathfrak{g}_j$, $j = 1, \ldots, k$. Then $n_j \geq 2$, $\dim_{\mathbb{C}} \mathfrak{g}_j \leq n_j^2 - 1$, and $n = n_1 \cdot \cdots \cdot n_k$.

Since $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} = n^2 - 2n - 1$, it follows from the claim in the proof of Lemma 2.1 in [8] that $k = 1$, i.e., $\mathfrak{g}^{\mathbb{C}}$ is simple. The minimal dimensions of irreducible faithful representations of complex simple Lie algebras are well-known. It then follows that a simple complex Lie algebra of dimension $n^2 - 2n - 1$ cannot have an n-dimensional irreducible representation. Hence, in fact, $m \neq 1$.

The lemma is proved. □

The following theorem is an easy consequence of Lemma 2.3.

THEOREM 2.4. *There exists no real manifold of dimension $2n \geq 4$ admitting an effective transitive action of SU_n.*

Proof. Let M be the manifold, $p \in M$ and I_p be as before the isotropy subgroup of p. Obviously, $\dim I_p = n^2 - 2n - 1$ (clearly, we have $n \geq 3$). Therefore, from Lemma 2.3 we see that either $n = 3$ and I_p^c is conjugate in SU_3 to $(U_1 \times U_1 \times U_1) \cap SU_3$ embedded in SU_3 in the standard way, or $n = 4$ and I_p^c is conjugate in SU_4 to $(U_2 \times U_2) \cap SU_4$ embedded in SU_4 in the standard way. In these cases, however, I_p^c clearly contains the center of SU_n for $n = 3, 4$, and hence the action of SU_n on M is not effective.

This contradiction proves the theorem. □

REMARK 2.5. Theorem 2.4 for $2n \geq 10$ for not necessarily effective actions follows from the classification in [16].

We will conclude this section with the following theorem.

THEOREM 2.6. *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by biholomorphic* *transformations with no fixed points. Let $p \in M$ and let $O(p)$ be the SU_n-orbit of p. Then $O(p)$ is either*

(i) *a strongly pseudoconvex compact real hypersurface in M, or*

(ii) *a compact complex hypersurface in M holomorphically equivalent to $\mathbb{CP}^{n-1}$.*

Moreover, there exist no more than two complex hypersurface orbits.

Proof. We will first show that a real hypersurface orbit has to be strongly pseudoconvex. The proof is similar to that of Proposition 2.2 in [8]. We show first that $O(p)$ is either Levi-flat or strongly pseudoconvex. This is obvious for $n = 2$ since $O(p)$ is a homogeneous real hypersurface and the corresponding Levi form has only one eigenvalue.

Assume now that $n \geq 3$. Since $O(p)$ is a real hypersurface in M, it arises in **II** of the proof of Proposition 2.1. Let W be the orthogonal complement to $V \cap iV$ in $T_p(M)$. Clearly, $\dim_{\mathbb{C}} V \cap iV = n - 1$ and $\dim_{\mathbb{C}} W = 1$. The group L_p is a subgroup of U_n and preserves both $V \cap iV$ and W. In addition, it preserves V and hence the line $W \cap V$. Therefore, it can only act as $\pm\mathrm{id}$ on W. Thus, the identity component L_p^c of L_p is a subgroup of the group of unitary transformations preserving $V \cap iV$ and acting trivially on W. Since $\dim L_p^c = n^2 - 2n$, L_p^c is isomorphic to SU_{n-1} and acts transitively on $V \cap iV$. Therefore, either all eigenvalues of the Levi form vanish or they all are of the same sign, which means that $O(p)$ is either Levi-flat, or strongly pseudoconvex.

Assume that $O(p)$ is Levi-flat. Then it is foliated by complex hypersurfaces in M. Let $\mathfrak{m}$ be the Lie algebra of all holomorphic vector fields on $O(p)$ corresponding to the automorphisms of $O(p)$ generated by our action of SU_n. Clearly, $\mathfrak{m}$ is isomorphic to $\mathfrak{su}_n$. Let M_p be the leaf of the foliation passing through p, and consider the subspace $\mathfrak{l} \subset \mathfrak{m}$ of vector fields tangent to M_p at p. The vector fields in $\mathfrak{l}$ remain tangent to M_p at each point $q \in M_p$, and therefore $\mathfrak{l}$ is in fact a Lie subalgebra of $\mathfrak{m}$. However, $\dim \mathfrak{l} = n^2 - 2$ and $\mathfrak{su}_n$ has no subalgebras of codimension 1.

