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A Moment Inequality on New Renewal Better Than Used in Expectation Class of Life Distributions with Hypothesis Testing Application

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Abstract. In the present work, a moment inequality is derived for new renewal better (worse) than used in expectation(NRBUE) (NRWUE) distributions. This inequality demonstrates that if the mean life is finite then all higher order moments exist. A new test statistics for testing exponentiality against NRBUE (NRWUE) is introduced based on this inequality. It is shown that the proposed test is simple and has high relative efficiency for some commonly used alternatives . Critical values are tabulated for sample sizes $n = 5(1)30$. A set of real data is used as an example to elucidate the use of the proposed test statistics for practical reliability analysis.

Key Words : *NRBUE (NRWUE), exponentiality, efficiency, moments, asymptotic normality.*

1. INTRODUCTION

Various classes of life distributions have been introduced in reliability, the applications of these classes of life distribution can be seen in engineering, social , biological sciences, maintenance and biometrics. Therefore, statisticians and reliability analysts have shown a growing interest in modelling survival data using classifications of life distributions based on some aspects of aging, see for example Barlow and Proschan (1981) and Zacks (1992). The most well known families of life distributions are the classes of increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU), new better than used in expectation (NBUE), harmonic new better than used in expectation (HNBUE) and new better

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than renewal used (NBRU). For some properties and interrelationships of these criteria, we refer to Bryson and Siddiqui (1969), Barlow and Proschan (1981), Rolski (1975) and Abouammoh and Qamber (2003).

Testing exponentiality aginst above classes of life distributions has seen a good deal of attention. For testing against IFR, we refer to Proschan and Pyke (1967), Barlow and Proschan (1981), Bickel and Doksum (1969) and Ahmad (1975) and others. For testing against IFRA, we refer to Ahmad (1975), Deshpand (1983) and Ahmad (1994) among others. Testing against NBU is discussed by Hollander and Proschan (1972), Koul (1977) and Ahmad (1994) among others; whereas testing against NBUE is considered by Hollander and Proschan (1972), Kanjo (1993) followed others. Finally, testing against HNBUE can be found in the work of Rolski (1975), Ahmad et al (1999) and Hendi et al (1998).

The thread that connects most work mentioned here is that a measure of departure from H_0, which is often some weighted functional of F, is developed which is strictly positive under H_1 and is zero under H_0. Then, a sample version of this measure is used as a test statistic and its properties are studied.

Let T be a nonnegative continuous random variable with distribution function $F(x)$, survival function $\overline{F} = 1 - F$, density function f, finite mean μ and finite variance σ^2. Consider a device with life length T life distribution F. The device is replaced instantly upon failure by a sequence of mutually independent devices. These devices are independent of the first unit and identically distributed with the same distribution F. When the renewal of the system is continued indefinitely, the (stationary) remaining life distribution of a device in operation at time t is $W_F(t) = \frac{1}{\mu}\int_0^t \overline{F}(u)du$, $t \geq 0$, where $\mu = \mu_F = \int_0^\infty \overline{F}(u)du$. The renewal distribution $W_F(t)$ and its comparison with parent life distribution F arise in construction of various preventive repair or maintenance policies, see Barlow and Proschan (1981) and Abouammah et al (1994).

Let T_w be a renewal random variable with distribution $W_F(t)$, survival function $\overline{W}_F(t) = \frac{1}{\mu}\int_t^\infty \overline{F}(u)du$.

The new renewal better than used in expectation (NRBUE) and its dual new renewal worse used in expectation (NRWUE) are defined as follows.

Definition 1. A life distribution F, with $F(0) = 0$ and its survival function $\overline{F}$ are said to have NRBUE if

$$\mu \int_x^\infty \overline{F}(u)du \leq \overline{F}(t) \int_0^\infty \overline{\nu}(t)dt, \qquad x \geq 1. \tag{1.1}$$

where

$$\nu(t) = \int_t^\infty \overline{F}(u)du$$

The dual class of life distributions that is NRWUE is defined by reversing the inequality sign of relation (1.1). This definition means that, F is NRBUE if

$$\mu(t) \leq (\geq)\mu_w. \tag{1.2}$$

where $\mu(t) = \frac{\int_t^\infty \overline{F}(u)du}{\overline{F}(t)}$, is the mean remaining life of the distribution $F(t)$ and μ_w is the mean of the renewal distribution $W_F(t)$ and this acount the name of NRBUE (NRWUE). Abouammoh et al (2000) studied some properties of NRBUE (NRWUE) classes of life distributions, they established their interrelationships and their relation to the existing classes.

