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NORMAL SYSTEMS OF COORDINATES ON MANIFOLDS OF CHERN-MOSER TYPE

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ABSTRACT. It is known that the CR geometries of Levi non-degenerate hypersurfaces in $\mathbb{C}^n$ and of the elliptic or hyperbolic CR submanifolds of codimension two in $\mathbb{C}^4$ share many common features. In this paper, a special class of normalized coordinates is introduced for any CR manifold M which is one of the above three kinds and it is shown that the explicit expression in these coordinates of an isotropy automorphism $f \in \mathrm{Aut}(M)_o \subset \mathrm{Aut}(M)$, $o \in M$, is equal to the expression of a corresponding element of the automorphism group of the homogeneous model. As an application of this property, an extension theorem for CR maps is obtained.

1. Introduction

In their fundamental paper [2], S.-S. Chern and J. Moser introduced two new methods to determine the CR invariants of Levi non-degenerate real hypersurfaces in $\mathbb{C}^{n+1}$.

The first one applies to any real analytic Levi non-degenerate real hypersurface $M \subset \mathbb{C}^{n+1}$ and it is based on the fact that each such hypersurface is locally equivalent to a hypersurface of a distinguished family, the so-called *hypersurfaces in normal form*. By this fact, the studies on CR invariants and on the automorphism groups of those hypersurfaces reduce to analysis of the equations of hypersurfaces in normal forms.

The second method applies to any smooth (and not only real analytic) Levi non-degenerate real hypersurface M. Chern and Moser proved that on M there exists a natural principal bundle $\pi : P_{CM}(M) \to M$ and a natural Cartan connection ω on $P_{CM}(M)$. Using this property, the CR invariants and the automorphism group of M can be explicitly expressed in terms of the curvature of ω and of the automorphisms of $P_{CM}(M)$, which leave ω invariant.

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In a sequence of other papers (see e.g. [7], [5], [10], [11], [4]), it was shown that there exist two more families of CR manifolds for which similar methods can be applied. These are the manifolds of dimension six with a 2-dimensional complex distribution $(\mathcal{D}, J)$ of either elliptic or hyperbolic type. We recall that, if M is one of such manifolds, the CR structure $(\mathcal{D}, J)$ is osculated by the CR structure of a quadric $\mathcal{Q}$ which is one of the following two models:

$$\begin{aligned} &\mathrm{Im}(w_1) = |z_1|^2, && \mathrm{Im}(w_2) = |z_2|^2 && \text{(hyperbolic)} \\ &\mathrm{Im}(w_1) = \mathrm{Re}(z_1\bar{z}_2), && \mathrm{Im}(w_2) = \mathrm{Im}(z_1\bar{z}_2) && \text{(elliptic)} \end{aligned}$$

An important feature which holds in common for the Levi non-degenerate CR structures of hypersurface type and the six dimensional manifolds with elliptic or hyperbolic complex distribution is that the symmetry group G of the associated homogeneous model (i.e. the osculating quadric) is semi-simple and with very large stability subgroup H. In fact, the CR structure of the mentioned CR manifolds is a *parabolic structure*, which admits a canonical Cartan connection (see [14], [3], [10]). This means that any of these manifolds has an associated H-principal bundle $\pi : P \to M$ and a Cartan connection $\omega : TP \to \mathfrak{g} = Lie(G)$, which is invariant under any automorphism of the CR structure. An explicit construction of such bundle P with a Cartan connection $\omega : TP \to \mathfrak{g}$ is given in [2] for Levi non-degenerate hypersurfaces and in [11] for elliptic and hyperbolic CR manifolds.

Another common property shared by the real-analytic Levi non-degenerate hypersurfaces in $\mathbb{C}^n$ and the 6-dimensional submanifolds of $\mathbb{C}^4$ of elliptic or hyperbolic type is that there exist special holomorphic coordinates in the ambient space (called *normal coordinates*), which are determined up to an action of the group H and in which the equation of the manifold takes a certain normal form.

So, the real analytic Levi non-degenerate hypersurfaces and the 6-dimensional submanifolds in $\mathbb{C}^4$ of elliptic or hyperbolic type share the following properties:

(a) their CR structures are parabolic structures and they admit a canonical Cartan connection;
(b) an explicit construction for such canonical Cartan connection is available;
(c) as real analytic submanifolds in $\mathbb{C}^N$, they all have normal forms.

For shortness, we will call these three families of manifolds *CM manifolds* ("CM" for "Chern-Moser").

So far, several results on the automorphism group of a real analytic CM manifold M have been obtained by working in normal coordinates, where the automorphisms of M assume a simplified form.

In our paper, we introduce a new system of coordinates, which we call *$\mathcal{Q}$-normalized coordinates*. Roughly speaking, such coordinates can be described as follows. Consider the osculating quadric $\mathcal{Q} = G/H$ of M, the principal bundle $\pi : P \to M$ and the Cartan connection $\omega : TP \to \mathfrak{g}$. Then, $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$ with $\mathfrak{g}_-$ being a nilpotent subalgebra and $\mathfrak{h} = Lie(H)$ and for some suitable neighborhood $\mathcal{U} \subset \mathfrak{g}_-$ the following maps are diffeomorphisms onto their images:

$$\alpha : \mathcal{U} \to \mathcal{Q} = G/H\ , \qquad \alpha(E) = \exp(E) \cdot H$$

$$\beta : \mathcal{U} \to M\ , \qquad \beta(E) = \Phi_1^{\tilde{E}}(u_o),$$

where u_o is a fixed point in P and $\Phi_1^{\tilde{E}}(u_o)$ is the flow along the vector field $\tilde{E}$ on P with $\omega(\tilde{E}) = E$. In fact, α^{-1} and β^{-1} are normal coordinates associated to the Cartan connections of Q and M (see [12]).

The $\mathcal{Q}$-normalized coordinates are given by the map

$$\alpha \circ \beta^{-1} : \beta(\mathcal{U}) \subset M \longrightarrow \mathcal{Q} \subset \mathbb{C}^n\ .$$

We show that this map is a system of coordinates in which the elements of the isotropy $\mathrm{Aut}(M)_{x_o}$, $x_o = \pi(u_o)$, are written by the expressions of some uniquely associated elements in the isotropy $H = \mathrm{Aut}(\mathcal{Q})_o$ of the quadric $\mathcal{Q}$. This implies that $\alpha \circ \beta^{-1}$ maps diffeomorphically the orbits of $\mathrm{Aut}(M)_{x_o}$ in $\beta(\mathcal{U}) \subset M$ into orbits of a suitable subgroup of H in $\alpha(\mathcal{U}) \subset Q$.

In contrast to the normal coordinates, these new coordinates are not determined by a holomorphic change of coordinates in the ambient space. However, in many applications, this fact does not have any influence. For example, in studying topological properties of the orbits of a given group A fixing a point $x \in M$ the $\mathcal{Q}$-normalized coordinates allow to reduce to the case $M = \mathcal{Q}$ and $A \subset H = \mathrm{Aut}(\mathcal{Q})_o$.

We believe that several known results, proved just in the real analytic case, can be extended to the category of all smooth CM manifolds by using $\mathcal{Q}$-normalized coordinates.

Finally, we show that any real analytic CM manifold, embedded in normal form, admits a *distinguished* system of $\mathcal{Q}$-normalized coordinates. Using this fact, we are able to show that the expression of a normalization map in $\mathcal{Q}$-normalized coordinates is again equal to the expression of a corresponding automorphism of the osculating quadric. This can be used to obtain a simplified proof of several properties of the normalization maps, like e.g. the fact that they constitute a group isomorphic to the stability subgroup $\mathrm{Aut}(\mathcal{Q})_0$ of the osculating quadric.

The structure of the paper is the following. In Section 2, we review some basic definitions on CM manifolds, needed in the rest of the paper. In Section 3 and Section 4, we introduce the $\mathcal{Q}$-normalized coordinates and we prove the above described properties. In Section 5, as an immediate application of the concept of $\mathcal{Q}$-normalized coordinates, we prove an extension theorem for CR maps which generalizes a theorem by Vitushkin on real analytic Levi non-degenerate hypersurfaces to all CM manifolds.

In all what follows, we will denote a CR structure on a manifold M by the pair $(\mathcal{D}, J)$, where $\mathcal{D}$ denotes the real distribution in TM underlying the holomorphic distribution of the CR structure and J is the family of complex structures $J_x : \mathcal{D}_x \to \mathcal{D}_x$, which makes the real distribution $\mathcal{D}$ a holomorphic distribution.

For any vector field A on a manifold N, we will use the symbol "Φ_t^A" to indicate the flow of A. For any Lie group G with Lie algebra $\mathfrak{g} = Lie(G)$, we will use the symbol $\exp : \mathfrak{g} \to G$ to indicate the exponential map of G.

2. Some basic facts on CM manifolds

In the following three subsections, we describe in detail what particular properties of CM manifolds will be used.

2.1. Quadric automorphisms

Below we list the Lie groups G_Q of rational automorphisms of the associated osculating quadric $\mathcal{Q}$ for the three types of CM manifolds. The stability subgroup at the origin $0 \in \mathcal{Q}$ is denoted by $H_Q = (G_Q)_0$ and, for any element $a \in H_Q$, we will denote by $F_a : \mathbb{C}^N \to \mathbb{C}^N$ the associated birational transformation of $\mathbb{C}^N$ which induces on $\mathcal{Q}$ the transformation a.

