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Some good extensions of compactness

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Abstract

The aim of this paper is to introduce good definitions of compactness, almost compactness, near compactness, weak compactness, and S-closedness in fuzzy topological spaces in Sostak's sense. These compactness related concepts are defined for arbitrary fuzzy sets and some of their properties studied.

Key Words : Fuzzy compactness, Fuzzy almost compactness, Fuzzy near compactness, Fuzzy weak compactness, Good extension

1. Introduction

Sostak [20] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [4], in the sense that not only the objects are fuzzified, but also the axiomatics. Sostak [21,22] gave some rules and showed how such an extension can be realized. Chattopadhyay[5,6] and Ramadan17,9] have redefined the same concept. Hohle and Sostak [14] introduce the concept of an L-fuzzy topologies and establish their corresponding convergence theory for any lattice L.

Compactness is one of the most important notions in topology. Since fuzzy topological spaces were introduced in [20,14], many papers on this problem have been written and a lot of different kinds of fuzzy compactness have been introduced and studied [7,10,11,12,19].

The aim of this paper is to introduce some good types of compactness in fuzzy topological spaces in view of the definition of Sostak, namely, compactness, almost compactness, near compactness, weak compactness and S-closedness, and study some of its interesting properties and characterizations.

2. Preliminaries

In this paper, I stands for the unit interval [0,1], $I_0=(0,1]$, $I_1=[0,1)$, $I^X=$ the set of all fuzzy subsets of X. For $\alpha\in I$, $\underline{1}\in I^X$ where $\underline{\alpha}(x)=x$ for all $x\in X$. For $\lambda\in I^X$, $\lambda^c=\underline{1}-\lambda$. For an ordinary subset A of X, we denote by χ_A, the characteristic function of A.

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Definition 2.1 [4.12] A mapping $\tau: I^X\rightarrow I$ is called a fuzzy topology on X if it satisfies the following conditions:

(O1) $\tau(\underline{0})=\tau(\underline{1})=1$,

(O2) $\tau(\mu_1\wedge\mu_2)\geq\tau(\mu_1)\wedge\tau(\mu_2)$, for any $\mu_1,\mu_2\in I^X$,

(O3) $\tau(\bigvee_{i\in\Gamma}\mu_i)\geq\bigwedge_{i\in\Gamma}\tau(\mu_i)$, for any $\{\mu_i\}_{i\in\Gamma}\subset I^X$.

The pair (X,τ) is called a fuzzy topological space (fts, for short).

Theorem 2.2 Let (X,τ) be a fts. A mapping $F_\tau: I^X\rightarrow I$ defined by $F_\tau(\lambda)=\tau(\lambda^c)$ is called a fuzzy cotopology on X if it satisfies the following conditions:

(F1) $F_\tau(\underline{0})=F_\tau(\underline{1})=1$,

(F2) $F_\tau(\mu_1\vee\mu_2)\geq F_\tau(\mu_1)\wedge F_\tau(\mu_2)$, for $\mu_1,\mu_2\in I^X$,

(F3) $F_\tau(\bigwedge_{i\in\Gamma}\mu_i)\geq\bigwedge_{i\in\Gamma}F_\tau(\mu_i)$, for any $\{\mu_i\}_{i\in\Gamma}\subset I^X$.

Theorem 2.3[8] Let (X,τ) be a fts. For each $\lambda\in I^X$, we define the operators $\mathrm{cl},\mathrm{int}: I^X\rightarrow I^X$ as

$$\mathrm{cl}(\lambda)=\bigwedge\{\mu\in I^X \mid \lambda\leq\mu,\ F_\tau(\mu)>0\}.$$

$$\mathrm{int}(\lambda)=\bigvee\{\mu\in I^X \mid \lambda\geq\mu,\ \tau(\mu)>0\}.$$

For each $\lambda,\mu\in I^X$, they satisfy the followings:

(1) $\mathrm{cl}(\underline{0})=\underline{0}$.

(2) $\lambda\leq\mathrm{cl}(\lambda)$.

(3) $\mathrm{cl}(\lambda)\vee\mathrm{cl}(\mu)=\mathrm{cl}(\lambda\vee\mu)$.

(4) $\mathrm{cl}(\mathrm{cl}(\lambda))=\mathrm{cl}(\lambda)$.

