International Journal of Fuzzy Logic and Intelligent Systems, vol. 3, no. 2, December 2003 pp. 258-261

γ-Connectedness in fuzzy topological spaces

I. M. Hanafy

Department of Mathematics, Faculty of Education, Suez Canal University, El-Arish, Egypt

Abstract

The aim of this paper is to introduce the concept $\gamma-$connectedness in fuzzy topological spaces. We also investigate some interre lations between this types of fuzzy connectedness together with the preservation properties under some types of fuzzy continuity. A comparison between some types of connectedness in fuzzy topological spaces is of interest.

Key words : *Fuzzy topological spaces, fuzzy* $\gamma-$continuity, $\gamma-$separated fuzzy sets, $\gamma-$connected fuzzy sets, $\delta-$connected fuzz ysets, $\theta-$ connected fuzzy sets.

1. Introduction and preliminaries

In [5], we introduce a new class of fuzzy functions called fuzzy $\gamma-$continuous functions, which contains the class of fuzzy continuous functions. We investigated several properties concerning such functions. The purpose of this paper is to introduce and study the concept of $\gamma-$connected fuzzy sets by the help of fuzzy $\gamma-$closure [5], and study the relationships between $\gamma-$connected fuzzy sets and such subsets in fuzzy topological spaces.

Throughout the paper, X and Y always represent fuzzy topological spaces (fts, for short) in Chang's sense [1]. For a fuzzy set U in X, $clU, \delta clU, \theta clU$ and U' will respectively denote, the closure, $\delta-$closure, $\theta-$ closure and complement of U. A fuzzy set U in a fts X is said to be quasi-coincident with a fuzzy set V, denoted by UqV, if there exists $x \in X$ such that $U(x)+V(x)>1$ [6]. It is Known that $U \leq V$ iff U and V′ are not quasi-coincident, denoted by $U\bar{q}V'$ [6].

Definition 1.1 [5]. A fuzzy set U in a fts X is called $\gamma-$open ($\gamma-$closed) if
$U \leq cl\,int U \vee int\,cl U$ ($U \geq cl\,int U \wedge int\,cl U$).

Definition 1.2 [5]. For a fuzzy set U in a fts X, fuzzy $\gamma-$closure (γcl, for short) is defined as follows:
$\gamma cl U = \wedge\{G : G$ is $\gamma-$closed fuzzy set, $U \leq G\}$.

Definition 1.3 [5]. A function $f: X \to Y$ is called fuzzy $\gamma-$continuous if for each open fuzzy set V in Y, $f^{-1}(V)$ is $\gamma-$open fuzzy set in X.

Definition 1.4 [5]. A function $f: X \to Y$ is called fuzzy $\gamma-$closed if the image of each closed fuzzy set in X is $\gamma-$closed fuzzy set in Y.

Definition 1.5 [3,4]. Two non-empty fuzzy sets U and V in a fts X are called separated (resp. $\delta-$separated, $\theta-$ separated) fuzzy sets if $U\bar{q}clV$ and $V\bar{q}clU$ (resp. $U\bar{q}\delta clV$ and $V\bar{q}\delta clU$, $U\bar{q}\theta clV$ and $V\bar{q}\theta clU$).

Remark 1.6 [8]. For any two non-empty fuzzy sets U and V the following implications hold:
$\theta-$ separated fuzzy set $\Rightarrow$ $\delta-$separated fuzzy set $\Rightarrow$ separated fuzzy set.

Definition 1.7 [3,4]. A fuzzy set U in a fts X is called connected (resp. $\delta-$connected, $\theta-$ connected) fuzzy set if U cannot be expressed as the union of two separated (resp. $\delta-$separated, $\theta-$ separated) fuzzy sets.

Remark 1.8 [8]. For a fuzzy set U , the following implications hold:
connected fuzzy set $\Rightarrow$ $\delta-$ connected fuzzy set $\Rightarrow$ $\theta-$ connected fuzzy set.

Definition 1.9 [7]. A fts is called fuzzy regular if for each open fuzzy set V and each fuzzy point pqV, there exists an open fuzzy set U such that $pqU \leq clU \leq V$.

