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Modeling and Optimization of RMS Pulse Width for Transmission in Dispersive Nonlinear Fibers

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Simple algebraic expressions are derived to approximate the optimal input RMS pulse width and the resulting output RMS pulse width in single-mode fibers. The results are compared with the previously published methods and with numerical results by the split-step Fourier method. In addition, for a transform-limited Gaussian input pulse, it is shown that there is no optimum input pulse width to minimize the output spectrum width. Finally, with fiber nonlinearity, it is shown mathematically that there is not an optimum input pulse width to minimize the product, $\sigma_t\sigma_\omega$, which is inversely proportional to the transmission capacity of WDM systems.

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I. INTRODUCTION

Root-mean-square (RMS) pulse width is of interest since it provides a useful metric for assessing performance limitations in fiber-optic communication systems. The RMS pulse width is directly related to the maximum data rate through the commonly used design criterion [1]

$$\sigma_t R_b < \frac{1}{4} \tag{1}$$

where σ_t is the RMS pulse width at the output of the fiber, and R_b is the bit rate.

Also the RMS spectral width (σ_ω) determines basic design parameters of a wavelength division multiplexed (WDM) system such as channel spacing and bandwidth of optical filters. In a WDM system, the total transmission capacity (C_T) is defined by $C_T = N_{ch} \cdot R_b$, where N_{ch} = the number of channels and R_b = bit rate per channel. To maximize C_T, it is required to have the largest possible N_{ch}, which can be achieved by having the smallest RMS spectrum width at a given distance. Additionally, it is also required to have the largest bit rate, but bit rate and RMS pulse width at the output of fiber should satisfy some condition like Eq. (1), which says that the output RMS pulse width should be decreased to increase bit rate, R_b. Then, the question, how can we maximize the total transmission capacity, C_T, is equivalent to the question, how can we minimize the product, $\sigma_\omega\sigma_t$ at a given distance? Therefore, the product, $\sigma_t(z)\sigma_\omega(z)$, should be inversely proportional to the capacity of WDM systems.

In a fiber transmission system where dispersion is dominant (negligible fiber nonlinearities), it is known that there exists an optimum input RMS pulse width, σ_o, to minimize the output width, $\sigma_t(z)$, when a transform-limited pulse is transmitted. The optimum input RMS pulse width, $\sigma_{o,opt}$, and the resulting minimum output pulse width, σ_{min}, is given as a function of transmission distance, z, by [1]

$$\sigma_{o,opt} = \sqrt{|\beta_2|z/2}, \quad \sigma_{min} = \sqrt{|\beta_2|z} \tag{2}$$

where z is the transmission distance and β_2 is the second order group-velocity dispersion parameter.

In the case of dispersion alone, the magnitude of the pulse spectrum is invariant, and consequently $\sigma_w(z)$ remains constant at its initial value, σ_{w_o}. Therefore, the product $\sigma_t(z)\sigma_w(z)$ will have the same functional form as $\sigma_t(z)$. In this case, the optimum input pulse width given by Eq. (2) will also minimize the product, $\sigma_t(z)\sigma_w(z)$.

However, there appear to be no published results on maximizing C_T in terms of RMS width when fiber nonlinearities are no longer negligible. The main objective of this paper is to study the possible existence and the functional form of the optimum input pulse width to minimize the RMS width, $\sigma_t(z)$, and the product of the two RMS quantities, $\sigma_\omega(z)\sigma_t(z)$ in the presence of fiber nonlinearities. Even though the RMS quantities are strictly applicable only for the case of transmission of an isolated pulse, it is of interest to see how their functional forms compare to Eq. (2) in the presence of nonlinearities. The derived results can provide basic design parameters for optimum performance of WDM systems.

In the following section, more accurate modeling than prior treatments [2,3] will be attempted first. The functional forms of the optimum input pulse width based on the developed RMS models follow, and the results will be compared with numerical results obtained by the split-step Fourier method.