(5) $\mathrm{cl}(\lambda)^c=\mathrm{int}(\lambda^c)$ and $\mathrm{int}(\lambda)^c=\mathrm{cl}(\lambda^c)$

(6) If $\tau(\lambda)>0$, then $\lambda=\mathrm{int}(\lambda)$,

(7) If $F_\tau(\lambda)>0$, then $\lambda=\mathrm{cl}(\lambda)$.

Let (X,τ_1) and (Y,τ_2) be fts's. A function $f:(X,\tau_1)\rightarrow(Y,\tau_2)$ is called continuous iff $\tau_2(\lambda)\leq\tau_1(f^{-1}(\lambda))$ for all $\lambda\in I^Y$.

Definition 2.4 Let (X,τ) be a fts and $\lambda\in I^X$.

(1) λ is fuzzy semi-open iff there exists $\mu\in I^X$ with $\tau(\mu)>0$ such that $\mu\leq\lambda\leq \mathrm{cl}(\mu)$.

(2) λ is fuzzy regular open iff $\lambda=\mathrm{int}(\mathrm{cl}(\lambda))$.

(3) λ is fuzzy regular closed iff $\lambda=\mathrm{cl}(\mathrm{int}(\lambda))$.

3. Some good extensions of compactness

Definition 3.1 Let (X,T) be an ordinary topological space and $\alpha\in I$. A mapping $\lambda:(X,T)\to I$, where I has its usual topology, is said to be α-lower semi-continuous if and only if for every $t\in I_1$ with $\alpha>t$, $\lambda^{-1}(t,1]\in T$

If λ is lower semi-continuous, then λ is α-lower semi-continuous for every $\alpha\in I$. Moreover, λ is 1-lower semi-continuous iff λ is lower semi-continuous. Naturally, every mapping from (X,T) to I is 0-lower semi-continuous.

Definition 3.2 [1,18] Let (X,T) be an ordinary topological space.

(1) Define the mapping $W(T):I^X\to I$ by

$$W(T)(\lambda)=\bigvee\{\alpha\in I_1 \mid \alpha-\text{lower semi}-\text{continuous}\}$$
$$=\bigvee\{\alpha\in I_1 \mid \lambda^{-1}(\alpha,1]\in T\}$$

Then $W(T)$ is a fuzzy topology on X.

(2) Define the mapping $W(T_c):I^X\to I$ by

$$W(T_c)(\lambda)=W(T)(\lambda^c)$$
$$=\bigvee\{\alpha\in I_1 \mid \lambda^{-1}[1-\alpha,1]\in T_c\},$$

where $T_c=\{A \mid A^c\in T\}$.

Then $W(T_c)$ is a fuzzy cotopology on X.

This provides a goodness of extension criterion for fuzzy topological properties. Recall that a fuzzy extension of a topological property of (X,T) is said to be good when it is possessed by $W(T)$ iff the original property is possessed by T.

Lemma 3.3 Let (X,T) be an ordinary topological space and $\mathrm{Int}(A)$ (resp. $\mathrm{Cl}(A)$) denoted interior (resp. closure) of A in (X,T). For each $\lambda\in I^X$ and $t\in I_1$, $\mathrm{int}(\lambda)$ (resp. $\mathrm{cl}(\lambda)$) denote interior (resp. closure) of A in $(X,W(T))$.

(1) $\mathrm{cl}(\lambda)^{-1}(t,1]\subseteq \mathrm{Cl}(\lambda^{-1}(t,1])$
$\subseteq \mathrm{Cl}(\lambda^{-1}[t,1])\subseteq \mathrm{cl}(\lambda)^{-1}[t,1]$.

(2) $\mathrm{int}(\lambda)^{-1}(t,1]\subseteq \mathrm{Int}(\lambda^{-1}(t,1])$
$\subseteq \mathrm{Int}(\lambda^{-1}[t,1])\subseteq \mathrm{int}(\lambda)^{-1}[t,1]$.

Proof (1) We will prove that any closed set C in (X,T) with $\lambda^{-1}(t,1]\subseteq C$ satisfies $\mathrm{cl}(\lambda)^{-1}(t,1]\subseteq C$. Now let $\lambda^{-1}(t,1]\subseteq C$, C is closed in (X,T) and let $\mu:X\to I$ defined by

$$\mu(x)=\begin{cases}1, & x\in C,\\ t, & x\notin C.\end{cases}$$

Then, for each $s\in I_1$, we have

$$\mu^{-1}[s,1]=\begin{cases}X, & s\leq t,\\ C, & s>t.\end{cases}$$

So, $\mu^{-1}[s,1]$ is closed set in (X,T), then μ^c is lower semi-continuous. Hence

$$W(T)(\mu^c)=W(T_c)(\mu)=1.$$

We also have that $\lambda\leq\mu$. Hence $\mathrm{cl}(\lambda)\leq\mu$.