Hence $O(p)$ must be strongly pseudoconvex, as required.

We will now show that a complex hypersurface orbit is holomorphically equivalent to $\mathbb{CP}^n$. Let $O(p)$ be a complex hypersurface orbit. As we saw in **I** of the proof of Proposition 2.1, L_p can be embedded into $U_1 \times U_{n-1}$. Clearly, $\dim L_p = (n-1)^2$, and by Lemma 2.1 of [8], L_p^c is conjugate in U_n to the group H^n. Hence the restriction of L_p to V is U_{n-1} and therefore L_p acts transitively on the unit sphere in V. Since $O(p)$ is compact, it now follows from [2] that $O(p)$ is holomorphically equivalent to $\mathbb{CP}^{n-1}$.

We will now show that there can be no more than two complex hypersurface orbits. First of all, there is a real hypersurface orbit in M. Indeed suppose that all orbits in M are complex hypersurfaces. Let $p \in M$ and consider the isotropy group I_p of p. Then $\dim I_p = (n-1)^2$ and therefore, by Lemma 2.1 of [8], I_p^c is conjugate to the group H^n defined in (2.2) and hence contains the center of SU_n. Therefore, if all orbits were complex hypersurfaces, the SU_n-action would not be effective on M. Hence, there is a real hypersurface orbit in M.

Let $O \subset M$ be a complex hypersurface orbit, and we assume that there is at least one other complex hypersurface orbit. Consider $M' := M \setminus O$. The manifold M' is non-compact, equipped with an action of the compact group SU_n and has a real hypersurface orbit. It now follows from the results in [13] (see Theorem 3 and Corollary 5.8 there) that M' has exactly one complex hypersurface orbit. Hence, M has exactly two such orbits.

The theorem is proved. □

To obtain a complete classification of effective SU_n-actions, one now needs to classify real hypersurface orbits and to glue them together with copies of $\mathbb{CP}^{n-1}$. This is not an objective of the current paper, and the final classification will appear in our future publication.

3. Examples of SU_n-actions

We start this section with examples showing that there are indeed more manifolds that admit effective actions of SU_n than those that admit effective actions of U_n. These examples are in dimension $n = 2$.

Recall the construction of a non-standard complex structure on $\mathbb{CP}^2 \setminus \{0\}$ given by Rossi in [14]. Let $(w_0 : w_1 : w_2 : w_3)$ be homogeneous coordinates in $\mathbb{CP}^3$. Consider in $\mathbb{CP}^3$ the variety V given by the equation

$$w_1 w_2 = w_3(w_3 + w_0). \tag{3.1}$$

Let $(z_0 : z_1 : z_2)$ be homogeneous coordinates in $\mathbb{CP}^2$. We consider the map $F : \mathbb{CP}^2 \setminus \{0\} \to V$ defined by the formulas

$$\begin{aligned} w_0 &= z_0^2, \\ w_1 &= z_1^2 - \frac{z_1 \overline{z_2}}{|z_1|^2 + |z_2|^2} z_0^2, \\ w_2 &= z_2^2 + \frac{\overline{z_1} z_2}{|z_1|^2 + |z_2|^2} z_0^2, \\ w_3 &= z_1 z_2 - \frac{|z_2|^2}{|z_1|^2 + |z_2|^2} z_0^2. \end{aligned}$$

The map F is everywhere 2-to-1, and its image is $V \setminus S$, where S is given by:

$$(3.2) \qquad w_0 = 1, \qquad , w_2 = -\overline{w_1}, \qquad w_3 \in \mathbb{R}.$$

The set $S \cap V$ is the limit set of the mapping F at 0.