II. RMS WIDTH VARIATION IN A DISPERSIVE NONLINEAR FIBER

If the pulse shape at the input of the fiber has a Gaussian form, the pulse shape at the output of the fiber as well as the RMS width can be easily found in analytical form in the case of dispersion alone. However, it is impossible, in general, to get analytical forms of output pulse shape and the RMS widths considering fiber nonlinearity. M. J. Potasek et al. derived the RMS pulse width by approximating the nonlinearity as a lumped effect at the input to the fiber [3]. The result can give a rough idea of how the interaction of dispersion and nonlinearity can increase the output pulse width, but it is found that the derived RMS pulse width in Ref. 3 seriously overestimates the output pulse width. Recently, a more elegant mathematical way, namely the variational method, has been reported to model the RMS pulse width more accurately [4,5]. For a Gaussian pulse, the square of the broadening factor is found to be [4]

$$\left(\frac{\sigma_t(z)}{\sigma_t(0)}\right)^2 = 1 + \left(1 + \frac{\mathrm{sgn}(\beta_2)N^2}{\sqrt{2}}\right)\left(\frac{z}{L_D}\right)^2 + \left(\frac{1}{24}N^4 + \mathrm{sgn}(\beta_2)\frac{\sqrt{2}}{24}N^2\right)\left(\frac{z}{L_D}\right)^4 \tag{3}$$

where L_D is the dispersion length ($L_D = \frac{t_o^2}{|\beta_2|}$, where t_o = half-width at $1/e$ intensity and β_2 = the second order group-velocity dispersion parameter), L_N is the nonlinear length ($L_N = \frac{1}{\gamma P_{avg}}$, where γ = fiber nonlinear coefficient and P_{avg} = average power of optical signal), and N is the normalized nonlinearity parameter defined as $N^2 = \frac{L_D}{L_N} = \frac{\gamma P_{avg} t_o^2}{|\beta_2|}$. However, the variational method involves complicated mathematical treatments to obtain the above result. For a simpler than the variational method yet more accurate result than the result in Ref. 3, we can model the nonlinearity as lumped at the center of the propagation distance. If the normalized input pulse, $U(0,t)$, has a Gaussian shape such that $U(0,t) = \exp(-\frac{t^2}{2t_o^2})$, the pulse shape at the middle of the fiber can be expressed as

$$U\left(\frac{z}{2},t\right) = U_D\left(\frac{z}{2},t\right)\exp\left[i\frac{z}{L_N}\left|U_D\left(\frac{z}{2},t\right)\right|^2\right] \tag{4}$$

where $U_D(\frac{z}{2},t) = \frac{t_o}{(t_o^2 - i\beta_2 z/2)^{1/2}}\exp\left[-\frac{t^2}{2(t_o^2 - i\beta_2 z/2)}\right]$. Now the output pulse shape will be determined by propagating $U(\frac{z}{2},t)$ in the remaining half of the distance by assuming dispersion alone.

Using Parseval's theorem and the property of the Fourier transform, $\Im[t^n w(t)] = (-j2\pi)^{-n}\frac{d^n W(f)}{df^n}$, we can obtain the RMS width with assumption of the lumped fiber nonlinearity at the middle of the propagation distance as follows.

$$\left(\frac{\sigma_t(z)}{\sigma_t(0)}\right)^2 = 1 + \left(1 + \frac{\mathrm{sgn}(\beta_2)N^2}{\sqrt{2}\sqrt{1+1/4(z/L_D)^2}}\right)\left(\frac{z}{L_D}\right)^2 + \frac{N^4}{3\sqrt{3}(1+1/4(z/L_D)^2)^2}\left(\frac{z}{L_D}\right)^4 \tag{5}$$

Since the method used in Ref. 3 and Eq. (5) is analogous to the numerical algorithm of the split-step Fourier method with the step size, z and $z/2$, respectively, we may call them the *one-step method* and the *two-step method*, respectively.

Fig. 1 compares RMS pulse width evolutions by the one-step, the two-step, and the variational method with the simulated results by the split-step Fourier method when $\beta_2 > 0$ (normal dispersion). While the one-step method overestimates the RMS pulse width significantly as the propagation distance is increased even with a modest nonlinearity (N=2), the two-step

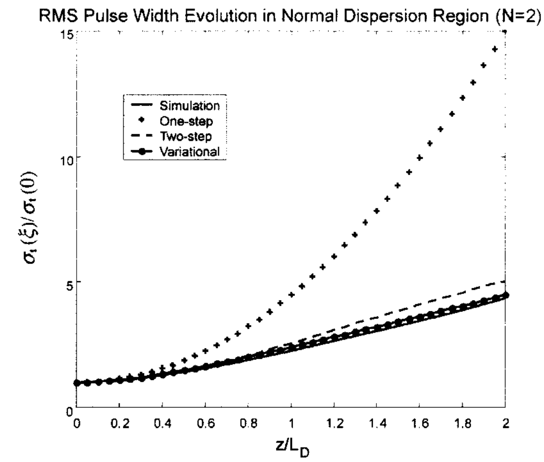

FIG. 1. Comparison of RMS pulse width models with the simulated one. Input pulse is a Gaussian shape and normal dispersion region is assumed. $\xi = z/L_D$.

method and the variational method follows the simulated result quite closely.