(a) If M is a real hypersurface, then $G_Q = \mathrm{SU}_{\mathrm{n,p}}/\mathbb{Z}_n$.
(b) If M is an elliptic manifold, then $G_Q = (\mathrm{SL}_3(\mathbb{C})/\mathbb{Z}_3) \rtimes \mathbb{Z}_2$.
(c) If M is an hyperbolic manifold, then $G_Q = (\mathrm{SU}_{3,1}/\mathbb{Z}_3 \times \mathrm{SU}_{3,1}/\mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

In all three cases, the Lie algebra $\mathfrak{g}_Q = Lie(G_Q)$ admits a natural structure of graded Lie algebra of depth two:

$$\mathfrak{g}_Q = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2 \tag{2.1}$$

and the subalgebra $\mathfrak{h}_Q = Lie(H_Q)$ is equal to

$$\mathfrak{h}_Q = \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2. \tag{2.2}$$

The rational automorphisms from the stability groups $H_Q = (G_Q)_0$ are particularly important for the construction of normal forms. Since we will use them later, we recall here their explicit description.

First of all, for the hyperquadrics $\operatorname{Im} w = \langle z, z\rangle$, let us denote any birational transformation $F_a : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ determined by an element $a \in G_Q$ by

$$F_a(z,w) = (F_a^z(z,w), F_a^w(z,w)) \ ,$$

where $F_a^w(z,w)$ is the last component of $F_a(z,w)$, while

$$F_a^z(z,w) = (F_a^{z_1}(z,w), \dots, F_a^{z_n}(z,w))$$

is the n-tuple of the first n components. Then the maps $F_a^z(z,w)$ and $F_a^w(z,w)$ are of the form:

$$F_a^z = \lambda_a U_a \frac{z + \alpha_a w}{1 - 2\,\mathrm{i}\langle z, \alpha_a\rangle - (r_a + \mathrm{i}\langle \alpha_a, \alpha_a\rangle) w} \ , \tag{2.3$_1$}$$

$$F_a^w = \lambda_a^2 \frac{w}{1 - 2\,\mathrm{i}\langle z, \alpha_a\rangle - (r_a + \mathrm{i}\langle \alpha_a, \alpha_a\rangle) w} \ , \tag{2.3$_2$}$$

where U_a is a pseudounitary endomorphism with respect to the hermitian form $\langle \cdot, \cdot\rangle$, λ_a is a positive real number, α_a is a $\mathbb{C}^n$-vector and r_a an arbitrary real number. In other words, each birational transformation F_a is uniquely determined by the associated parameters $(\lambda_a, U_a, \alpha_a, r_a)$. Notice also that the parameters $(\lambda_a, U_a, \alpha_a, r_a)$ can be recovered from the explicit expressions of the components of F_a by the following relations:

$$\lambda_a = \sqrt{\left.\frac{\partial F_a^w}{\partial w}\right|_0} \ , \qquad U_a = \frac{1}{\lambda_a} \left.\frac{\partial F_a^z}{\partial z}\right|_0 \ , \tag{2.4$_1$}$$

$$\alpha_a = \frac{1}{\lambda_a} U_a^{-1} \left.\frac{\partial F_a^z}{\partial w}\right|_0 \ , \qquad r_a = \frac{1}{2\lambda_a^2} \operatorname{Re} \left.\frac{\partial^2 F_a^w}{(\partial w)^2}\right|_0 \ . \tag{2.4$_2$}$$

For what concerns hyperbolic quadrics, from Belošapka's results [1] it follows that, for any $a \in H_Q$, the corresponding rational automorphism

$$F_a = (F_a^{z_1}, F_a^{z_2}, F_a^{w_1}, F_a^{w_2})$$

splits into the direct product of two sphere automorphisms and it is of the form

$$(2.5_1) \qquad F_a^{z_j} = c_a^j \frac{z_j + \alpha_a^j w_j}{1 - 2\,\mathrm{i}\,\overline{\alpha_a^j} z_j - (r_j + \mathrm{i}\,|\alpha_a^j|^2) w_j} ,$$

$$(2.5_2) \qquad F_a^{w_j} = |c_a^j|^2 \frac{w_j}{1 - 2\,\mathrm{i}\,\overline{\alpha_a^j} z_j - (r_j + \mathrm{i}\,|\alpha_a^j|^2) w_j} ,$$

for $j = 1, 2$, possibly followed by the map that interchanges z_1, w_1 with z_2, w_2. Here, c_a^j are arbitrary non-vanishing complex numbers, α_a^j are arbitrary complex numbers and r_a^j are arbitrary real numbers. Then F_a is uniquely determined by the parameters $(c_a^j, \alpha_a^j, r_a^j)$, $j = 1, 2$, and such parameters can be recovered from the components of F_a by the following formulae:

$$(2.6_1) \qquad c_a^j = \left.\frac{\partial F_a^{z_j}}{\partial z_j}\right|_0 , \qquad \alpha_a^j = (c_a^j)^{-1} \left.\frac{\partial F_a^{z_j}}{\partial w_j}\right|_0 ,$$

$$(2.6_2) \qquad r_a^j = \frac{1}{2|c_a^j|^2} \operatorname{Re} \left.\frac{\partial^2 F^{w_j}}{(\partial w_j)^2}\right|_0 .$$

And now, let us turn to the elliptic quadric. The simplest representation of the isotropic automorphisms of the elliptic quadric can be obtained in coordinates $z_1, z_2, \omega_1, \omega_2$ where $\omega_1 = w_1 + \mathrm{i}\, w_2$ and $\omega_2 = w_1 - \mathrm{i}\, w_2$. Using these coordinates, the quadric can be written by just one complex equation, namely

$$(2.7) \qquad \frac{\omega_1 - \bar{\omega}_2}{2\,\mathrm{i}} = z_1 \bar{z}_2 .$$

The isotropic automorphisms $F_a = (F_a^{z_1}, F_a^{z_2}, F_a^{\omega_1}, F_a^{\omega_2})$ take then the form

$$(2.8_1) \qquad F_a^{z_1} = c_a^1 \frac{z_1 + \alpha_a^1 \omega_1}{1 - 2\,\mathrm{i}\,\overline{\alpha_a^2} z_1 - (r_a^1 + \mathrm{i}\,\alpha_a^1 \overline{\alpha_a^2}) \omega_1} ,$$

$$(2.8_2) \qquad F_a^{\omega_1} = c_a^1 \overline{c_a^2} \frac{\omega_1}{1 - 2\,\mathrm{i}\,\overline{\alpha_a^2} z_1 - (r_a^1 + \mathrm{i}\,\alpha_a^1 \overline{\alpha_a^2}) \omega_1} .$$

The formulae for $F_a^{z_2}$, $F_a^{\omega_2}$ are analogous with indices 1 and 2 interchanged. As for the hyperbolic quadrics, these mappings again can be followed by a mapping that interchanges z_1, ω_1 with z_2, ω_2. As before, c_a^j are arbitrary non-vanishing complex numbers, α_a^j are arbitrary complex numbers, but the r_a^j are mutually conjugate complex numbers. Thus, the parameters $(c_a^j, \alpha_a^j, r_a^j)$, $j = 1, 2$, corresponding to an automorphism F_a are expressed in terms of its components by

$$(2.9_1) \qquad c_a^j = \left.\frac{\partial F_a^{z_j}}{\partial z_j}\right|_0 , \qquad \alpha_a^j = c_a^{j\,-1} \left.\frac{\partial F_a^{z_j}}{\partial w_j}\right|_0 ,$$

$$(2.9_2) \qquad r_a^1 = \overline{r_a^2} = \frac{1}{2c_a^1\overline{c_a^2}}\frac{1}{2}\left(\left.\frac{\partial^2 F^{\omega_1}}{(\partial\omega_1)^2}\right|_0 + \left.\frac{\partial^2 F^{\omega_2}}{(\partial\omega_2)^2}\right|_0\right) .$$

Notice that a "twisted real part" of the second derivatives appears in the equation (2.9_2). This phenomenon is due to the twisted imaginary part of the ω's in the equation of the quadric.

2.2. Chern-Moser bundle

For any CM manifold M, it has been proved that there exists a canonically associated H_Q-principal bundle $\pi : P_{CM}(M) \to M$, which is called *Chern-Moser bundle of M* (see [2], [11], [10]). In all cases, it is constructed as a bundle $\pi_o : P_{CM}(M) \to E$ over an auxiliary bundle $\hat{\pi} : E \to M$.

For manifolds of hypersurface type, the bundle $\hat{\pi} : E \to M$ is the conormal bundle of the distribution $\mathcal{D}$, i.e. E is the subbundle of $\hat{\pi} : T^*M \to M$, whose fibers $E_x = \hat{\pi}^{-1}(x) \subset T_x^*M$, $x \in M$, are given by all 1-forms θ_x which satisfy

$$\ker \theta_x = \mathcal{D}_x \ .$$

This is an $\mathbb{R}^*$- principal bundle. We recall that, for any $\theta_x \in E_x$ and any two $\mathcal{D}$-vector fields ξ, η, the value

$$\theta_x([\xi, \eta])$$

depends only on the values of ξ, η at the base point x of θ_x and therefore defines a bilinear form on $\mathcal{D}_x$. This form can be considered as the imaginary part of a Hermitian form (namely, it is the Levi form of the CR structure) that will be exploited for further reductions.