In this spirit, the moment inequality developed in Section 2 can be used to construct test statistics for NRBUE (NRWUE). In Section 3 this test statistic is based on sample moments of aging distributions. This test statistic is simple to derive, and has exceptionally high efficiency of some of well known alternatives relative to other tests. Monte Carlo null distribution critical points are obtained for sample sizes $n = 5(1)30$. An example using real data representing 40 patients suffering from blood cancer from the Ministry of Health Hospitals in Saudi Arabia is given as an application.

2. MOMENT INEQUALITY

In the spirit of the work of Ahmad (2001), we state and prove the following result.

Theorem 1. If F is NRBUE (NRWUE), then

$$\frac{1}{(r+2)}\mu\mu_{(r+2)} \leq (\geq)\frac{1}{2}\mu_{(r+1)}\mu_{(2)}, \qquad r \geq 1 \tag{2.1}$$

where

$$\mu_{r+1} = (r+1)\int_0^\infty x^r\overline{F}(u)du.$$

Proof. Since F is NRBUE, then

$$\mu\nu(x) \leq \overline{F}(x)\int_0^\infty \nu(t)dt. \tag{2.2}$$

Since

$$\int_0^\infty \nu(t)dt = \int_0^\infty t\overline{F}(t)dt = \frac{\mu_{(2)}}{2}, \tag{2.3}$$

the inequality in (2.2) becomes as follows:

$$\mu\nu(x) \leq \frac{1}{2}\mu_{(2)}\overline{F}(x). \tag{2.4}$$

Multiplying both sides in (2.4) by $x^r, r \geq 1$ and integrating over $(0,\infty)$, w.r.t. x, we obtain

$$\mu\int_0^\infty x^r\nu(x)dx \leq \int_0^\infty x^r\overline{F}(x)dx\frac{\mu_{(2)}}{2}. \tag{2.5}$$

The left hand side of (2.5) is

$$\mu \int_0^\infty x^r \nu(x)dx = \frac{\mu}{(r+1)} \int_0^\infty x^{r+1}\overline{F}(x)dx$$

$$= \frac{\mu\mu_{(r+2)}}{(r+1)(r+2)}. \tag{2.6}$$

Substituting from (2.6) in (2.5), we obtain the result.

3. APPLICATIONS TO HYPOTHESES TESTING

3.1 Testing against NRBUE (NWBUE) alternatives

The test presented here depends on a sample $X_1, X_2, \ldots, X_n$ from a population with distribution F. We wish to test the null hypothesis $H_0 : \overline{F}$ is exponential with mean μ against $H_1 : \overline{F}$ is NRBUE (NWBUE) and not exponential. Using Theorem 1, we may use the following quantity as a measure of departure from H_0 in favor of H_1:

$$\delta_r = \frac{1}{2}\mu_{(2)}\mu_{(r+1)} - \frac{1}{(r+2)}\mu\mu_{(r+2)}. \tag{3.1}$$

Note that under $H_0 : \delta_r = 0$, while under $H_1 : \delta_r > (<)0$. Thus to estimate δ_r by $\hat{\delta}_{r_n}$, let $X_1, X_2, \ldots, X_n$ be a random sample from F, let $dF_n(x) = \frac{1}{n}$ and μ is estimated by $\overline{X}$. Then $\hat{\delta}_{r_n}$ is given by using (3.1) as

$$\hat{\delta}_{r_n} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{X_i^2 X_j^{r+1}}{2} - \frac{X_i X_j^{r+2}}{(r+2)}\right\}. \tag{3.2}$$

Thus to make the test statistic scale invariant, we take

$$\hat{\Delta}_{r_n} = \frac{\hat{\delta}_{r_n}}{\overline{X}^{r+3}} \tag{3.3}$$

where $\overline{X} = \frac{1}{n}\sum X_k$ is the usual sample mean.