Thus $\mathrm{cl}(\lambda)(x)>t$ implies $\mu(x)>t$ and $x\in C$.

It follows that $\mathrm{cl}(\lambda)^{-1}(t,1]\subseteq \mathrm{Cl}(\lambda^{-1}(t,1])$..

Clearly, since $\lambda\leq \mathrm{cl}(\lambda)$,

$$\lambda^{-1}(t,1]\subseteq \mathrm{cl}(\lambda)^{-1}(t,1]\subseteq \mathrm{cl}(\lambda)^{-1}[t,1].$$

Let $x\notin \mathrm{cl}(\lambda)^{-1}[t,1]$. Then $\mathrm{cl}(\lambda)(x)<t$. By the definition of $\mathrm{cl}(\lambda)$, there exists $\mu\in I^X$ with $\lambda\leq\mu$ and $W(T)(\mu^c)>0$ such that

$$\mathrm{cl}(\lambda)(x)\leq\mu(x)<t.$$

Since $\lambda^{-1}[t,1]\leq\mu^{-1}[t,1]\in T_c$,

$$\mathrm{Cl}(\lambda^{-1}[t,1])\leq\mu^{-1}[t,1].$$

Thus, $x\notin \mathrm{Cl}(\lambda^{-1}[t,1])$. Hence,

$$\mathrm{Cl}(\lambda^{-1}[t,1])\subseteq \mathrm{cl}(\lambda)^{-1}[t,1].$$

(2) Similarly, for interiors clearly

$$\mathrm{int}(\lambda)^{-1}(t,1]\subseteq\lambda^{-1}(t,1]\subseteq\lambda^{-1}[t,1].$$

Since $\mathrm{int}(\lambda)^{-1}(t,1]\in T$, so

$$\mathrm{int}(\lambda)^{-1}(t,1]\subseteq \mathrm{Int}(\lambda^{-1}(t,1])\subseteq \mathrm{Int}(\lambda^{-1}[t,1]).$$

Secondly, let $U\subseteq\lambda^{-1}[t,1]$, where U is open in T, Defining $\mu\in I^X$ by

$$\mu(x)=\begin{cases}t, & x\in U,\\ 0, & x\notin U,\end{cases}$$

then, for all $s\in I_1$,

$$\mu^{-1}(s,1]=\begin{cases}\phi, & s\geq t,\\ U, & s<t.\end{cases}$$

Then, $\mu^{-1}(s,1]\in T$ for all $s\in I_1$, so, $W(T)(\mu)=1$ and $\mu\leq\lambda$. So, $\mu\leq \mathrm{int}(\lambda)$. Hence $x\in C$ so $\mathrm{int}(\lambda)(x)\geq t$. Thus $\mathrm{Int}(\lambda^{-1}[t,1])\subseteq \mathrm{Int}(\lambda)^{-1}[t,1]$.

Corollary 3.4 In above lemma, we have

$$\chi_{\mathrm{Cl}(A)}=\mathrm{cl}(\chi_A) \quad\text{and}\quad \chi_{\mathrm{Int}(A)}=\mathrm{int}(\chi_A).$$

Proof Taking $\lambda=\chi_A$ in Lemma 3.3(1), we have

$$\mathrm{cl}(\chi_A)^{-1}(t,1]\subseteq \mathrm{Cl}(\chi_A^{-1}(t,1])$$
$$\subseteq \mathrm{Cl}(\chi_A^{-1}[t,1])\subseteq \mathrm{cl}(\chi_A)^{-1}[t,1],$$

for all $t\in I_1$. Since

$$\mathrm{Cl}(A)=\mathrm{Cl}(\chi_A^{-1}(t,1])\subseteq \mathrm{cl}(\chi_A)^{-1}[t,1],$$

we have $\mathrm{cl}(\chi_A)(x) \geq t$ for all $x \in \mathrm{cl}(A)$ and $t \in I_1$. Hence $\mathrm{cl}(\chi_A)(x) = 1$ for all $x \in \mathrm{cl}(A)$. But for $x \notin \mathrm{cl}(A)$, then $x \notin \mathrm{cl}(\chi_A)^{-1}(t, 1]$.