For elliptic and hyperbolic manifolds, the conormal bundle $\hat{\pi} : \mathcal{E} \to M$ is the subbundle of $\hat{\pi} : (T^* \times T^*)M \to M$, whose fibers $\mathcal{E}_x = \hat{\pi}^{-1}(x) \subset T_x^*M \times T_x^*M$, $x \in M$, are given by all pairs of 1-forms (θ_x^1, θ_x^2) which satisfy

$$\ker \theta_x^1 \cap \ker \theta_x^2 = \mathcal{D}_x \ .$$

This conormal bundle $\hat{\pi} : \mathcal{E} \to M$ is a $\mathrm{GL}_2(\mathbb{R})$-principal bundle and it can be reduced to a subbundle $\hat{\pi} : E \to M$, which is a $\mathbb{C}^* \times \mathbb{Z}_2$-bundle or a $\mathbb{R}^* \times \mathbb{R}^* \times \mathbb{Z}_2$-bundle in the elliptic and hyperbolic case, respectively.

To obtain such a reduction in the hyperbolic case, one has to consider only pairs of 1-forms $(\theta^1, \theta^2) \in \mathcal{E}$, so that the bilinear forms $\theta^i([\xi, \eta])$ are both degenerate. This condition fixes the two forms (θ^1, θ^2) up to scale and order.

In the elliptic case, one has to consider only pairs of 1-forms $(\theta^1, \theta^2) \in \mathcal{E}$ such that the $\mathbb{C}$-valued bilinear form $(\theta^1 + \mathrm{i}\,\theta^2)([\xi, \eta])$ is degenerate. This condition fixes the $\mathbb{C}$-valued form (and hence the pair (θ^1, θ^2)) up to scale and conjugation.

In both cases, the reduction $\hat{\pi} : \hat{E} \to M$ obtained, is the auxiliary bundle we mentioned.

For any CM manifold, the Chern-Moser bundle $\pi : P_{CM}(M) \to M$ is defined as a bundle of so-called *adapted frames* (or, by duality, *adapted coframes*) over E. The exact definition of the adapted frames at a given point $\phi \in E$ depends only on the geometric data $(\mathcal{D}, J)$ of M and of ϕ itself and we refer the reader to [2] for the CR manifolds of hypersurface type and to [11] for the elliptic and hyperbolic manifold. Here, we only need to recall that it is proved that $\pi : P_{CM}(M) \to M$ has a natural structure of H_Q-principal bundle, where $H_Q = (G_Q)_0$ is the stability subgroup of the automorphism group G_Q of osculating quadric $\mathcal{Q}$ associated with M. Moreover, we want to stress the fact that the construction ensures that any (local) CR equivalence between two CM manifolds M_1 and M_2 of the same kind lifts automatically to a mapping of the corresponding Chern-Moser bundles $P_{CM}(M_1)$ to $P_{CM}(M_2)$.

From now on, for any (local) CR equivalence $f : M_1 \to M_2$ between to CM manifold, we will denote by $\hat{f} : P_{CM}(M_1) \to P_{CM}(M_2)$ the corresponding lifted map between the corresponding Chern-Moser bundles.

2.3. Cartan connection

The characteristic property of the Chern-Moser bundle as defined in [2] or [11] is the existence of a smooth field of distinguished frames at tangent spaces $T_u P_{CM}(M)$ of $P_{CM}(M)$, depending only on the CR structure of M. This means that there is a vector-valued 1-form ω on $P_{CM}(M)$ that assigns to any tangent vector its coordinates with respect to the distinguished frame at the base point of this vector. It is shown that the 1-form ω takes values in the Lie algebra $\mathfrak{g}_Q = Lie(G_Q)$ and satisfies the following conditions:

(1) for any $u \in P_{CM}(M)$ the mapping $\omega_u : T_u P_{CM}(M) \to \mathfrak{g}_Q$ is an isomorphism,
(2) $R_h^* \omega = \mathrm{Ad}_{h^{-1}}\, \omega$ for any $h \in H_Q$,

(3) for any vertical fundamental vector field $\hat{X}$, corresponding to the flow $R_{\exp tX}$ for $X \in \mathfrak{h}_Q = Lie(H_Q)$, one has $\omega_u(\hat{X}_u) = X$ at all points $u \in P_{CM}(M)$.

Thus, ω is a Cartan connection. Moreover, it is proved that ω satisfies the following crucial property. A diffeomorphism $h : P_{CM}(M) \to P_{CM}(M)$ is the lift $h = \hat{f}$ of some CR automorphism $f \in \mathrm{Aut}(M)$ if and only if

$$h^*\omega = \omega \ .$$

This $\mathfrak{g}_Q$-valued 1-form ω is called *Chern-Moser connection of* M.

2.4. The fundamental vector fields on the Chern-Moser bundle

Let M be a CM manifold, $\pi : P_{CM}(M) \to M$ the associated Chern-Moser bundle and $\omega : TP_{CM}(M) \to \mathfrak{g}_Q$ the Chern-Moser connection. For any $X \in \mathfrak{g}_Q$, we will call *fundamental vector field on* $P_{CM}(M)$ *associated with* X the unique vector field such that, at any $u \in P_{CM}(M)$

$$\omega_u(\hat{X}) = X \ . \tag{2.2}$$

Notice that, for any $X \in \mathfrak{h}_Q \subset \mathfrak{g}_Q$, the associated fundamental vector field $\hat{X}$ coincides exactly with the vector field corresponding to the flow $R_{\exp tX}$ on $P_{CM}(M)$.

About the fundamental vector fields, we have the following technical lemma, which will be used in the next sections.

LEMMA 2.1. *Let M be a CM manifold. Then:*

(i) *for any $a \in H_Q$ and any $X \in \mathfrak{g}_Q$, we have that $R_{a*}(\hat{X}) = \widehat{\mathrm{Ad}_{a^{-1}}(X)}$;*

(ii) *for any CR automorphism $f : M \to M$ and any $X \in \mathfrak{g}_Q$, we have that $\hat{f}_*(\hat{X}) = \hat{X}$;*

(iii) *consider the linear projection map $p_- : \mathfrak{g} \to \mathfrak{g}^{-2} + \mathfrak{g}^{-1}$, determined by the decomposition (2.1); then, for any $X \in \mathfrak{g}$, $t \in \mathbb{R}$ and $\mathcal{U} \subset P_{CM}(M)$, such that the flow $\Phi_t^{\hat{X}}\big|_{\mathcal{U}}$ is defined,*

$$\pi \circ \Phi_t^{\hat{X}}\Big|_{\mathcal{U}} = \pi \circ \Phi_t^{\widehat{p_-(X)}}\Big|_{\mathcal{U}} \ .$$

Proof. (i) Since ω is a Cartan connection, $(R_a)^*\omega = \mathrm{Ad}_{a^{-1}} \circ \omega$ for any $a \in H_Q$. Hence, if $\hat{X}$ is a fundamental vector field,

$$\omega(R_{a*}(\hat{X})) = \mathrm{Ad}_{a^{-1}} \circ \omega(\hat{X}) = \mathrm{Ad}_{a^{-1}}(X) \ .$$

This shows that $R_{a*}(\hat{X})$ is the fundamental vector field associated with $\mathrm{Ad}_{a^{-1}}(X)$.

(ii) Since $\hat{f}^*\omega = \omega$, for any fundamental vector field $\hat{X}$,

$$\omega(\hat{f}_*(\hat{X})) = \hat{f}^*\omega(\hat{X}) = \omega(\hat{X}) = X \ .$$

(iii) Notice that $\pi \circ \Phi_t^{\hat{X}} = \Phi_t^{\pi_*(\hat{X})}$ and $\pi \circ \Phi_t^{\widehat{p_-(X)}} = \Phi_t^{\pi_*(\widehat{p_-(X)})}$. On the other hand, if we denote by $\tilde{X}$ the element in $\mathfrak{h}_Q = \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$ equal to $\tilde{X} = X - p_-(X)$, we have that

$$\hat{X} = \widehat{p_-(X)} + \hat{\tilde{X}} \ .$$

Since $\hat{\tilde{X}}$ is tangent to the fibers at all points, it follows that $\pi_*(\hat{X}) = \pi_*(\widehat{p_-(X)})$. So,

$$\pi \circ \Phi_t^{\hat{X}} = \Phi_t^{\pi_*(\hat{X})} = \Phi_t^{\pi_*(\widehat{p_-(X)})} = \pi \circ \Phi_t^{\widehat{p_-(X)}} \ ,$$

as we needed to prove. □

2.5. Normal forms

For any CM manifold, which is embedded as a real analytic submanifold of $\mathbb{C}^n$, it is possible to determine a new embedding, the so-called embedding in *normal form*. Here we recall the constructions of such normal forms introduced by Chern and Moser [2] for the real analytic hypersurfaces, by Loboda [6] for the hyperbolic manifolds, and by Ežov and Schmalz [5] for elliptic manifolds.

DEFINITION 2.2. If M is a real analytic, Levi non-degenerate hypersurface in $\mathbb{C}^{n+1}$, containing the origin, it is said to be *embedded in normal form* if it is defined by an equation of the form

$$\operatorname{Im} w = \langle z, z\rangle + \sum_{k,\ell \geq 2} \rho_{k,\ell}(z, \bar{z}, \operatorname{Re} w)$$

where:

(a) $\langle z, z\rangle$ is the Hermitian form $\langle z, z\rangle = \sum_{j=1}^{p} |z^j|^2 - \sum_{j=p+1}^{n} |z^j|^2$ where (n, p) is the signature of the Levi form of M;
(b) each function $\rho_{k,\ell}$ is a polynomial of degree k in the variable z and a polynomial of degree ℓ in the variable $\bar{z}$;
(c) ρ_{22}, ρ_{32} and ρ_{33} satisfy the equations

$$\operatorname{tr} \rho_{22} = \operatorname{tr}^2 \rho_{32} = \operatorname{tr}^3 \rho_{33} = 0 \ ,$$

where the operator tr is

$$\operatorname{tr}\rho_{k\ell} = \frac{1}{k\ell}\Delta\rho_{k\ell}$$

and Δ is the Laplacian associated with the Hermitian form $\langle z, z\rangle$.