2. $\gamma-$separated fuzzy sets

Definition 2.1. Two non-empty fuzzy sets U and V in a fts X are called $\gamma-$ separated fuzzy sets if $U\bar{q}\gamma clV$ and $V\bar{q}\gamma clU$.

Remark 2.2. For any two non-empty fuzzy sets U and V, since $\gamma clU \leq clU \leq \delta clU \leq \theta clU$ and so V, the following implications hold:

θ – separated fuzzy set $\Rightarrow \delta$ – separated fuzzy set $\Rightarrow$ separated fuzzy set $\Rightarrow$ γ – separated fuzzy set.

Theorem 2.3. Let U,V be non-empty fuzzy sets in a fts X.
(i) if U and V are γ – separated fuzzy sets and U_1, V_1 are non-empty fuzzy sets such that $U_1 \leq U$ and $V_1 \leq V$, then U_1 and V_1 are also γ – separated fuzzy sets.

(ii) if $U \bar{q} V$ and either both are γ – open fuzzy sets or both are γ – closed fuzzy sets, then U and V are γ – separated fuzzy sets.

(iii) if U and V are either both γ – open fuzzy sets or both are γ – closed fuzzy sets and if $A = U \cap V'$ and $B = V \cap U'$,then A and B are γ – separated fuzzy sets.

Proof. (i) Since $U_1 \leq U$, we have $\gamma cl U_1 \leq \gamma cl U$, then $V \bar{q} \gamma cl U \Rightarrow V_1 \bar{q} \gamma cl U \Rightarrow V_1 \bar{q} \gamma cl U_1$. Similarly $U_1 \bar{q} \gamma cl V_1$. Hence U_1 and V_1 are γ – separated fuzzy sets.

(ii) When U and V are γ – closed fuzzy sets, then $U = \gamma cl U$ and $V = \gamma cl V$. Since $U \bar{q} V$, we have $\gamma cl U \bar{q} V$. When U and V are γ – open fuzzy sets, U' and V' are γ – closed fuzzy sets. Then $U \bar{q} V \Rightarrow U \leq V' \Rightarrow \gamma cl U \leq \gamma cl V' = V' \Rightarrow \gamma cl U \bar{q} V$. Similarly, $\gamma cl V \; \bar{q} U$. Hence U and V are γ – separated fuzzy sets.

(iii) When U and V are γ – open fuzzy sets, U' and V' are γ – closed fuzzy sets. Since $A \leq V'$, $\gamma cl A \leq \gamma cl V' = V'$ and so $\gamma cl A \bar{q} V$. Thus $B \bar{q} \gamma cl A$. Similarly , $A \bar{q} \gamma cl B$. Hence A and B are γ – separated fuzzy sets.
When U and V are γ – open fuzzy sets, $U = \gamma cl U$ and $V = \gamma cl V$. Since $A \leq V'$, $\gamma cl V \; \bar{q} A$ and hence $\gamma cl B \; \bar{q} A$. Similarly $\gamma cl A \; \bar{q} B$. Hence A and B are γ – separated fuzzy sets.

Theorem 2.4. Let U and V be non-empty fuzzy sets of a fts X, then U and V are γ – separated fuzzy sets iff there exist two γ – open fuzzy sets A and B such that $U \leq A$, $V \leq B$, $U \bar{q} B$ and $V \bar{q} A$.

Proof. Let U and V be γ – separated fuzzy sets. Putting $B = (\gamma cl U)'$ and $A = (\gamma cl V)'$, then A and B are γ – open fuzzy sets such that $U \leq A$, $V \leq B$, $U \bar{q} B$ and $V \bar{q} A$.

Conversely, let A and B be γ – open fuzzy sets such that $U \leq A$, $V \leq B$, $U \bar{q} B$ and $V \bar{q} A$. Since B' and A' are γ – closed fuzzy sets, we have $\gamma cl U \leq B' \leq V'$ and $\gamma cl V \leq A' \leq U'$. Thus $\gamma cl U \bar{q} V$ and $\gamma cl V \bar{q} U$. Hence U and V are γ – separated fuzzy sets.