Since $\mathrm{cl}(\chi_A)(x) \leq t$ for all $t \in I_1$, then $\mathrm{cl}(\chi_A) = \chi_{\mathrm{Cl}(A)}$.

We can prove the dual argument for interiors.

Lemma 3.5 In Lemma 3.3, let A be a subset of X.

(a) If A is semiopen in (X, T), then χ_A is fuzzy semiopen in $(X, W(T))$.

(b) A set A is regular open in (X, T) iff χ_A is fuzzy regular open in $(X, W(T))$.

Proof (a) Let A be semiopen in (X, T). Then there exists an open set G in (X, T) such that $G \subseteq A \subseteq \mathrm{Cl}(G)$. It follows that $W(T)(\chi_G) = 1$ and $\chi_G \leq \chi_A \leq \chi_{\mathrm{Cl}(G)} = \mathrm{cl}(\chi_G)$, so χ_A is fuzzy semiopen.

(b) From Corollary 3.4, it easily proved that

$A = \mathrm{Int}(\mathrm{Cl}(A))$ iff $\chi_A = \chi_{\mathrm{Int}(\mathrm{Cl}(A))} = \mathrm{int}(\mathrm{cl}(\chi_A))$

Definition 3.6 A fts (X, T) is fuzzy compact iff for each family $\beta \subset I^X$ and $\alpha \in I_1$ with $\tau(\lambda) > \alpha$ for all $\lambda \in \beta$ such that $\sup_{\lambda \in \beta} \lambda \geq \alpha$, and for each $\varepsilon \in (0, \alpha]$, there exists a finite subfamily $\beta_0 \subseteq \beta$ such that $\sup_{\lambda \in \beta_0} \lambda \geq \alpha - \varepsilon$.

In the crisp case of τ, fuzzy compactness coincides with Lowen's fuzzy compactness [16].

Theorem 3.7 $(X, W(T))$ is fuzzy compact iff (X, T) is compact. Thus fuzzy compactness is a good extension of compactness.

Proof Let $\{G_i \mid i \in \Gamma\}$ be an open cover of X in T. Then the family $\{\chi_{G_i} \mid i \in \Gamma\}$ is a family of fuzzy sets in X with $\sup_{i \in \Gamma} \chi_{G_i} = 1 > \alpha$ for each $\alpha \in I_1$. Since the characteristic function of every open set is 1-lower semi-continuous, then $W(T)(\chi_{G_i}) = 1 > \alpha$, for all $i \in \Gamma$ and $\alpha \in I_1$. Since $(X, W(T))$ is fuzzy compact, for all $\varepsilon \in I_0$, there exist $i_1, i_2, \ldots, i_n \in \Gamma$ such that

$$\sup_{i \in \{i_1, \ldots, i_n\}} (\chi_{G_i}) \geq 1 - \varepsilon$$

i.e., $\sup_{i \in \{i_1, \ldots, i_n\}} (\chi_{G_i}) = 1$ So $\bigcup_{i \in \{i_1, \ldots, i_n\}} G_i = X$ and (X, T) is compact.

Conversely, let $\beta \subset I^X$ with $W(T)(\lambda) > \alpha$ for each $\lambda \in \beta$ and $\alpha \in I_1$ such that

$$\sup_{\lambda \in \beta} \lambda \geq \alpha$$

and let $0 < \varepsilon \leq \alpha$. Take t such that $\alpha - \varepsilon < t < \alpha$. From $W(T)(\lambda) > \alpha > t$, for all $\lambda \in \beta$, we have $\lambda^{-1}(t, 1] \in T$ for each $\lambda \in \beta$.

Then $\{\lambda^{-1}(t, 1]\}_{\lambda \in \beta}$ is a collection of open sets covering X. Otherwise, there exists $x \in X$ such that, $(\sup_{\lambda \in \beta} \lambda)(x) \leq t < \alpha$ contradicting the fact that $\sup_{\lambda \in \beta} \lambda \geq \alpha$. Since (X, T) is compact, there exists a finite subfamily $\beta_0 = \{\lambda_i \mid i = 1, 2, \ldots, n\}$ such that $\{\lambda_i^{-1}(t, 1] \mid \lambda_i \in \beta_0\}$ covers X, i.e. $\sup_{\lambda_i \in \beta_0} \lambda_i \geq \alpha - \varepsilon$.