DEFINITION 2.3. If M is a real analytic, hyperbolic submanifold of $\mathbb{C}^4$, containing the origin, it is said to be *embedded in normal form* if it is defined by two equations of the form

$$\operatorname{Im}(w^i) = |z^i|^2 + \sum_{k_1,k_2,\ell_1,\ell_2\geq 1} \rho^i_{k_1,k_2,\ell_1,\ell_2}(z,\bar z,\operatorname{Re} w)\ , \qquad i = 1,2$$

where:

(b') each function $\rho^i_{k_1,k_2,\ell_1,\ell_2}$ is a polynomial of degree k_i in the variable z^i and a polynomial of degree ℓ_i in the variables $\bar z^i$ whose coefficients are real-analytic functions of $\operatorname{Re} w$;

(c') the sum $\sum_{k_1,k_2,\ell_1,\ell_2\geq 2}\rho^i_{k,\ell}$ belongs to the subspace P of power series, defined as follows:

$$\begin{aligned}
\rho^1_{k_1,k_2,1,0} &= 0 & \rho^2_{k_1,k_2,0,1} &= 0, \quad \text{for } k_1+k_2\geq 2\\
\rho^1_{1,1,0,1} &= 0 & \rho^2_{1,1,1,0} &= 0\\
\rho^1_{1,1,1,1} &= 0 & \rho^2_{1,1,1,1} &= 0\\
\rho^1_{2,0,2,0}|_{\operatorname{Re} w_2=0} &= 0 & \rho^2_{0,2,0,2}|_{\operatorname{Re} w_1=0} &= 0\\
\rho^1_{3,0,2,0}|_{\operatorname{Re} w_2=0} &= 0 & \rho^2_{0,3,0,2}|_{\operatorname{Re} w_1=0} &= 0\\
\rho^1_{3,0,3,0}|_{\operatorname{Re} w_2=0} &= 0 & \rho^2_{0,3,0,3}|_{\operatorname{Re} w_1=0} &= 0.
\end{aligned}$$

Notice that the two defining equations of an hyperbolic manifold embedded in normal form are real. Therefore, the conditions above have consequences for their conjugate pendants.

DEFINITION 2.4. If M is a real analytic, elliptic submanifold $\mathbb{C}^4$, containing the origin, it is reasonable to combine the defining equations into one complex equation

$$V = z_1\bar z_2 + \sum_{k_1,k_2,\ell_1,\ell_2\geq 1} \rho_{k_1,k_2,\ell_1,\ell_2}(z_1,z_2,\bar z_1,\bar z_2,U,\bar U),$$

where $U = \operatorname{Re} w_1 + \mathrm{i} \operatorname{Re} w_2$ and $V = \operatorname{Im} w_1 + \mathrm{i} \operatorname{Im} w_2$. It is said to be *embedded in normal form* if

$$\begin{aligned} \rho_{k_1,k_2,0,1} &= 0 & \rho_{1,0,\ell_1,\ell_2} &= 0, \quad \text{for } k_1+k_2, \ell_1+\ell_2 \geq 2 \\ \rho_{1,1,1,0} &= 0 & \rho_{0,1,1,1} &= 0 \\ \rho_{1,1,1,1} &= 0 & \rho_{2,0,0,2}|_{\bar U=0} &= 0 \\ \rho_{3,0,0,2}|_{\bar U=0} &= 0 & \rho_{2,0,0,3}|_{\bar U=0} &= 0 \\ \rho_{3,0,0,3}|_{\bar U=0} &= 0. \end{aligned}$$

Notice that if $M \subset \mathbb{C}^N$ is embedded in normal form, there exists only one defining equation $\rho = 0$ for such embedding of M, which satisfies the conditions given in Definitions 2.2, 2.3 or 2.4. In fact, $\rho = 0$ is the equation of a graph over the tangent plane $\{\operatorname{Im} w = 0\}$ at the point 0. The graph M determines the equation uniquely.

From now on, any defining equation $\rho = 0$, which satisfies the conditions given in Definitions 2.2 - 2.4, will be said to be *in normal form*. The previous remark means that *for any embedding in normal form of a real analytic CM manifold M there is exactly one defining equation in normal form.*

We recall that, by the results of [2], [6] and [5], the following theorem holds, which explains the importance of the embeddings in normal form and the associated defining equations in normal form.

THEOREM 2.5. *Let $M \subset \mathbb{C}^N$ be a real analytic CM submanifold of $\mathbb{C}^N$ containing the origin. Then, for any element $a \in H_Q$, there exists an open neighborhood $\mathcal{V} \subset \mathbb{C}^N$ of 0 and a so-called normalizing map*

$$\underset{[M,a]}{\mathbb{N}} : \mathcal{V} \subset \mathbb{C}^N \to \mathbb{C}^N ,$$

which is a holomorphic map, depending on the manifold M, such that

(α) *$M' = \underset{[M,a]}{\mathbb{N}}(\mathcal{V} \cap M)$ is a submanifold of $\mathbb{C}^N$, which is embedded in normal form;*

(β) *$\underset{[M,a]}{\mathbb{N}}$ is uniquely determined by the same first and second order derivatives at 0 that determine F_a (see Section 2.1) and possesses the same 2-jet as F_a.*

In particular, if $M \subset \mathbb{C}^N$ is a real analytic CM submanifold embedded in normal form and $M' = F(M)$ for some local biholomorphism $F : \mathcal{V} \subset \mathbb{C}^N \to \mathbb{C}^N$ fixing the origin, then $F = \underset{[M',a]}{\mathbb{N}}^{-1}$ for some $a \in H_Q$.

Finally, if $M \subset \mathbb{C}^N$ is embedded in normal form, a local biholomorphism F, which fixes the origin and induces a CR equivalence between

M and another CM manifold $M' = F(M)$ in normal form, is of the form $F = \underset{[M,a]}{\mathbb{N}} = \underset{[M',a^{-1}]}{\mathbb{N}}^{-1}$ for some $a \in H_Q$.

3. Systems of Q-normalized coordinates

First of all, we need the following two technical lemmas.

LEMMA 3.1. *Let M be a CM manifold of dimension n. For any $u \in P_{CM}(M)$, there exists an open neighborhood $\hat{\mathcal{U}}_u \subset \mathfrak{g}^{-2} + \mathfrak{g}^{-1}$ of 0, such that:*

1. *the restriction of the exponential map $\exp : \mathfrak{g} \to G_Q$ to $\hat{\mathcal{U}}_u$ is a diffeomorphism between $\hat{\mathcal{U}}_u$ and a n-dimensional submanifold $\exp(\mathcal{U}_u) \subset G_Q$;*
2. *for any $X \in \hat{\mathcal{U}}_u$, the flow $\Phi_t^{\hat{X}}$ of the corresponding fundamental vector field $\hat{X}$ is defined for any $t \in [0,1]$ and any u' in a suitable neighborhood $\tilde{\mathcal{U}}_u \subset P_{CM}(M)$ of u;*
3. *the map*
$$\widetilde{\exp}_u : \hat{\mathcal{U}}_u \to M \ , \qquad \widetilde{\exp}_u(X) = \pi(\Phi_1^{\hat{X}}(u))$$
is a diffeomorphism between $\hat{\mathcal{U}}_u$ and a neighborhood $\mathcal{U}_u$ of the point $x = \pi(u)$.

Proof. Consider a system of coordinates $\xi : \mathcal{V} \subset P_{CM}(M) \to \mathbb{R}^N$ on a neighborhood $\mathcal{V}$ of $u \in P_{CM}(M)$ and, for any $X \in \mathfrak{g}$ and any $v \in \mathcal{V}$, denote by $\gamma_{X,v}$ the maximal integral curve in $\mathcal{V}$ of the system of ordinary differential equations
$$\frac{d\gamma_{X,v}}{dt} = \hat{X}|_{\gamma_{X,v}} \ , \qquad \gamma_{X,v}(0) = v \ .$$
Such a system depends smoothly on X and v and satisfies the property
$$\gamma_{(sX),v}(t) = \gamma_{X,v}(s \cdot t) \ .$$
By classical facts on the smooth dependence of ordinary differential equations on initial data, there exists a relatively compact neighborhood $u \in \tilde{\mathcal{V}} \subset \mathcal{V}$ and a relatively compact neighborhood $0 \in \hat{\mathcal{V}} \subset \mathfrak{g}$, so that the curves $\gamma_{X,v}$ are defined for $t \in [0,1]$ whenever $X \in \hat{\mathcal{V}}$ and $v \in \tilde{\mathcal{V}}$. Moreover, choosing a smaller neighborhood $\hat{\mathcal{V}} \subset \mathfrak{g}$, we may always assume that $\exp|_{\hat{\mathcal{V}}}$ is a diffeomorphism onto its image. It follows that $\hat{\mathcal{U}}_u = \hat{\mathcal{V}} \cap (\mathfrak{g}^{-2} + \mathfrak{g}^{-1})$ satisfies (1) and (2). Finally, observe that the subspace
$$\text{span}\{ \hat{X}_u \ , \ X \in \mathfrak{g}^{-2} + \mathfrak{g}^{-1} \} \subset T_u P_{CM}(M)$$

is supplementary to the vertical subspace $\ker \pi_*|_u \subset T_u P_{CM}(M)$ and it has the same dimension of M. On the other hand,

$$\begin{aligned} &\operatorname{span}\{ \tilde{V} \in T_x M \ : \ \tilde{V} = \widetilde{\exp}_{u*}(X) \ , \ X \in T_0 \hat{\mathcal{U}}_u = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} \ \} \\ &= \operatorname{span}\{ \ \pi_*(\hat{X}_u) \ , \ X \in \mathfrak{g}^{-2} + \mathfrak{g}^{-1} \ \} \ . \end{aligned}$$