3. γ – connected fuzzy sets

Definition 3.1. A fuzzy set which cannot be expressed as the union of two γ – separated fuzzy sets is said to be a γ – connected fuzzy set.

Remark 3.2. From the above definition and Remark 1.8, the following implications hold:

γ – connected fuzzy set $\Rightarrow$ connected fuzzy set $\Rightarrow$ δ – connected fuzzy set $\Rightarrow$ θ – connected fuzzy set.

Lemma 3.3. A fts X is fuzzy regular iff $\gamma cl U = cl U = \delta cl U = \theta cl U$, for each fuzzy set U in X.

Proof. Obvious (see [2]).

Theorem 3.4. For a fuzzy set U in a fuzzy regular space X, the following are equivalent:
(i) U is γ – connected fuzzy set .
(ii) U is connected fuzzy set .
(iii) U is δ – connected fuzzy set .
(iv) U is θ – connected fuzzy set .

Proof. Follows directly by virtue of Lemma 3.3.

Theorem 3.5. In a fts X, if U is γ – connected fuzzy set, then $\gamma cl U$ is so.

Proof. Suppose that $\gamma cl U$ is not γ – connected fuzzy set, then there are two non-empty γ – separated fuzzy sets A and B in X such that $\gamma cl \; U = A \cup B$. Now, from $U = (A \cap U) \cup (B \cap U)$ and $\gamma cl (A \cap U) \leq \gamma cl A$, $\gamma cl (B \cap U) \leq \gamma cl B$ and $A \bar{q} B$, we obtain $\gamma cl (A \cap B) \bar{q} B$. Hence $\gamma cl (A \cap U) \bar{q} (B \cap U)$. Similarly $\gamma cl (B \cap U) \bar{q} (A \cap U)$. Therefore U is not γ – connected fuzzy set. Hence the result.

Theorem 3.6. Let U be a non-empty γ – connected fuzzy set in a fts X. If U is contained in the union of two γ – separated fuzzy sets A and B, then exactely one of the following conditions (i) and (ii) holds:
(i) $U \leq A$ and $U \cap B = 0_X$.
(ii) $U \leq B$ and $U \cap A = 0_X$.

Proof. We first note that when $U \cap B = 0_X$, Then $U \leq A$, since $U \leq A \cup B$. Similarly, when $U \cap A = 0_X$, we have $U \leq B$. Since $U \leq A \cup B$, both $U \cap A = 0_X$ and $U \cap B = 0_X$ cannot hold simultaneously. Again if $U \cap A \neq 0_X$ and $U \cap B \neq 0_X$, then by Theorem 2.3 (i), $U \cap A$ and $U \cap B$ are γ – separated fuzzy sets such that $U = (U \cap A) \cup (U \cap B)$, contradicting the $\gamma-$connectedness of a fuzzy set U. Hence exactly one of the conditions (i) and (ii) must hold.

Theorem 3.7. Let $\{U_j : j \in J\}$ be a collection of $\gamma-$connected fuzzy sets in a fts X. If there exists $k \in J$ such that $U_j \cap U_K \neq 0_X$ for each $j \in J$, then $U = \{U_j : j \in J\}$ is $\gamma-$connected fuzzy set.

Proof. Suppose that U is not $\gamma-$connected fuzzy set. Then there exist $\gamma-$separated fuzzy sets A and B such that $U = A \cup B$. By Theorem 3.4, we have either (i) $U_j \leq A$ with $U_j \cap B = 0_X$ or (ii) $U_j \leq B$ with $U_j \cap A = 0_X$ for each $j \in J$. Similarly, either (iii) $U_k \leq B$ with $U_k \cap A = 0_X$ or (iv) $U_k \leq A$ with $U_k \cap B = 0_X$. We may assume, without loss of generality, that U_j is non-empty for each $j \in J$, and hence exactly one of (iii) and (iv) will hold. Since $U_j \cap U_k \neq 0_X$ for each $j \in J$, the conditions (i) and (iii) cannot hold, and similarly (ii) and (iv) cannot hold simultaneously. If (i) and (iv) hold , then $U_j \leq A$ with $U_j \cap B = 0_X$ for each $j \in J$. Then $U \leq A$ and $U \cap B = 0_X$ and thus $B = 0_X$, a contradiction. Similarly, if (ii) and (iii) hold , then we have $A = 0_X$, a contradiction. Hence the result.