Consider the unique complex structure on $\mathbb{CP}^2 \setminus \{0\}$ making F locally biholomorphic. We denote $\mathbb{CP}^2 \setminus \{0\}$ with this new complex structure by X. Clearly, SU_2 acts effectively on X by diffeomorphisms in the usual way: for $(z_0 : z_1 : z_2) \in X$ and $g \in SU_2$ we have

$$g(z_0 : z_1 : z_2) := (z_0 : u_1 : u_2),$$

where $(u_1, u_2) := g(z_1, z_2)$. It can be verified directly that this is an action by biholomorphic automorphisms of X. Let $\mathfrak{S}^3_R$ be the sphere of radius R in X. It is an SU_2-orbit in X and therefore the CR-structure it has as a real hypersurface in X is invariant under the standard action of SU_2. It follows from the results of [14] (see also [15]) that none of the $\mathfrak{S}^3_R$ is CR-equivalent to the ordinary sphere S^3 and hence none of the $\mathfrak{S}^3_R$ is spherical.

Further, it can be shown (directly or using an approach based on classifying algebras, as in [10]) that a CR-structure on S^3 invariant under the standard action of SU_2 is equivalent to either the standard CR-structure or to the CR-structure of one of $\mathfrak{S}^3_R$ by means of an SU_2-equivariant CR-diffeomorphism, and the manifolds $\mathfrak{S}^3_R$, $0 < R < \infty$, are pairwise non-CR-equivalent.

Further, it can be checked directly that the hyperplane P at infinity in X is in fact a complex curve in X equivalent to $\mathbb{CP}^1$ and at the same time is an SU_n-orbit.

Denote by $\mathfrak{S}^2_{r,R}$, $0 \leq r < R \leq \infty$, the spherical layer $S^2_{r,R}$ equipped with the non-standard complex structure induced by the complex structure of X and by $\widetilde{\mathfrak{S}^2_{r,\infty}}$, $r \geq 0$ a spherical layer with infinite outer radius with added hyperplane at infinity P. Then $\mathfrak{S}^2_{r,R}$, $0 \leq r < R \leq \infty$ is an example of a manifold that admits an effective SU_2-action, does not admit an effective U_2-action and for which all orbits are real hypersurfaces. Clearly, $\widetilde{\mathfrak{S}^2_{r,\infty}}$ is an analogous example with a unique complex hypersurface orbit.

We now give examples of manifolds with non-effective SU_n-actions that have a discrete kernel. Let $z = (z_1, z_2, z_3)$ be coordinates in $\mathbb{C}^3$. Consider the quadric $Q \subset \mathbb{C}^3$ given by

$$z_1^2 + z_2^2 + z_3^2 = 1.$$

Clearly, the group $SO_3(\mathbb{C})$ acts transitively on Q. It is known that $SO_3(\mathbb{C})$ is isomorphic to $SL_2(\mathbb{C})/\mathbb{Z}_2$ (here we write $\mathbb{Z}_2$ for the center of $SL_2(\mathbb{C})$). Hence, $SL_2(\mathbb{C})/\mathbb{Z}_2$ acts transitively on Q. The isotropy subgroup under this action is $GL_1(\mathbb{C})/\mathbb{Z}_2$. Hence

$$Q = (SL_2/\mathbb{Z}_2) \,/\, (GL_1(\mathbb{C})/\mathbb{Z}_2) = SL_2(\mathbb{C})/GL_1(\mathbb{C}),$$

and hence $SL_2(\mathbb{C})$ acts transitively on Q with isotropy subgroup $GL_1(\mathbb{C})$.

Consider the induced action of SU_2 on Q. This action is not effective since the isotropy subgroup of any point in Q contains the center of SU_2. Let $z^0 \in Q$. If $\operatorname{Im} z^0 \neq 0$, then the SU_2-orbit of z^0 is a real hypersurface in Q. All points z^0 with $\operatorname{Im} z_0 = 0$ lie on a single orbit O that has codimension 2 in Q. However, this orbit is not a complex curve in Q, it is in fact totally real in Q.

We remark here that quadric Q by an affine change of coordinates is equivalent to the finite part of quadric V given by (3.1). Under this change of coordinates $Q \setminus O$ is mapped onto $V \setminus S$. Then the SU_2-action on $Q \setminus O$ can be lifted to $X \setminus P$. The result of this lift is precisely the restriction to $X \setminus P$ of the SU_2-action on X introduced above.