For this reason, the rank of the differential of $\widetilde{\exp}_u|_{\hat{\mathcal{U}}_u}$ is maximal at the point u. So, by choosing a smaller neighborhood $\hat{\mathcal{U}}_u$, we may always suppose that $\widetilde{\exp}_u|_{\hat{\mathcal{U}}_u}$ is a diffeomorphism onto its image, i.e. that (3) holds. □

The inverse map $\widetilde{\exp}^{-1} : \mathcal{U}_u \to \hat{\mathcal{U}}_u$ to $\widetilde{\exp}$ as defined in Lemma 3.1, will be called *system of normal coordinates, associated with the frame* $u \in P_{CM}(M)$. The neighborhood $\mathcal{U}_u$ of x, will be called *normalizable neighborhood of* x*, associated with the frame* $u \in P_{CM}(M)|_x$.

As before, for any $a \in G_Q = \operatorname{Aut}(Q)$, we denote by $F_a : \mathbb{C}^N \to \mathbb{C}^N$ the corresponding birational transformation and by $\hat{F}_a$ the lift of F_a to the Chern-Moser bundle of Q of the automorphism a.

LEMMA 3.2. *Let Q be an osculating quadric of a CM manifold, $P_{CM}(Q)$ the associated Chern-Moser bundle and u_o a fixed point of $P_{CM}(Q)$. Finally, consider the G_Q-equivariant identification map*

$$\imath : G_Q \longrightarrow P_{CM}(Q) \ , \qquad \imath(g) \stackrel{\text{def}}{=} \hat{F}_g(u_o) \ .$$

Then the Chern-Moser connection $\omega : TP_{CM}(Q) \to \mathfrak{g}_Q$ coincides with $(\imath^{-1})^*\psi$, where $\psi : G_Q \to \mathfrak{g}_Q$ is the Maurer-Cartan form of G_Q.

Moreover, for any $X \in \mathfrak{g}_Q$ and any $u = \hat{F}_g(u_o)$, $g \in \mathfrak{g}_Q$, the following properties hold:

(a) $\Phi_t^{\hat{X}}(u) = \hat{F}_{\exp(t\,\mathrm{Ad}_g(X))}(u)$;
(b) $F_{\exp(tp_-(X))}(\pi(u_o)) = F_{\exp(tX)}(\pi(u_o))$ (*for the definition of p_-, see Lemma 2.1* (iii)).

Proof. The first claim follows from the fact that the differential $d\omega = (\imath^{-1})^* d\psi$ and the 2-form $[\omega,\omega] = (\imath^{-1})^*[\psi,\psi]$ satisfy all linear relations which uniquely characterize the Chern-Moser connection, as defined in [2] and [11].

To prove (a) and (b), first of all, notice that since $\omega = (\imath^{-1})^*\psi$, for any left invariant vector field $X \in \mathfrak{g}_Q$, the associated fundamental vector field $\hat{X}$ is equal to

$$\hat{X} = \imath_*(X) \ .$$

So, for any $X \in \mathfrak{g}_{\mathcal{Q}}$ and any $u = \hat{F}_g(u_o) \in P_{CM}(\mathcal{Q})$, the curve

$$\alpha_t = \hat{F}_{g\cdot\exp(tX)}(u_o) = \hat{F}_{g\cdot\exp(tX)\cdot g^{-1}}(u)$$

is an integral curve of the vector field

$$\hat{F}_{g*}\imath_*(X) = \hat{F}_{g*}(\hat{X}) = \hat{X}$$

passing through the point u. This implies that $\Phi_t^{\hat{X}}(u) = \hat{F}_{\exp(t\,\mathrm{Ad}_g(X))}(u)$ and proves (a).

(b) follows from (a) with $u = u_o$, Lemma 2.1 (iii) and the identities

$$\pi\left(\hat{F}_{\exp(tX)}(u_o)\right) = F_{\exp(tX)}(\pi(u_o))$$

and $\pi\left(\hat{F}_{\exp(tp_-(X))}(u_o)\right) = F_{\exp(tp_-(X))}(\pi(u_o))$. □

Now, with the help of the previous two lemmas, we may introduce the main concept of this section and establish their first properties in the next Propositions 3.5 and 3.6.

DEFINITION 3.3. Consider a point $x \in M$ and a frame $u \in P_{CM}(M)|_x \subset P_{CM}(M)$, over the point x, and let $\mathcal{U}_u$ be a normalizable neighborhood of x associated with u. Let also $\hat{\mathcal{U}}_u = \widetilde{\exp}_u^{-1}(\mathcal{U}_u)$.

We call *$\mathcal{Q}$-normalization of $\mathcal{U}_u$ associated with* u the map

$$\underset{[u]}{\mathcal{N}} : \mathcal{U}_u \subset M \to \mathcal{Q} \subset \mathbb{C}^4 , \qquad \underset{[u]}{\mathcal{N}}(y) = F_{\exp(\widetilde{\exp}_u^{-1}(y))}(0) .$$

The complex coordinates of any point $\underset{[u]}{\mathcal{N}}(y) \in \mathcal{Q}$ are called *$\mathcal{Q}$-normalized complex coordinates of y determined by* u. The point $x = \pi(u)$ has clearly normalized complex coordinates equal to 0 and it is called the *center of the $\mathcal{Q}$-normalization* $\underset{[u]}{\mathcal{N}}$.

REMARK 3.4. Notice that, by Lemma 3.2 (b), if $M = \mathcal{Q}$ and $u \in P_{CM}(M)|_0$, a $\mathcal{Q}$-normalization associated with u is simply the identity map of $\mathcal{Q}$.

Observe also that a $\mathcal{Q}$-normalization $\mathcal{N}$ is almost never a CR map. In fact, it is simply a diffeomorphism between an open set of M and an open set of the osculating quadric $\mathcal{Q}$, which turns out to be a CR transformation if and only if M coincides or it is locally equivalent with $\mathcal{Q}$.

In the proof of Proposition 3.6 below we will use the following algebraic Lemma.

LEMMA 3.5. *Let $\mathfrak{g} = \mathfrak{g}_- + \mathfrak{p}$ be a decomposition of the Lie algebra $\mathfrak{g}$ into subalgebras, $p_- : \mathfrak{g} \to \mathfrak{g}_-$ the corresponding projection and P the subgroup of G with $Lie(P) = \mathfrak{p}$. Then for any $a \in P$ the mapping*

$$p_- \circ \mathrm{Ad}_{a^{-1}}|_{\mathfrak{g}_-} : \mathfrak{g}_- \to \mathfrak{g}_-$$

is a linear isomorphism.

Proof. Since $Lie(P) = \mathfrak{p}$, the subspace $\mathfrak{p} \subset \mathfrak{g}$ is invariant under $\mathrm{Ad}_{a^{-1}}$, thus $\mathrm{Ad}_{a^{-1}}$ has block form

$$\begin{pmatrix} p_- \circ \mathrm{Ad}_{a^{-1}}|_{\mathfrak{g}_-} & 0 \\ * & \mathrm{Ad}_{a^{-1}}|_{\mathfrak{p}} \end{pmatrix}.$$

On the other hand $\mathrm{Ad}_{a^{-1}}$ is an isomorphism of $\mathfrak{g}$. This is only possible if $p_- \circ \mathrm{Ad}_{a^{-1}}|_{\mathfrak{g}^-}$ is a linear isomorphism. □

PROPOSITION 3.6. *Let $\underset{[u]}{\mathcal{N}}, \underset{[u']}{\mathcal{N}} : \mathcal{V} \subset M \to \mathbb{C}^4$ be two Q-normalizations, centered at $x = \pi(u) = \pi(u')$, and let $a \in H_Q$ be the unique element such that*

$$u' = u \cdot a\ .$$

Then there exists an open subset $\mathcal{V}' \subset \mathcal{V}$, so that

$$\left.\underset{[u']}{\mathcal{N}}\right|_{\mathcal{V}'} = F_a \circ \left.\underset{[u]}{\mathcal{N}}\right|_{\mathcal{V}'}\ .$$

Proof. Let us denote by $\hat{\mathcal{U}} = \widetilde{\exp}_u^{-1}(\mathcal{V})$ and $\hat{\mathcal{U}}' = \widetilde{\exp}_{u'}^{-1}(\mathcal{V})$ the open sets in $\mathfrak{g}^{-2} + \mathfrak{g}^{-1}$, which are image of $\mathcal{V}$ under the normal systems of coordinates $\widetilde{\exp}_u^{-1}$ and $\widetilde{\exp}_{u'}^{-1}$, respectively.