Setting $\phi(X_1, X_2) = \frac{1}{2}X_1^2 X_2^{r+1} - \frac{1}{r+2}X_1 X_2^{r+2}$, then $\hat{\Delta}_{r_n}$ in (3.3) is equivalent to the following U-statistic.

$$U_n = \frac{1}{\binom{n}{2}}\sum_{i<j}\psi(X_i, X_j) \tag{3.4}$$

The following theorem summarizes the large sample properties of $\hat{\Delta}_{r_n}$ or U_n.

Theorem 2. As $n \to \infty, \sqrt{n}(\hat{\Delta}_{r_n} - \Delta_r)$ is asymptotically normal with mean 0 and variance

$$\sigma^2 = \mu^{-(r+3)} Var \left\{ \frac{X_1^2 \mu_{(r+1)} + X_1^{r+1} \mu_{(2)}}{2} - \frac{X_1^{r+2} \mu + X_1 \mu_{(r+2)}}{(r+2)} \right\} \tag{3.5}$$

Under $H_0, \Delta_r = 0$ and variance σ_0^2 is given by

$$\sigma_0^2 = \frac{(2r+4)!}{(r+2)^2} + 2(r+2)! - \frac{2(2r+3)!}{(r+2)} - 2(r+1)!^2(r+1), \quad r \geq 1. \tag{3.6}$$

Proof. Since $\hat{\Delta}_{r_n}$ and $\frac{\hat{\delta}_{r_n}}{\mu^{r+3}}$ have the same limiting distribution, we use $\sqrt{n}(\hat{\delta}_{r_n} - \delta_{r_n})$. Now this is asymptotically normal with mean 0 and variance $\sigma^2 = var[\phi(X_1)]$, where

$$\phi(X_1) = E[\phi(X_1, X_2)|X_1] + E[\phi(X_2, X_1)|X_1]. \tag{3.7}$$

But

$$\phi(X_1) = \frac{X_1^2 \mu_{(r+1)} + X_1^{r+1} \mu_{(2)}}{2} - \frac{X_1^{r+1} \mu + X_1 \mu_{(r+2)}}{(r+2)} \tag{3.8}$$

Hence (3.5) follows.

Under H_0

$$\phi(X_1) = \frac{(r+1)! X_1^2 + 2X_1^{r+1}}{2} - \frac{X_1^{r+2} + (r+2)! X_1}{(r+2)} \tag{3.9}$$

Hence (3.6) follows. The theorem is proved.

When $r = 1$,

$$\delta_1 = \frac{1}{2}\mu_{(2)}^2 - \frac{1}{3}\mu\mu_{(3)}. \tag{3.10}$$

In this case $\sigma_0^2 = 8$ and the test statistic is

$$\hat{\delta}_{1_n} = \frac{2}{n(n-1)} \sum_{i<j}\sum \left\{ \frac{X_i^2 X_j^2}{2} - \frac{X_i X_j^3}{3} \right\}. \tag{3.11}$$

$$\hat{\Delta}_{1_n} = \frac{\hat{\delta}_{1_n}}{\overline{X}^4} \tag{3.12}$$

which is quite simple for applications.

To use the above test, calculate $\sqrt{n}\hat{\Delta}_{1_n}/\sigma_0$ and reject H_0 if this exceeds the normal variate value $Z_{1-\alpha}$. To illustrate the test, we calculate, via Monte Carlo Method, the empirical critical points of $\hat{\Delta}_{1_n}$ in (3.12) for samples 5(1)50. Table 1 gives the upper percentile points of statstic $\hat{\Delta}_{1_n}$ for 95%, 98%, 99%.