III. OPTIMIZATION OF RMS WIDTHS

1. Optimum input pulse width to minimize $\sigma_t(z)$

In the normalized nonlinear Schrödinger equation, transmission distance is often normalized by the dispersion distance, L_D [7]. However, L_D is defined in terms of the input pulse width. Since we want to optimize the RMS widths with respect to the input pulse width, it is more convenient to normalize distance by the nonlinear distance, L_N, which is independent of the initial pulse width. Additionally, it is convenient to define a new normalized quantity, s, such that

$$s(\zeta) = \frac{\sigma_t(\zeta)}{\sqrt{|\beta_2|L_N/2}} \tag{6}$$

where $\zeta = z/L_N$.

Notice that $s_o = s(0) = \frac{\sigma_o}{\sqrt{|\beta_2|L_N/2}} = \sqrt{\frac{L_D}{L_N}}$ is the same as the nonlinear parameter, N. Now the broadening factor, $\sigma_t(\zeta)/\sigma_o$, is the same as the ratio, $s(\zeta)/s_o$, and the optimization of $\sigma_t(\zeta)$ with respect to σ_o is the same as the optimization of $s(\zeta)$ with respect to s_o. With fixed physical parameters, the optimum input pulse will indicate the optimum nonlinearity constant, $s_{o,opt}(= N_{opt})$ in the system.

In the following, the functional forms of the optimum σ_o will be derived based on the two-step method.

Eq. (5) can be rewritten in terms of $\zeta = z/L_N$ using the relationship, $\frac{z}{L_D} = \frac{z}{L_N}\frac{L_N}{L_D} = \frac{\zeta}{N^2} = \frac{\zeta}{s_o^2}$, such as

$$\left(\frac{\sigma_t(z)}{\sigma_t(0)}\right)^2 = \frac{s^2}{s_o^2} = 1 + \frac{\mathrm{sgn}(\beta_2)}{\sqrt{2}\sqrt{1+\frac{1}{4}\frac{\zeta^2}{s_o^4}}}\frac{\zeta^2}{s_o^2} + \left[1 + \frac{\zeta^2}{3\sqrt{3}\left(1+\frac{1}{4}\frac{\zeta^2}{s_o^4}\right)^2}\right]\frac{\zeta^2}{s_o^4} \tag{7}$$

By differentiating s^2 with respect to s_o^2 and setting to zero, we get the quartic equation below. Here $x = s_o^2$.

$$x^4 - \zeta^2 x^2 + \frac{\mathrm{sgn}(\beta_2)\zeta^4}{4\sqrt{2}}\frac{x}{\left(1+\frac{1}{4}\frac{\zeta^2}{x^2}\right)^{3/2}} - \frac{\zeta^4}{3\sqrt{3}}\frac{x^2}{\left(1+\frac{1}{4}\frac{\zeta^2}{x^2}\right)^2} + \frac{\zeta^6}{3\sqrt{3}}\frac{1}{\left(1+\frac{1}{4}\frac{\zeta^2}{x^2}\right)^3} = 0 \tag{8}$$

In the extreme case of $\zeta \ll 1$, the above equation is simplified greatly such that $x^4 \approx \zeta^2 x^2$. From the simplified relation,

$$s_{o,opt} \approx \sqrt{\zeta}, \quad \sigma_{o,opt} \approx s_{o,opt}\sqrt{\frac{|\beta_2|L_N}{2}} = \sqrt{\frac{|\beta_2|z}{2}} \tag{9}$$

which gives

$$s_{min} \approx \sqrt{2\zeta}, \quad \sigma_{min} \approx \sqrt{|\beta_2|z} \tag{10}$$

Eqs. (9) and (10) lead to the case of dispersion alone. This is not a surprising result since the condition of $\zeta \ll 1$ indicates the propagation distance is much smaller than the nonlinear distance, L_N, which means the nonlinearity has little effect on the transmission of the pulse.

In the other extreme case of $\zeta \gg 1$, Eq. (8) reduces to

$$x^4 - \frac{\zeta^4}{3\sqrt{3}}x^2 + \frac{\zeta^6}{3\sqrt{3}} \approx 0 \tag{11}$$

In Eq. (11), it is assumed $\frac{1}{4}\frac{\zeta^2}{x^2} = \frac{1}{4}\frac{\zeta^2}{s_o} \ll 1$, the validity of which will be checked later.