From Lemma 2.1, for any $X \in \mathfrak{g}_Q$ and any t, for which the flow $\Phi_t^{\hat{X}}$ is defined, we have that

$$R_a \circ \Phi_t^{\hat{X}} = \Phi_t^{R_{a*}(\hat{X})} \circ R_a = \Phi_t^{\widehat{\mathrm{Ad}_{a^{-1}}(X)}} \circ R_a\ .$$

It follows that, for any $X \in \hat{\mathcal{U}}$,

$$\begin{aligned} \widetilde{\exp}_u(X) &= \pi(\Phi_1^{\hat{X}}(u)) = \pi(R_a \circ \Phi_1^{\hat{X}}(u)) \\ &= \pi(\Phi_1^{\widehat{\mathrm{Ad}_{a^{-1}}(X)}}(u \cdot a)) = \pi(\Phi_1^{\widehat{p_- \circ \mathrm{Ad}_{a^{-1}}(X)}}(u')) \end{aligned} \tag{3.2}$$

where we applied Lemma 2.1 (iii).

By Lemma 3.5

$$p_- \circ \mathrm{Ad}_{a^{-1}}|_{\mathfrak{g}^{-2}+\mathfrak{g}^{-1}} : \mathfrak{g}^{-2} + \mathfrak{g}^{-1} \to \mathfrak{g}^{-2} + \mathfrak{g}^{-1}$$

is a linear isomorphism. It follows that the set

$$\hat{\mathcal{U}}'' = p_- \circ \mathrm{Ad}_{a^{-1}}(\hat{\mathcal{U}})$$

is an open neighborhood of 0 and, for any X that belongs to $(p_- \circ \mathrm{Ad}_{a^{-1}})^{-1}(\hat{\mathcal{U}}'' \cap \hat{\mathcal{U}}')$, we may compute both values $\widetilde{\exp}_u(X)$ and $\widetilde{\exp}_{u'}(p_- \circ \mathrm{Ad}_{a^{-1}}(X))$. Then, by (3.2),

$$\widetilde{\exp}_u(X) = \widetilde{\exp}_{u'}(p_- \circ \mathrm{Ad}_{a^{-1}}(X)) .$$

This amounts to say that, for any y in a suitable neighborhood $\mathcal{V}'$ of x,

$$\widetilde{\exp}_{u'}^{-1}(y) = p_- \circ \mathrm{Ad}_{a^{-1}}(\widetilde{\exp}_u^{-1}(y)) .$$

and hence that

$$\underset{[u']}{\mathcal{N}}(y) = F_{\exp(p_- \circ \mathrm{Ad}_{a^{-1}}(\widetilde{\exp}_u^{-1}(y)))}(0) .$$

Using Lemma 3.2 (b), we get that

$$\underset{[u']}{\mathcal{N}}(y) = F_{a \circ \exp(\widetilde{\exp}_u^{-1}(y)) \circ a^{-1}}(0) = F_a \circ F_{\exp(\widetilde{\exp}_u^{-1}(y))} \circ F_{a^{-1}}(0) = F_a(\underset{[u]}{\mathcal{N}}(y)),$$

where we used the fact that $F_{a^{-1}}(0) = 0$. □

PROPOSITION 3.7. *Let $f : M \to M'$ be a CR transformation, mapping the point x into $x' = f(x)$ and let $\mathcal{N} : \mathcal{U} \subset M \to \mathbb{C}^4$, $\mathcal{N}' : \mathcal{U}' \subset M' \to \mathbb{C}^4$ be two $\mathcal{Q}$-normalizations associated with $u \in P_{CM}(M)$, with $x = \pi(u)$, and with $u' \in P_{CM}(M')$, with $x' = \pi(u')$, respectively. Let also $a = a(f, u, u') \in H_{\mathcal{Q}}$ be the unique element such that*

$$\hat{f}(u) = u' \cdot a ,$$

where $\hat{f} : P_{CM}(M) \to P_{CM}(M')$ is the lifted map of f.

Then

$$\mathcal{N}' \circ f \circ \mathcal{N}^{-1} = F_a .$$

In other words, the function $\tilde{f} = \mathcal{N}' \circ f \circ \mathcal{N}^{-1}$, which represents f in the systems of $\mathcal{Q}$-normalized complex coordinates $\mathcal{N}$ and $\mathcal{N}'$, coincides with the restriction on $\mathcal{Q}$ of the birational transformation $F_a : \mathbb{C}^4 \to \mathbb{C}^4$.

Proof. Consider the lifted automorphism $\hat{f} : P_{CM}(M) \to P_{CM}(M')$ and recall that, for any $X \in \mathfrak{h}_{\mathcal{Q}}$, $\hat{f}_*(\hat{X}) = \hat{X}'$, where $\hat{X}$ and $\hat{X}'$ represent the fundamental vector fields associated with X on $P_{CM}(M)$ and $P_{CM}(M')$, respectively (see Lemma 2.1). So, for any $X \in \widetilde{\exp}_u^{-1}(\mathcal{U})$, from Lemma 2.1 (iii) it follows that

$$\begin{aligned} f \circ \widetilde{\exp}_u(X) &= f \circ \pi(\Phi_1^{\hat{X}}(u)) = \pi(\hat{f} \circ \Phi_1^{\hat{X}}(u)) = \pi(\Phi_1^{\hat{f}_*(\hat{X})}(\hat{f}(u))) \\ &= \pi(\Phi_1^{\hat{X}'}(u' \cdot a)) = \pi(R_a \circ \Phi_1^{\widehat{\mathrm{Ad}_a(X)}'}(u')) \\ &= \pi(\Phi_1^{\widehat{\mathrm{Ad}_a(X)}'}(u')) = \pi(\Phi_1^{\widehat{p_- \circ \mathrm{Ad}_a(X)}'}(u')). \end{aligned}$$

Therefore, for any $X \in \widetilde{\exp}_u^{-1}(\mathcal{U} \cap f^{-1}(\mathcal{U}'))$ and any $y \in \mathcal{U} \cap f^{-1}(\mathcal{U}')$

$$f \circ \widetilde{\exp}_u(X) = \widetilde{\exp}_{u'}(p_- \circ \mathrm{Ad}_a(X)) \ \Rightarrow\ \widetilde{\exp}_{u'}^{-1}(f(y)) = p_- \circ \mathrm{Ad}_a(\widetilde{\exp}_u^{-1}(y)).$$

Using Lemma 3.2 (b), we get that at all points of $\mathcal{U} \cap f^{-1}(\mathcal{U}')$,

$$\begin{aligned}\mathcal{N}'(f(y)) &= F_{\exp(\widetilde{\exp}_{u'}^{-1}(f(y)))}(0) \\ &= F_{\exp(p_- \circ \mathrm{Ad}_a(\widetilde{\exp}_u^{-1}(y)))}(0) = F_{\exp(\mathrm{Ad}_a(\widetilde{\exp}_u^{-1}(y)))}(0) \\ &= F_a \circ F_{\exp(\widetilde{\exp}_u^{-1}(y))} \circ F_{a^{-1}}(0) = F_a \circ \mathcal{N}(y) ,\end{aligned}$$

where we used again the fact that $F_a(0) = 0$. □

From the previous proposition, we get the following immediate corollary.

COROLLARY 3.8. *Let $\mathcal{N} : \mathcal{U} \subset M \to \mathbb{C}^4$ be a $\mathcal{Q}$-normalization associated with $u \in P_{CM}(M)$, with $x = \pi(u)$. Then for any CR transformation $f : M \to M$ fixing the point x, there exists an element $a \in H_{\mathcal{Q}}$ such that $\mathcal{N} \circ f \circ \mathcal{N}^{-1}$ (i.e. the expression of f in the $\mathcal{Q}$-normalized coordinates) is equal to*

$$\mathcal{N} \circ f \circ \mathcal{N}^{-1} = F_a.$$

REMARK 3.9. Notice that Corollary 3.8 implies the following important fact. Let $H \subset \mathrm{Aut}_{x_o}(M) \subset \mathrm{Aut}(M)$ be a subgroup of the isotropy subgroup of automorphisms of the CM manifold M, fixing a point x_o, and let $\tilde{H}$ the subgroup of $H_{\mathcal{Q}}$ defined by

$$\tilde{H} = \{ \ \tilde{h} \in H_{\mathcal{Q}} \ : \ \hat{h}(u) = u \cdot \tilde{h} \ , \ h \in H \ \} \ .$$

Then, for any y in a $\mathcal{Q}$-normalizable neighborhood of x_o, the $\mathcal{Q}$-normalization $\mathcal{N}$ sends diffeomorphically any orbit

$$H(y) = \{ \ y' \in \mathcal{U} \ : \ y' = h(y) \ , \ h \in H \ \} \subset M$$

onto the corresponding orbit of $\mathcal{Q}$

$$\tilde{H}(\mathcal{N}(y)) = \{ \ z \in \mathcal{N}(\mathcal{U}) \ : \ z = \tilde{h}(\mathcal{N}(y)) \ , \ \tilde{h} \in \tilde{H} \ \} \subset \mathcal{Q} \ .$$

In this way, up to a diffeomorphism, one may analyze completely the local orbits of a stability group H just by looking at the orbits in $\mathcal{Q}$ of the corresponding automorphism group $\tilde{H}$.

This gives considerable simplifications of quite a few proofs on existence of non-compact automorphisms of strongly pseudoconvex real hypersurfaces, of hyperbolic manifolds and of elliptic manifolds (see e.g. [1], [6], [13] and [9]). A detailed discussion of such applications and new results will be given in a forthcoming paper.