Another example of a non-effective action in dimension $n = 2$ can be constructed as follows. Let $M = \mathbb{CP}^2$. Consider the adjoint representation of SU_2, $\mathrm{Ad} : SU_2 \to GL(\mathfrak{su}_2)$, and for every $g \in SU_2$ view $\mathrm{Ad}(g)$ as a transformation of $\mathbb{C}^3$. Define an action of SU_2 on M by applying $\mathrm{Ad}(g)$ to vectors of homogeneous coordinates. This action is not effective since $\mathrm{Ad}(-\mathrm{id}) = \mathrm{Ad}(\mathrm{id})$ is the identity transformation, and thus the center of SU_2 acts trivially on M. In this example, as in the preceding one, there is an orbit of codimension 2 that is not a complex curve in M, but is totally real.

More examples of manifolds with non-effective SU_n-actions for any $n \geq 2$ can be constructed by setting $M = \mathbb{CP}^{n-1} \times C$, where C is a complex curve. Let SU_n act on $\mathbb{CP}^{n-1}$ by applying matrices from SU_n to vectors of homogeneous coordinates, and let SU_n act on C trivially. The resulting action is not effective on M since every element of the center of SU_n acts by the identity transformation. The orbit of every point is equivalent to $\mathbb{CP}^{n-1}$.

References

[1] D. Barrett, E. Bedford and J. Dadok, $\mathbb{T}^n$*-actions on holomorphically separable complex manifolds*, Math. Z. **202** (1989), 65–82.

[2] J. Bland, T. Duchamp and M. Kalka, *A characterization of* $\mathbb{CP}^n$ *by its automorphism group*, Complex Analysis (University Park, Pa, 1986), 60–65, Lecture Notes In Mathematics **1268**, Springer-Verlag, 1987.

[3] J. A. Gifford, A. V. Isaev and S. G. Krantz, *On the dimensions of the automorphism groups of hyperbolic Reinhardt domains*, Illinois J. Math. **44** (2000), 602–618.

[4] R. E. Greene and S. G. Krantz, *Characterization of complex manifolds by the isotropy subgroups of their automorphism groups*, Indiana Univ. Math. J. **34** (1985), 865–879.

[5] W. C. Hsiang and W. Y. Hsiang, *Some results on differentiable actions*, Bull. Amer. Math. Soc. **72** (1966), 134–138.

[6] W. Y. Hsiang, *On the principal orbit type and P. A. Smith theory of* $SU(p)$ *actions*, Topology **6** (1967), 125–135.

[7] A. V. Isaev and S. G. Krantz, *On the automorphism groups of hyperbolic manifolds*, J. Reine Angew. Math. **534** (2001), 187–194.

[8] A. V. Isaev and N. G. Kruzhilin, *Effective actions of the unitary group on complex manifolds*, to appear in Canad. J. Math. in 2002.

[9] W. Kaup, *Reelle Transformationsgruppen und invariante Metriken auf komplexen Räumen*, Invent. Math. **3** (1967), 43–70.

[10] A. Krüger, *Homogeneous Cauchy-Riemann structures*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **18** (1991), no. 4, 193–212.

[11] A. Klimyk and K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer, Berlin, 1997.

[12] K. Mukoyama, *Smooth* $SU(p,q)$*-actions on* $(2p+2q-1)$*-sphere and on the complex projective* $(p+q-1)$*-space*, Kyushu J. Math. **55** (2001), 213–236.

[13] T. Nagano, *Transformation groups with* $(n-1)$*-dimensional orbits on non-compact manifolds*, Nagoya Math. J. **14** (1959), 25–38.

[14] H. Rossi, *Attaching analytic spaces to an analytic space along a pseudoconcave boundary*, Proc. Conf. Complex Analysis, Minneapolis, 1964, Springer-Verlag, 1965, 242–256.

[15] ———, *Homogeneous strongly pseudoconvex hypersurfaces*, Proc. Conf. Complex Analysis Rice Univ., Houston, 1972, Rice Univ. Studies, **59** (1973), no. 1, 131–145.

[16] F. Uchida, *Smooth actions of special unitary groups on cohomology complex projective spaces*, Osaka J. Math. **12** (1975), 375–400.

[17] E. Vinberg and A. Onishchik, *Lie Groups and Algebraic Groups*, Springer-Verlag, 1990.

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