Then the solution is

$$x^2 = s_{o,opt}^4 \approx \frac{1}{2}\left(\frac{\zeta^4}{\sqrt{27}} \pm \sqrt{\frac{\zeta^8}{27} - \frac{4}{\sqrt{27}}\zeta^6}\right) \tag{12}$$

Since we used the assumption $\frac{1}{4}\frac{\zeta^2}{x^2} \ll 1$, the positive sign (+) is appropriate in Eq. (12). Then

TABLE 1. Summary of the optimum input pulse widths and the minimum output pulse widths in the normal dispersion region by the various methods. ($\zeta = \frac{z}{L_N} \gg 1$)

	One-step Method	Two-step Method	Variational Method [4,5]	Split-step Fourier Simulation and Curve Fitting
$s_{o,opt}(= N_{opt})$	$\frac{\sqrt{2}}{27^{1/8}}\zeta = 0.94\zeta$	$\frac{1}{27^{1/8}}\zeta = 0.66\zeta$	$\frac{1}{27^{1/8}}\zeta = 0.452\zeta$	$0.2897\zeta+0.9056$
s_{min}	$1.786\times\zeta$	$1.254\times\zeta$	$1.056\times\zeta$	$0.9751\zeta+0.6427$
$\sigma_{o,opt}$	$0.662z\sqrt{\frac{\vert\beta_2\vert}{L_N}}$	$0.468z\sqrt{\frac{\vert\beta_2\vert}{L_N}}$	$0.32z\sqrt{\frac{\vert\beta_2\vert}{L_N}}$	
$\frac{\sigma_{min}}{\sigma_{o,opt}} = \frac{s_{min}}{s_{o,opt}}$	1.9	1.9	2.34	3.35

$$s_{o,opt} \approx \frac{\zeta}{27^{1/8}} = 0.66\zeta, \quad \sigma_{o,opt} \approx s_{o,opt}\sqrt{\frac{|\beta_2|L_N}{2}} = 0.468z\sqrt{\frac{|\beta_2|}{L_N}} \tag{13}$$

From Eq. (13), $\frac{1}{4}\frac{\zeta^2}{s_{o,opt}^4} = \frac{1}{4}\frac{\sqrt{27}}{\zeta^2} \ll 1$ when $\zeta \gg 1$, which validates the assumption made in Eq. (11). With Eqs.(7) and (13),

$$\begin{aligned} s_{min}^2 &= \frac{2}{27^{1/4}}\zeta^2 + \frac{\mathrm{sgn}(\beta_2)}{\sqrt{2}}\zeta^2 + 27^{1/4} \\ &\approx \zeta^2\left(\frac{2}{27^{1/4}} + \frac{\mathrm{sgn}(\beta_2)}{\sqrt{2}}\right) \end{aligned}$$

Although Eq. (13) predicts that the optimum pulse width is independent of the sign of β_2, the RMS pulse width is more meaningful for estimating distortion effects in fiber transmission in the case of normal dispersion. In this case, $\mathrm{sgn}(\beta_2) = +1$. Then

$$\begin{aligned} s_{min} &\approx 1.259\zeta = 1.9s_{o,opt}, \\ \sigma_{min} &\approx 1.9\sigma_{o,opt} = 0.89z\sqrt{\frac{|\beta_2|}{L_N}} \end{aligned} \tag{14}$$

When $\zeta \gg 1$, which is the more interesting case, the analytical results are summarized in Table 1. For comparison purposes, the previously published analytical results by one step method and by the variational method and the simulation result by the split-step Fourier method are also included. Normal dispersion is assumed in all the cases. It is interesting to observe that all the analytical methods predict that $\sigma_{o,opt}$ and σ_{min} are linearly proportional to the propagation distance, z unlike the case of dispersion alone where $\sigma_{o,opt}$ and σ_{min} are proportional to the square root of the propagation distance, z. As the analytical methodology gets more sophisticated, the proportionality constants get smaller and closer to the simulated values. This is because, in simpler models, the interaction of nonlinearity and dispersion is underestimated such that a larger nonlinearity (larger $s_o = N$) gives a narrower output pulse width (nonlinearity alone makes the output pulse width invariant.).