4. Comparisons between normal forms and $\mathcal{Q}$-normalizations in the real analytic case

In this section, we make explicit the relation between the embeddings in normal form of a real analytic CM manifold and the $\mathcal{Q}$-normalizations, as introduced in the previous section.

Let $M \subset \mathbb{C}^N$ be a real analytic CM submanifold of $\mathbb{C}^N$, which is embedded in normal form and let $\rho = 0$ be the uniquely associated defining equation in normal form. We want to show that there exists a naturally distinguished element u in the fiber of the Chern-Moser bundle $P_{CM}(M)$ over the point $0 \in M$ which we will call the *canonical element*. Notice that the canonical element allows to identify the fibers of $P_{CM}(M)$ with the set of Q-normalizations. We proceed considering each possibility for M separately.

First of all, let us assume the $M \subset \mathbb{C}^{n+1}$ is a real analytic Levi non-degenerate hypersurface embedded in normal form and let

$$\rho(z, \bar{z}, \operatorname{Re} w, \operatorname{Im} w) = \operatorname{Im} w - \langle z, z\rangle - \sum_{k,\ell \geq 2} \rho_{k,\ell}(z, \bar{z}, \operatorname{Re} w) = 0$$

be its uniquely associated defining equation in normal form (see remark after Definition 4.1). Recall that, by the definition in [2], the elements of $\pi^{-1}(0) \subset P_{CM}(M)$ are adapted linear frames $(e_0, \ldots, e_{2n+1})$ at the tangent spaces $T_\theta \hat{E}$ of the conormal bundle $\hat{\pi} : \hat{E} \to M$ at the points $\theta \in \hat{E}$, which lie above $\hat{\pi}(\theta) = 0$. The points $\theta \in \hat{\pi}^{-1}(0)$ are non-vanishing 1-forms such that

$$\ker \theta = \mathcal{D}_0 \ , \tag{4.1}$$

where $\mathcal{D}_0$ is the tangent complex space at 0. In order to simplify the notation, we will represent any point $u \in \pi^{-1}(0)$ as $(2n + 3)$-tuple $(\theta; e_0, \ldots, e_{2n+1})$, where θ satisfies (4.1) and $(e_0, \ldots, e_{2n+1})$ is an adapted frame at θ.

Now, consider the element $u_o = (\theta; e_0, \ldots, e_{2n+1}) \in \pi^{-1}(0)$ defined as follows:

(1) $\theta \overset{\text{def}}{=} d\rho \circ J|_{T_0M} = d(\operatorname{Im} w) \circ J|_{T_0M}$; here J denotes the standard complex structure of $\mathbb{C}^{n+1}$ given by the multiplication by $\sqrt{-1}$;

(2) the first $2n + 1$ elements of the coframe, which is dual to $(e_0, \ldots, e_{2n+1})$, are $e^0 \overset{\text{def}}{=} \hat{\pi}^*(\theta)$ and

$$e^{2i-1} \overset{\text{def}}{=} \hat{\pi}^* \left(d(\operatorname{Re} z^i)|_{T_0M}\right) , \ e^{2i} \overset{\text{def}}{=} \hat{\pi}^* \left(d(\operatorname{Im} z^i)|_{T_0M}\right) \quad \text{for } \ 1 \leq i \leq n \ ;$$

(3) the last element e^{2n+1} is defined as follows: for any $x \in M$, any element $\theta \in \hat{\pi}^{-1}(x)$ can be written as $\theta = t_\theta \cdot (d\rho \circ J)|_{T_xM}$, for some $t_\theta \in \mathbb{R}^*$; so we have a natural real function τ

$$\tau : \hat{E} \to \mathbb{R}^* , \qquad \tau(\theta) = t_\theta ;$$

we set

$$e^{2n+1} \overset{\text{def}}{=} \frac{1}{\tau(\theta)} d\tau|_\theta .$$

From the definition in [2], we have that $u_o = (\theta; e_0, \ldots, e_{2n+1})$ defined by (1)-(3) is an adapted coframe. Notice also that u_o is uniquely determined by the defining function ρ in normal form and hence it is uniquely associated to the embedded manifold M. We will call such u_o the *canonical element associated with* M.

Secondly, let us assume the $M \subset \mathbb{C}^4$ is a real analytic hyperbolic or elliptic submanifold embedded in normal form and let

$$\mathrm{Im}(w^i) - \langle z, z\rangle^i - \sum_{k_1,k_2,\ell_1,\ell_2 \geq 1} \rho^i_{k_1,k_2,\ell_1,\ell_2}(z, \bar{z}, \mathrm{Re}\, w) = 0 , \qquad i = 1,2$$

be its uniquely associated defining equations in normal form. As above, any element $u \in \pi^{-1}(0) \subset P_{CM}(M)$ can be denoted by $u = (\theta^1, \theta^2; e_1, \ldots, e_8)$, where: (θ^1, θ^2) are two 1-forms such that

$$\ker \theta^1 \cap \ker \theta^2 = \mathcal{D}_0 \tag{4.2}$$

and which satisfy certain additional conditions (see [11] for the exact definition); $(e_1, \ldots, e_8)$ is an adapted frame in $T_{(\theta^1,\theta^2)}\hat{E}$.

Then, we consider the element $u_o = (\theta^1, \theta^2; e_1, \ldots, e_8) \in \pi^{-1}(0)$ defined as follows:

(1') $\theta^i \overset{\text{def}}{=} d\rho^i \circ J|_{T_0M} = d(\mathrm{Im}\, w^i) \circ J|_{T_0M}$;

(2') the first 6 elements of the coframe, which is dual to $(e_0, \ldots, e_6)$, are $e^1 = \hat{\pi}^*(\theta^1)$, $e^2 = \hat{\pi}^*(\theta^2)$ and

$$e^{2i+1} \overset{\text{def}}{=} \hat{\pi}^* \left(d(\mathrm{Re}\, z^i)|_{T_0M}\right) , \ e^{2i+2} \overset{\text{def}}{=} \hat{\pi}^* \left(d(\mathrm{Im}\, z^i)|_{T_0M}\right) \quad \text{for} \ \ i = 1,2 ;$$

(3') the last two elements e^7 and e^8 are defined as follows: for any $x \in M$, any pair $(\theta^1, \theta^2) \in \hat{\pi}^{-1}(x)$ can be written as $\theta = A_\theta \cdot (d\rho \circ J)|_{T_xM}$, for some $A_\theta \in G$; where G is the structure group of $\hat{\pi} : \hat{E} \to M$; so we have a natural function τ

$$\tau : \hat{E} \to G , \qquad \tau(\theta^1, \theta^2) = A_\theta ;$$

if we fix a suitable basis E_1, E_2 for $Lie(G)$, we may write the differential $d\tau$ as $d\tau = E_1 \otimes d\tau^1 + E_2 \otimes d\tau^2$ and, using the same arguments for (4.5) in [11], one can check that there exists exactly two real numbers $0 \neq \lambda^i$, $i = 1, 2$ so that $(e^1, \ldots, e^8)$ with

$$e^7 \stackrel{\text{def}}{=} \frac{1}{\lambda^1} d\tau^1|_{(\theta^1,\theta^2)}, \qquad e^8 \stackrel{\text{def}}{=} \frac{1}{\lambda^2} d\tau^2|_{(\theta^1,\theta^2)}$$

is an adapted coframe as defined in [11], Section 4.

Also in this case, the element $u_o = (\theta; e_0, \ldots, e_{2n+1}) \in P_{CM}(M)$ defined by (1')-(3') is uniquely determined by the defining function ρ in normal form and hence it is uniquely associated to the embedded manifold M. Again, we will call such u_o the *canonical element associated with M*.

The existence of the canonical element brings us to introduce the following definition.

DEFINITION 4.1. Let $M \subset \mathbb{C}^n$ be a real analytic CM manifold embedded in normal form and let $u_M \in \pi^{-1}(0) \subset P_{CM}(M)$ be its canonical element. The $\mathcal{Q}$-normalization at 0 associated with u_M will be called *canonical $\mathcal{Q}$-normalization of M* the $\mathcal{Q}$-normalization at 0 and it will be denoted by $\underset{[M]}{\mathcal{N}} : \mathcal{U} \subset M \to \mathbb{C}^n$.

The following proposition determines how the canonical elements change if one replaces an embedding in normal form with another.

PROPOSITION 4.2. *Let $M \subset \mathbb{C}^n$ be a CM manifold embedded in normal form and let $u_M \in \pi^{-1}(0) \subset P_{CM}(M)$ be its canonical element. For any normalizing map $\underset{[M,a]}{\mathbb{N}} : \mathcal{V} \subset \mathbb{C}^n \to \mathbb{C}^n$, let us denote by $M_a \stackrel{\text{def}}{=} \underset{[M,a]}{\mathbb{N}}(M)$ the new embedded CM manifold in normal form and by $\underset{[M,a]}{\widehat{\mathbb{N}}} : P_{CM}(M) \to P_{CM}(M_a)$ denote the lifted map between the Chern-Moser bundles of M and M_a, respectively.*

Then, for any $a \in H_{\mathcal{Q}}$, the canonical element u_{M_a} of M_a is equal to

$$u_{M_a} = \underset{[M,a]}{\widehat{\mathbb{N}}}(u_M) \cdot a^{-1} .$$

Proof. We will prove the claim just for the hypersurface case, since for the other two cases the arguments are analogous. To simplify the notation, let us write the components of $\underset{[M,a]}{\mathbb{N}}$ and $\underset{[M,a]}{\mathbb{N}}^{-1} = \underset{[M_a,a^{-1}]}{\mathbb{N}}$ as

$$\underset{[M,a]}{\mathbb{N}} = (f^{z_1}, \ldots, f^{z_n}, f^w) , \qquad \underset{[M,a]}{\mathbb{N}}^{-1} = (g^{z_1}, \ldots, g^{z_n}, g^w) .$$

and let $u_M = (\theta; e_0, \dots, e_{2n+1})$ be the canonical element of M. Finally let

$$(\lambda_a, U_a, \alpha_a, r_a)$$

be the parameter which determine uniquely the element F_a, as defined in Section 2.1.