Otherwise, there exists $x \in X$ such that

$$\sup_{\lambda_i \in \beta_0} \lambda_i(x) < \alpha - \varepsilon < t.$$

Then $x \notin (\sup_{\lambda_i \in \beta_0} \lambda_i)^{-1}(t, 1]$ contradicting the covering property of $\{\lambda_i^{-1}(t, 1] \mid \lambda_i \in \beta_0\}$.

Definition 3.8 A fts (X, T) is fuzzy almost compact iff for each family $\beta \subset I^X$ and $\alpha \in I_1$ with $\tau(\lambda) > \alpha$ for all $\lambda \in \beta$ such that $\sup_{\lambda \in \beta} \lambda \geq \alpha$, and for each $\varepsilon \in (0, \alpha]$, there exists a finite subfamily $\beta_0 \subseteq \beta$ such that $\sup_{\lambda \in \beta_0} \mathrm{cl}(\lambda) \geq \alpha - \varepsilon$.

In the crisp case of τ, fuzzy almost compactness coincides with L_0-fuzzy almost compactness introduced by Bulbul and Warner [3].

Theorem 3.9 $(X, W(T))$ is fuzzy almost compact iff (X, T) is almost compact. Thus fuzzy almost compactness is a good extension of almost compactness.

Proof Let $\{G_i \mid i \in \Gamma\}$ be an open cover of (X, T). Then $\{\chi_{G_i} \mid i \in \Gamma\}$ is a family of fuzzy sets in X with $\sup_{i \in \Gamma} \chi_{G_i} = 1 > \alpha$ and $W(T)(\chi_{G_i}) = 1 > \alpha$ for each $\alpha \in I_1$. Since $(X, W(T))$ is fuzzy almost compact, there exist $i_1, i_2, \ldots, i_n \in \Gamma$ such that

$$\sup_{i \in \{i_1, \ldots, i_n\}} (\mathrm{cl}(\chi_{G_i})) \geq 1 - \varepsilon$$

for all $\varepsilon \in I_0$, i.e.,

$$\sup_{i \in \{i_1, \ldots, i_n\}} (\mathrm{cl}(\chi_{G_i})) = 1.$$

By Corollary 3.4, $\mathrm{cl}(\chi_{G_i}) = \chi_{\mathrm{Cl}(G_i)}$ So,

$$\sup_{i \in \{i_1, \ldots, i_n\}} \chi_{\mathrm{Cl}(G_i)} = 1.$$

Then $\bigcup_{i \in \{i_1, \ldots, i_n\}} \mathrm{Cl}(G_i) = X$, and (X, T) is almost compact.

Conversely, let $\beta \subset I^X$ with $W(T)(\lambda) > \alpha$ for $\lambda \in \beta$ and $\alpha \in I_1$ such that $\sup_{\lambda \in \beta} \lambda \geq \alpha$. Let $0 < \varepsilon \leq \alpha$ and take t such that $\alpha - \varepsilon < t < \alpha$. Since $W(T)(\lambda) > \alpha > t$, for all $\lambda \in \beta$, we have $\lambda^{-1}(t, 1] \in T$ for each $\lambda \in \beta$. Then $\{\lambda^{-1}(t, 1]\}_{\lambda \in \beta}$ is a collection of open sets covering X. Since (X, T) is almost compact, there exists a finite subfamily $\beta_0 = \{\lambda_i \mid i = 1, 2, \ldots, n\}$ such that

$$\{\mathrm{Cl}(\lambda_i^{-1}(t, 1]) \mid \lambda_i \in \beta_0\}$$

covers X. Then by Lemma 3.3(1),

$$\{\mathrm{cl}(\lambda_i)^{-1}(t, 1] \mid \lambda_i \in \beta_0\}$$

covers X. And $\sup_{\lambda_i\in\beta_0}\mathrm{cl}(\lambda_i)\geq\alpha-\varepsilon$.

Otherwise, there exists $x\in X$ such that

$$\sup_{\lambda_i\in\beta_0}\mathrm{cl}(\lambda_i)(x)<\alpha-\varepsilon<t.$$

Then $x\notin(\sup_{\lambda_i\in\beta_0}\mathrm{cl}(\lambda_i))^{-1}[t,1]$ contradicting the covering property of $\{\mathrm{cl}(\lambda_i)^{-1}[t,1]\mid\lambda_i\in\beta_0\}$.