Theorem 3.8. If a function $f : X \to Y$ is fuzzy $\gamma-$continuous and U is $\gamma-$connected fuzzy set relative to X , then $f(U)$ is connected fuzzy set relative to Y.

Proof. Suppose that $f(U)$ is not connected fuzzy set in Y, there exists two separated fuzzy sets A and B relative to Y such that $f(U) = A \cup B$. Then there exist open fuzzy sets G and H in Y such that $A \leq G, B \leq H, A\bar{q}H$ and $B\bar{q}G$. Since f is fuzzy $\gamma-$continuous , $f^{-1}(G)$ and $f^{-1}(H)$ are $\gamma-$ open fuzzy sets in X such that $f^{-1}(A) \leq f^{-1}(G)$, $f^{-1}(B) \leq f^{-1}(H)$, $f^{-1}(A)\bar{q}$ $f^{-1}(H)$ and $f^{-1}(B)\bar{q}$ $f^{-1}(G)$. Thus $f^{-1}(A)$ and $f^{-1}(B)$ are $\gamma-$ separated fuzzy sets in X from Theorem 2.4, and $U = f^{-1}(f(U)) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. Hence this is contrary to the fact that U is $\gamma-$connected fuzzy set. Therefore $f(U)$ must be connected fuzzy set.

Theorem 3.9. If a bijective function $f : X \to Y$ is fuzzy $\gamma-$closed and U is $\gamma-$connected fuzzy set relative to Y, then $f^{-1}(U)$ is connected fuzzy set relative to X.

Proof. Suppose that $f^{-1}(U)$ is not connected fuzzy set in X, there exists two separated fuzzy sets A and B relative to X such that $f^{-1}(U) = A \cup B$. Then there exist open fuzzy sets G and H in X such that $A \leq G, B \leq H, A\bar{q}H$ and $B\bar{q}G$. Since f is fuzzy $\gamma-$closed function , $f(clG)$ and $f(clH)$ are $\gamma-$closed fuzzy sets in Y such that $f(clA) \leq f(clG)$, $f(clB) \leq f(clH)$, $f(clA)\bar{q}$ $f(clH)$ and $f(clB)\bar{q}$ $f(clG)$. Since $f(clB)\bar{q}$ $f(clG)$ implies $f(B)\bar{q}$ $f(G)$. Thus $f(A)$ and $f(B)$ are $\gamma-$separated fuzzy sets in Y from Theorem 2.3 (ii), and $U = f(f^{-1}(U)) = f(A \cup B) = f(A) \cup f(B)$. Hence this is contrary to the fact that U is $\gamma-$connected fuzzy set. Therefore $f^{-1}(U)$ must be connected fuzzy set.

Definition 3.10. A function $f : X \to Y$ is called fuzzy $\gamma-$irresolute (fuzzy strongly $\gamma-$irresolute) if the inverse image of each $\gamma-$open fuzzy set in Y is $\gamma-$open (open) in X.

Lemma 3.11. Let $f : X \to Y$ be a function then the following are equivalent:

(i) f is fuzzy $\gamma-$irresolute.

(ii) The inverse image of each $\gamma-$closed fuzzy set in Y is $\gamma-$closed in X.

(iii) $\gamma cl\, f^{-1}(V) \leq f^{-1}(\gamma cl\, V) \leq f^{-1}(cl\, V)$.

Proof. Obvious.

Theorem 3.12. [7] . If a function $f : X \to Y$ is fuzzy $\gamma-$irresolute and U is $\gamma-$connected fuzzy set relative to X, then $f(U)$ is $\gamma-$connected relative to Y.

Proof. By using Definition 3.10 and Lemma 3.11, it is a direct consequence of Theorem 3.8.