By definition, the 1-form θ is mapped by $\underset{[M,a]}{\mathbb{N}}$ into the 1-form

$$\theta' = (\underset{[M,a]}{\mathbb{N}}^{-1})^* \theta = d(\operatorname{Im} g^w) \circ J|_{T_0 M_a} ,$$

where we used the fact that $(\underset{[M,a]}{\mathbb{N}})_* \circ J \circ (\underset{[M,a]}{\mathbb{N}}^{-1})_* = J$, since $\underset{[M,a]}{\mathbb{N}}$ is holomorphic. On the other hand, by the definition of normalizing map given in [2], we have that $d(\operatorname{Im} g^w) = \lambda_a^2 d \operatorname{Im} w$ and that $T_0 M_a$ and $T_0 M$ can be both identified with the subspace $\{\operatorname{Im} w = 0\} \subset \mathbb{C}^{n+1}$. Hence θ and θ' can be identified with two 1-forms on $\{\operatorname{Im} w = 0\}$ and they satisfy:

$$\theta' = \lambda_a^2 \theta . \tag{4.3}$$

Similarly, we obtain that the 1-forms $(e^0, e^1, \dots, e^{2n})$ are transformed by $\underset{[M,a]}{\mathbb{N}}$ into the 1-forms $(e'^0, e'^1, \dots, e'^{2n})$ on the auxiliary bundle $\hat{\pi}' : \hat{E}' \to M_a$ equal to

$$e'^0 = \hat{\pi}'^*(\theta') , \qquad e'^i = \lambda_a \left[(U_a)_i^j \hat{\pi}'^*(x_j) + \alpha_a^i \hat{\pi}'^*(\theta) \right] ,$$

where by x_j we denote the real coordinates $(\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Im} z_n)$. The computation of the image of the 1-form e^{2n+1}, even if straightforward, is more complicate and we omit it.

In all cases, by comparison with the properties (1)-(3), which define the canonical element of M_a, and by the explicit formulae for the right action of a, we get that

$$\widehat{\underset{[M,a]}{\mathbb{N}}}(u_M) = u_{M_a} \cdot a$$

and this concludes the proof. □

From Proposition 4.2, we get the following theorem which allows to express any normalizing map $\underset{[M,a]}{\mathbb{N}}$ in terms of $\mathcal{Q}$-normalization maps.

THEOREM 4.3. *Let $M \subset \mathbb{C}^n$ be a CM manifold in normal form and, for any normalization map $\underset{[M,a]}{\mathbb{N}} : \mathcal{V} \subset \mathbb{C}^n \to \mathbb{C}^n$, let $M_a \overset{\text{def}}{=} \underset{[M,a]}{\mathbb{N}}(M)$.*

Then, there exists a neighborhood $\mathcal{U} \subset M$ of the origin, such that for any $a \in H_Q$, the normalization map $\underset{[M,a]}{\mathbb{N}}$ *can be written as*

$$\underset{[M,a]}{\mathbb{N}}\Big|_{\mathcal{U}} = \underset{[M_a]}{\mathcal{N}}^{-1} \circ F_a \circ \underset{[M]}{\mathcal{N}} .$$

In particular, any local automorphism of M fixing the origin is of the form

$$\underset{[M,a]}{\mathbb{N}}\Big|_{\mathcal{U}} = \underset{[M]}{\mathcal{N}}^{-1} \circ F_a \circ \underset{[M]}{\mathcal{N}} ,$$

for some $a \in H_Q$.

Proof. It follows immediately from Proposition 4.2 and Proposition 3.7. □

Theorem 4.3 gives a geometrical interpretation of several crucial analogies between the normalizing map $\underset{[M,a]}{\mathbb{N}}$ and the associated transformations F_a of the osculating quadric. It shows that each normalizing maps $\underset{[M,a]}{\mathbb{N}}$ is identifiable with the corresponding F_a if expressed in a special system of real coordinates, namely the $\mathcal{Q}$-coordinates $\underset{[M]}{\mathcal{N}}$ on the starting points and the $\mathcal{Q}$-coordinates $\underset{[M_a]}{\mathcal{N}}$ on the target points.

5. Extensions of local CR equivalences

Proposition 3.7 and Theorem 4.3 give immediate results on the extendibility of local CR equivalences between smooth and real analytic CM manifolds. To give an example of such possible applications, we present a proposition on the local CR equivalences between real analytic CM manifolds in normal form, which is an analogue of the theorem for real hypersurfaces given at p.185 in [15].

Let us denote by $\mathcal{M}(\delta, m)$ the class of real analytic CM manifolds $M \subset \mathbb{C}^n$, which are embedded in normal form and such that the $\mathcal{Q}$-normalization $\underset{[M]}{\mathcal{N}}$ satisfies the following two requirements:

(1) the domain of definition of $\underset{[M]}{\mathcal{N}}$ contains the ball of radius δ:

$$\mathcal{B}_\delta(0) \cap M = \{ z \in \mathbb{C}^n \ : \ |z| < \delta \} \cap M \ ;$$

(2) for any $0 \le r \le \frac{\delta}{m}$,

$$\underset{[M]}{\mathcal{N}}^{-1}(\mathcal{B}_r(0) \cap \mathcal{Q}) \subset \mathcal{B}_{mr}(0) \cap M \ .$$

Notice that, by the mean value theorem, in order to see if a given CM manifold M is in the class $\mathcal{M}(\delta, m)$, it is enough to check that (1) holds and that, for any i, j, the following inequality holds. Let $\mathcal{N}^i$ be the components of the map $\underset{[M]}{\mathcal{N}}$ and $\tilde{\mathcal{N}}^i$ the components of the inverse map $\underset{[M]}{\mathcal{N}}^{-1}$. Then (2) holds if for any i, j

$$\sup_{p \in \underset{[M]}{\mathcal{N}}(\mathcal{B}_{\frac{\delta}{m}}(0) \cap M)} \left| \frac{\partial \tilde{\mathcal{N}}^i}{\partial z_j} \right| \le m$$

and, by the derivation rule of inverse functions, this is guaranteed if

$$\frac{\sup_{p \in \mathcal{B}_\delta(0) \cap M} \left| \frac{\partial \mathcal{N}^i}{\partial z_j} \right|^{\dim M - 1}}{\inf_{p \in \mathcal{B}_\delta(0) \cap M} |\det J(\mathcal{N})|} \le m \ .$$

Now, from Theorem 4.3 we get directly the following corollary.

COROLLARY 5.1. *Let $M, M' \subset \mathbb{C}^n$ be two CM manifolds in normal form in the class $\mathcal{M}(\delta, m)$ and let $H : \mathbb{C}^n \to \mathbb{C}^n$ be a local biholomorphism, which fixes the origin, maps $M \cap \mathcal{U}$ into a subset of M' and so that the moduli of the components of the second order jet $j^2(H)_0$ and the moduli of the first order jet $j^1(H^{-1})_0$ are less or equal to $k > 0$. Then H extends holomorphically to the open set $\mathcal{U}' = \underset{[M]}{\mathcal{N}}^{-1}(\mathcal{B}_{\delta_*}(0) \cap \mathcal{Q})$, where $\delta_* = \delta_*(\delta, m, k)$ is a constant depending only on δ, m and k.*

Proof. By Theorems 2.5 and 4.3, H is of the form

$$H = \underset{[M,a]}{\mathbb{N}}\Big|_{\mathcal{U} \cap M} = \underset{[M_a]}{\mathcal{N}}^{-1} \circ F_a \circ \underset{[M]}{\mathcal{N}} \ , \tag{5.1}$$

for some $a \in H_{\mathcal{Q}}$. Since the equation of M is in normal form and H is a normalization the 2-jet $j^2(H)_0$ equals $j^2(F_a)_0$.

Now, for any $a \in H_{\mathcal{Q}}$, let us denote by $\epsilon(a)$ the supremum of the radii $r \le \frac{\delta}{m}$ such that the set $\mathcal{B}_r(0) \cap \mathcal{Q}$ is included in the domain of definition of F_a and it is contained in

$$F_a^{-1}(\mathcal{B}_{\frac{\delta}{m}} \cap \mathcal{Q}) \ .$$

From (5.1) and the definition of δ and m, it is clear that H extends holomorphically on

$$\mathop{\mathcal{N}}_{[M]}^{-1}\left(\mathcal{B}_{\epsilon(a)}(0)\cap\mathcal{Q}\right).$$

If we take the minimum of $\epsilon(a)$ on the compact subset of $H_{\mathcal{Q}}$, corresponding to those elements whose components of the second order jet and the first order inverse jet are bounded by k, we get the constant $\delta^* = \delta^*(\delta, m, k)$ we were looking for. □

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