Definition 3.10 A fts (X,T) is fuzzy nearly compact iff for each family $\beta\subset I^X$ and $\alpha\in I_1$ with $\tau(\lambda)>\alpha$ for all $\lambda\in\beta$ such that $\sup_{\lambda\in\beta}\lambda\geq\alpha$, and for each $\varepsilon\in(0,\alpha]$, there exists a finite subfamily $\beta_0\subseteq\beta$ such that $\sup_{\lambda\in\beta_0}\mathrm{int}(\mathrm{cl}(\lambda))\geq\alpha-\varepsilon$.

In the crisp case of τ, fuzzy near compactness coincides with L_0-fuzzy near compactness introduced by Bulbul and Warner [3].

Theorem 3.11 $(X,W(T))$ is fuzzy nearly compact iff (X,T) is nearly compact. Thus fuzzy near compactness is a good extension of nearcompactness.

Proof The proof follows closely the argument of the preceding theorem. Assuming $(X,W(T))$ to be fuzzy nearly compact, proceed as before to obtain $i_1,i_2,\ldots,i_n\in\Gamma$ such that

$$\sup_{i\in\{i_1,\ldots,i_n\}}\mathrm{int}(\mathrm{cl}(\chi_{G_i}))\geq 1-\varepsilon$$

for all $\varepsilon\in I_0$, i.e.,

$$\sup_{i\in\{i_1,\ldots,i_n\}}\mathrm{int}(\mathrm{cl}(\chi_{G_i}))=1$$

By Corollary 3.4, $\mathrm{int}(\mathrm{cl}(\chi_{G_i}))=\chi_{\mathrm{Int\,Cl}(G_i)}$

$$\sup_{i\in\{i_1,\ldots,i_n\}}\chi_{\mathrm{Int}(\mathrm{Cl}(G_i))}=1.$$

Then

$$\bigcup_{i\in\{i_1,\ldots,i_n\}}\mathrm{Int}(\mathrm{Cl}(G_i))=X$$

Hence (X,T) is nearly compact.

Conversely, proceeding again as before, we obtain a finite subcollection $\{\lambda_i\mid i=1,\ldots,n\}$ such that $\{\mathrm{Int\,Cl}(\lambda_i^{-1}(t,1])\mid\lambda_i\in\beta_0\}$ covers X. Then by Lemma 3.3(1) and (2),

$$\mathrm{Int}(\mathrm{Cl}(\lambda_i^{-1}(t,1]))\subseteq\mathrm{Int}(\mathrm{cl}(\lambda)^{-1}[t,1])$$
$$\subseteq\mathrm{int}(\mathrm{cl}(\lambda))^{-1}[t,1].$$

So, $\{(\mathrm{int}(\mathrm{cl}(\lambda_i)))^{-1}[t,1]\mid i=1,\ldots,n\}$ covers X. Thus, $\sup_{i\in\{1,\ldots,n\}}\mathrm{int}(\mathrm{cl}(\lambda_i))\geq\alpha-\varepsilon$.

Definition 3.12 A fts (X,T) is fuzzy weakly compact iff for each family $\beta\subset I^X$ of regular open sets and $\alpha\in I_1$ such that $\sup_{\lambda\in\beta}\lambda\geq\alpha$, and for each $\varepsilon\in(0,\alpha]$, there exists a finite subfamily $\beta_0\subseteq\beta$ such that $\sup_{\lambda\in\beta_0}\mathrm{cl}(\lambda)\geq\alpha-\varepsilon$.

In the crisp case of τ, fuzzy weak compactness coincides with L_0-fuzzy weak compactness introduced by Bulbul and Warner [3].

Theorem 3.13 $(X,W(T))$ is fuzzy weakly compact iff (X,T) is weakly compact. Thus fuzzy weak compactness is a good extension of weak compactness.

Proof Replacing the open sets of Theorem 3.9, by regular open sets, again there exist $i_1,i_2,\ldots,i_n\in\Gamma$ such that $\sup_{i\in\{i_1,\ldots,i_n\}}\mathrm{cl}(\chi_{G_i})=1$

So, $\bigcup_{i\in\{i_1,\ldots,i_n\}}\mathrm{Cl}(G_i)=X$ and (X,T) is weakly compact.