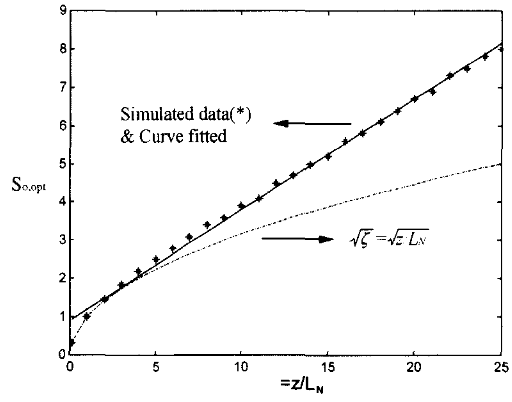

FIG. 2. Comparison of simulated $s_{o,opt}$ with curve- fitting and square-root of ζ.

Since we obtained analytical expressions in two extreme cases, $\zeta \ll 1$ and $\zeta \gg 1$, it is of interest to find the critical distance, ζ_c, which divides the two regions. Fig. 2 compares the simulated $s_{o,opt}$ with the dispersion dominant case, $s_{o,opt} = \sqrt{\zeta}$. When ζ is relatively small, the dependence of $s_{o,opt}$ on ζ is pretty well predicted by the square root of ζ. As a rule of thumb, when $\zeta < 3$, $s_{o,opt} \approx \zeta$. Otherwise, $s_{o,opt}$ can be more accurately predicted by the curve fitting result. Therefore the critical distance, $\zeta_c \approx 3$.

2. Optimum Input Pulse Width to Minimize the Product of $\sigma_t(z)$ and $\sigma_\omega(z)$

Recently, it was also reported that the spectrum width could be modeled accurately by the variational method [6]. For a transform-limited Gaussian input pulse, the spectral width can be expressed in terms of the normalized distance $\zeta(= z/L_N)$ using the relation, $\sigma_o^2\sigma_{\omega o}^2 = \sigma_t^2(z=0)\sigma_\omega^2(z=0) = 1/4$, such as

$$\sigma_\omega^2(\zeta) = \frac{1}{4\sigma_o^2} + \frac{1}{\sqrt{2}|\beta_2|L_N}\left(1 - \frac{1}{\sqrt{\frac{\sigma_t^2(\zeta)}{\sigma_o^2}}}\right) \quad (15)$$

The square of the pulse broadening factor by the variational analysis [4] can be rewritten in terms of ζ as below .

$$\frac{\sigma_t^2(\zeta)}{\sigma_o^2} = \frac{s^2(\zeta)}{s_o^2} = 1 + C_1\frac{1}{s_o^2} + C_2\frac{1}{s_o^4} + C_3\frac{1}{s_o^6} \quad (16)$$

where $C_1 = \frac{1}{\sqrt{2}}\zeta^2$, $C_2 = \zeta^2 + \frac{1}{24}\zeta^4$, and $C_3 = \frac{\sqrt{2}}{24}\zeta^4$. Now consider the product of output pulse width and output spectrum width. Using Eqs. (15) and (16),

$$\sigma_t^2\sigma_\omega^2 = \frac{1}{4}\frac{s^2(\zeta)}{s_o^2}\left[1 + \sqrt{2}s_o^2\left(1 - \frac{1}{s(\zeta)/s_o}\right)\right] \quad (17)$$

If we define $T(\zeta) = 4\sigma_t^2(\zeta)\sigma_\omega^2(\zeta)$,

$$T(\zeta) = \frac{s^2(\zeta)}{s_o^2} + \sqrt{2}s_o^2\left(\frac{s^2(\zeta)}{s_o^2} - \frac{s(\zeta)}{s_o}\right) \quad (18)$$

For simplicity of notation, define $x = s_o^2$ and $y = s^2/s_o^2$. Then Eq. (18) can be expressed in terms of x and y as below.

$$T(\zeta) = 4\sigma_t^2(\zeta)\sigma_\omega^2(\zeta) = y + \sqrt{2}x(y - \sqrt{y}) \quad (19)$$

To have an optimum input pulse width, the derivative of $T(\zeta)$ with respect to x should have zero value(s).

$$\frac{\partial T}{\partial x} = \left(1 + \sqrt{2}x - \frac{1}{\sqrt{2y}}\right)\frac{\partial y}{\partial x} + \sqrt{2}(y - \sqrt{y}) \quad (20)$$

where

$$\frac{\partial y}{\partial x} = -\frac{C_1}{x^2} - \frac{2C_2}{x^3} - \frac{3C_3}{x^4}\ (< 0)$$
$$\text{and}\ \ y = 1 + \frac{C_1}{x^1} + \frac{C_2}{x^2} + \frac{C_3}{x^3}(> 1)$$