Definition 3.13. A function $f : X \to Y$ is called fuzzy $M\gamma-$open (fuzzy $M\gamma-$ closed) if the image of each $\gamma-$open ($\gamma-$closed) fuzzy set in X is $\gamma-$open ($\gamma-$closed) in Y.

Definition 3.14. Two fts's X and Y are called fuzzy $\gamma-$ homeomorphic if there exists a bijective function $f : X \to Y$ such that f is fuzzy $\gamma-$irresolute and fuzzy $M\gamma-$ open. Such function f is called a fuzzy $\gamma-$homeomorphism.

Remark 3.15. It is clear that each fuzzy homeomorphism is fuzzy $\gamma-$homeomorphism while the converse may not be true.

Example 3.16. An identity fuzzy function f from any fts X to a Discrete fts Y is fuzzy $\gamma-$homeomorphism but not fuzzy homeomorphism.

Definition 3.17 . A function $f : X \to Y$ is called fuzzy strongly $M\gamma-$open (fuzzy strongly $M\gamma-$closed) if the image of each $\gamma-$open ($\gamma-$closed) fuzzy set in X is open (closed) in Y.

From the above definitions one can easily obtain the following theorems.

Theorem 3.18. Let a bijective function $f: X \to Y$ be a fuzzy $M\gamma$ – closed and U be a γ – connected fuzzy set relative to Y, then $f^{-1}(U)$ is γ – connected relative to X.

Proof. Obvious.

Theorem 3.19. Let a function $f: X \to Y$ be a fuzzy γ – homeomorphism, then G is γ – connected fuzzy set relative to X (G is γ – connected fuzzy set relative to Y) iff $f(G)$ is γ – connected fuzzy set relative to Y ($f^{-1}(G)$ is γ – connected fuzzy set relative to X).

Proof. Obvious.

Corollary 3.20. Let a function $f: X \to Y$ be a fuzzy homeomorphism, then G is γ – connected fuzzy set relative to X (G is γ – connected fuzzy set relative to Y) iff $f(G)$ is γ – connected fuzzy set relative to Y ($f^{-1}(G)$ is γ – connected fuzzy set relative to X).

Proof. It is obvious from Remark 3.15.

Theorem 3.21. Let X and Y be fuzzy γ – homeomorphic, then X is fuzzy γ – connected iff Y is fuzzy γ – connected.

Proof. Obvious.

Theorem 3.22. If a function $f: X \to Y$ is a fuzzy strongly $M\gamma$ – continuous and G is connected fuzzy set relative to X, then $f(G)$ is γ – connected fuzzy set relative to Y.

Proof. Obvious.

Theorem 3.23. Let a bijective function $f: X \to Y$ be a fuzzy strongly $M\gamma$ – closed and G is connected fuzzy set relative to Y, then $f^{-1}(G)$ is γ – connected fuzzy set relative to X.

References

[1] C.L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24(1968) 182-190.

[2] J.R. Choi, B.Y.Lee and J.H. Park, On fuzzy θ – continuous mappings, Fuzzy Sets and Systems, 54(1993)107-114.

[3] S. Ganguly and S. Saha, On separation axioms and separation of connected sets in fuzzy topological spaces, Bull. Calcutta Math. Soc. 79(1987)215-225.

[4] S. Ganguly and S. Saha, A note on δ – continuity and δ – connected sets in fuzzy set theory, Simon Stevin 62(1988)127-141.

[5] I.M.Hanafy, Fuzzy γ – open sets and fuzzy γ – continuity, J. Fuzzy Math., Vol.7 No 2.(1999),419-430.

[6] P. P. Ming and L.Y. Ming, Fuzzy topology I Neighborhood structure of a fuzzy point and Moore Smith convergence, J. Math. Anal. Appl. 76(1980),571-599.

[7] M.N. Mukherjee and B. Ghosh, Fuzzy semiregularization topologies and fuzzy submaximal spaces, Fuzzy Sets and Systems 44(1991)283-294.

[8] J.H. Park, B.Y. Lee and J.R. Choi, Fuzzy θ – connectedness, Fuzzy Sets and Systems 59(1993)237-244.