Table 1. Critical Values of $\hat{\Delta}_{1_n}$.

n	95%	98%	99%
5	0.5154	0.5926	0.6345
6	0.5943	0.6774	0.7363
7	0.6315	0.7548	0.8100
8	0.6535	0.7973	0.9076
9	0.6553	0.8413	0.9627
10	0.6405	0.8605	0.9859
11	0.6813	0.8838	0.1.0174
12	0.6829	0.8753	0.1.0275
13	0.6773	0.8984	0.1.1033
14	0.6767	0.8928	0.1.0081
15	0.6968	0.9013	0.1.0733
16	0.6895	0.9234	0.1.0663
17	0.5295	0.6920	0.1.0616
18	0.6581	0.8804	0.1.0884
19	0.6852	0.9220	0.1.1270
20	0.7074	0.9345	0.1.0394
21	0.6692	0.8828	0.1.0849
22	0.6656	0.8830	0.1.0630
23	0.6907	0.8980	0.1.0514
24	0.6650	0.8690	0.1.0514
25	0.6735	0.8667	0.1.0448
26	0.6535	0.8666	0.1.0620
27	0.6756	0.8630	0.1.0527
28	0.6485	0.8550	0.9868
29	0.6551	0.8412	0.1.0074
30	0.6588	0.8517	0.1.0187

The calculations are based on 5000 simulated samples $n = 5(1)30$. These percentiles values changes slowly as n increases.

Since the test statistic $\hat{\Delta}_{r_n}$ in (3.3) is new and no other tests are known for these class NRBUE. We compare these to some other classes. Here we choose the test $\hat{\Delta}_n$ presented by Kanjo (1993) . We use the concept of "Pitman's asymptotic efficiency" (PAE). To do this we need to evaluate the PAE of the proposed test and compare it with the PAE of the test $\hat{\Delta}_n$. Note that PAE of $\hat{\Delta}_{r_n}$ is given by

$$PAE(\Delta_r(\theta)) = \left\{ \frac{d}{d\theta}\Delta_r(\theta)|_{\theta\to\theta_0} \right\} /\sigma 0 \tag{3.13}$$

In our case this reduces to

$$\begin{aligned} PAE(\Delta_r(\theta_0)) &= \frac{1}{2}(r+1)!\mu'_{(2)}(\theta_0) + \mu'_{(r+1)}(\theta_0) - (r+1)!\mu'(\theta_0) \\ &\quad - \frac{1}{(r+2)}\mu'_{(r+1)}(\theta 0). \end{aligned} \tag{3.14}$$

Two of the most commonly used alternatives (cf. Hollander and Proschan (1972)) are:

(i)	Linear failure rate family	: $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}$,	$x > 0, \theta > 0$
(ii)	Makeham family	: $\bar{F}_\theta = e^{-x-\theta(x+e^{-x}-1)}$,	$x \geq 0, \theta > 0$

The null hypothesis is at $\theta = 0$ for linear failure rate family and Makeham family. The PAE's of these alternatives of our procedure are, respectively:

$$PAE(\Delta_r, LFR) = r(r+1)!, \qquad r \geq 1 \tag{3.15}$$

$$PAE(\Delta_r, Makeham) = (r+1)![\frac{1}{4} + 2^{-(r+2)} - 2^{-(r+2)}] \tag{3.16}$$

Direct calculations of PAE of $\hat{\Delta}_n$ and $\hat{\Delta}_{1_n}$ are summarized in Table 2.

Table 2. PAE of $\hat{\Delta}_n$ and $\hat{\Delta}_{1_n}$.

Distribution	$\hat{\Delta}_n$	$\hat{\Delta}_{1_n}$
F_1 Linear failure rate	0.433	0.707
F_2 Makham	0.144	0.088

From Table 2, it appears that the test statistics $\hat{\Delta}_{1_n}$ is more efficient than $\hat{\Delta}_n$ for linear failure rate F_1.

Note that: Since $\hat{\Delta}_{r_n}$ defines a class (with parameter) r of test statistics, we choose by one of two possibilities. Either to choose r small enough to keep calculation simple (such as what did by choosing $r = 1$) or in cases when the alternatives of interest is known, we can choose r that the maximizes the PAE of that alternatives.

4. NUMERICAL EXAMPLE

Consider the data in Abouammoh et al (1994). These data represent 40 patients suffering from blood cancer from one of the Ministry of Health Hospital in Saudi Arabia and the ordered life times (in days) are 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1169, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1604, 1696, 1735, 1799, 1815, 1852.

Using equation (3.12), the value of test statistics, based on the above data is $\hat{\Delta}_{1_n} = 0.05119$ This value leads to the acceptance of H_0 at the signifcace level $\alpha = 0.95$(see Table 1).The data therefore do not satisfy NRBUE Property.

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