We can rearrange Eq. (20) such that

$$\frac{\partial T}{\partial x} = \left(1 - \frac{1}{\sqrt{2y}}\right)\frac{\partial y}{\partial x} + \sqrt{2}(1 - \sqrt{y}) - \sqrt{2}\left(\frac{C_2}{x^2} + \frac{2C_3}{x^3}\right)$$

Since $y = s^2/s_o^2 > 1$ and $\frac{\partial y}{\partial x} < 0$, the first and the second terms are always negative. Furthermore, because x, C_2 and C_3 are all positive quantities, the third term is also negative, which means that the derivative of $T(\zeta)$ with respect to x is always negative regardless of the initial pulse width. This result leads to the conclusion that the variational method predicts there is no optimum input pulse width which minimizes the product of $\sigma_t(z)$ and $\sigma_\omega(z)$ when the input pulse is a transform-limited Gaussian. This is mainly because $\sigma_\omega(z)$ is a monotonically decreasing function of the initial pulse width, σ_o (or the normalized initial pulse width, s_o).

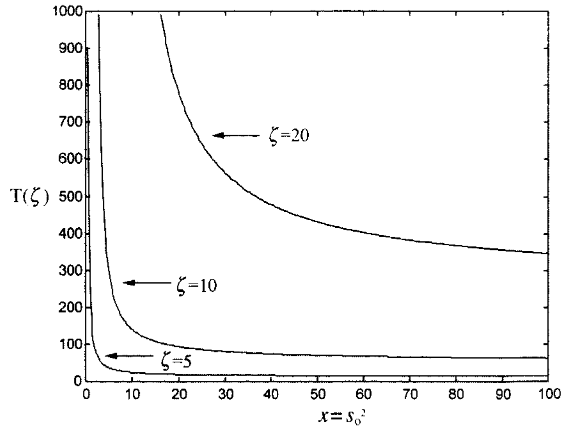

FIG. 3. $T(\zeta)$ as a function of s_o^2 in the normal dispersion region with a Gaussian input.

In Fig. 3, $T(\zeta)$, as calculated by the split-step Fourier method at a few fixed distances, is plotted as a function of input pulse width. It is seen from Fig. 3 that $T(\zeta)$ has no optimum value but decreases monotonically as the input pulse width (s_o) increases.

IV. CONCLUSION

Output RMS pulse width is modeled by lumping the fiber nonlinearity at the middle of the propagation distance. The methodology is fairly simple and the resulting two-step model predicts the output RMS pulse width much closer to the simulated one compared to the previous one-step model, in which the fiber nonlinearity is lumped at the input of the fiber. The two-step model is also used to derive the optimum input pulse width to minimize the output pulse width at a given distance. While the two-step model is not as good as the variational model (it gives a little larger deviation from the simulation results), the two-step model is much simpler to derive and easier to understand. It is also interesting to see that all of the analytical models including the one-step model predict the same functional form of $\sigma_{o,opt}$, which is linearly proportional to the propagation distance, z. If the maximum bit rate is taken to be $1/(4\sigma_t)$ (see Eq. (1)), all the analytical models predict the maximum bit rate-transmission distance product has a functional form of

$$R_b z \sim \sqrt{\frac{L_N}{|\beta_2|}} = \sqrt{\frac{1}{\gamma P_{avg}|\beta_2|}} \quad (21)$$

if $\zeta \gg 1 (z \gg L_N)$.

Eq. (21) predicts that the maximum bit rate-transmission distance product is inversely proportional to the square roots of both the average power of the signal and the fiber dispersion coefficient.

When $\zeta \ll 1 (z \ll L_N)$, $\sigma_{o,opt}$ degenerates to the case of dispersion alone, where $\sigma_{o,opt}$ is proportional to the square root of z. In this case, $R_b z \sim \sqrt{z/|\beta_2|}$. The simulation results (Fig. 2) show that the boundary between the two extreme cases is near ζ=3 (The transmission distance is 3 times the nonlinear distance L_N).

When we desire to optimize the product of $\sigma_t(z)$ and $\sigma_\omega(z)$ in the case of dispersion alone, the initial pulse width which minimizes the output pulse width will also be the optimum value to minimize $\sigma_t(z) \times \sigma_\omega(z)$ because $\sigma_\omega(z)$ is invariant. However, with fiber nonlinearity, it is shown mathematically that there is not an optimum input pulse width regardless of the propagation distance. The reason is that the output spectrum is a monotonically decreasing function of input pulse width σ_o and the optimum pulse width is not a strong function of σ_o.

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