Again, conversely, taking $\beta\subseteq I^X$ to consist of fuzzy regular open sets, we obtain $\{\lambda^{-1}(t,1]\}_{\lambda\in\beta}$ as a collection of open sets covering X. Since $\lambda^{-1}(t,1]$ is open, then $\lambda^{-1}(t,1]\subseteq\mathrm{Int}(\mathrm{Cl}(\lambda^{-1}(t,1]))$. Since

$$\mathrm{Int}(\mathrm{Cl}(\lambda^{-1}(t,1]))=\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda^{-1}(t,1])))),$$

then $\{\mathrm{Int}(\mathrm{Cl}(\lambda^{-1}(t,1]))\mid\lambda\in\beta\}$ is a collection of regular open sets covering X. Since (X,T) is weakly compact, there exists a finite subcollection $\{\mathrm{Int}(\mathrm{Cl}(\lambda_i^{-1}(t,1]))\mid i=1,2,\ldots,n\}$ such that

$$X=\bigcup_{i\in\{1,2,\ldots,n\}}\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\lambda_i^{-1}(t,1]))).$$

Since

$$\mathrm{Int}(\mathrm{Cl}(\lambda_i^{-1}(t,1]))\subseteq\mathrm{int}(\mathrm{cl}(\lambda_i))^{-1}[t,1]=\lambda_i^{-1}[t,1],$$

$\{\mathrm{Cl}(\lambda_i^{-1}[t,1])\mid i=1,2,\ldots,n\}$ covers X.

By Lemma 3.3(1),

$$X=\bigcup_{i\in\{1,2,\ldots,n\}}\mathrm{cl}(\lambda_i)^{-1}[t,1].$$

Hence, $\sup_{i\in\{i=1,2,\ldots,n\}}\mathrm{cl}(\lambda_i)\geq\alpha-\varepsilon$.

Definition 3.14 A fts (X,T) is S-closed iff for each family $\beta\subset I^X$ of fuzzy semiopen sets and $\alpha\in I_1$ such that $\sup_{\lambda\in\beta}\lambda\geq\alpha$, and for each $\varepsilon\in(0,\alpha]$, there exists a finite subfamily $\beta_0\subseteq\beta$ such that

$$\sup_{\lambda\in\beta_0}\mathrm{cl}(\lambda)\geq\alpha-\varepsilon.$$

Theorem 3.15 $(X,W(T))$ is S-closed iff (X,T) is S-closed. So it is a good extension.

Proof Taking $\{G_i\}_{i\in\Gamma}$ to be a semiopen cover, we get $\{\chi_{G_i}\}_{i\in\Gamma}$ a family of fuzzy semiopen sets from Lemma 3.5(a). The rest of the proof in this direction is exactly as before.

Conversely, let β be a collection of fuzzy semiopen sets and the previous notation for α,ε and t. Since λ is fuzzy semiopen for each $\lambda\in\beta$, there exists $\mu\in I^X$ with $W(T)(\mu)\geq\alpha$ such that $\mu\leq\lambda\leq\mathrm{cl}(\mu)$. Then $\mu^{-1}(t,1]\in T$ and

$$\mu^{-1}(t,1]\subseteq\lambda^{-1}(t,1]$$
$$\subseteq\mathrm{cl}(\mu)^{-1}(t,1]\subseteq\mathrm{Cl}(\mu^{-1}(t,1]).$$

Then $\lambda^{-1}(t,1]$ is semiopen in T for each $\lambda\in\beta$.

So, $\{\lambda^{-1}(t,1] \mid \lambda\in\beta\}$ covers X. Since (X, T) is S-closed, there exists a finite family $\{\lambda_i ; i=1,2,\ldots,n\}$ such that

$\{Cl(\lambda_i^{-1}[t,1]) \mid i=1,2,\ldots,n\}$ covers X.

The remainder of the proof follows that of Theorem 3.9.

4. Some properties of several fuzzy compactness types

Theorem 4.1 In a fts (X,τ), the following are equivalent:

(a) (X,τ) is fuzzy nearly compact.

(b) Each family $\beta\subset I^X$ with $F_\tau(\lambda)>\alpha$ for all $\lambda\in\beta$, having the property that there exists $\varepsilon\in(0,\alpha)$ such that

$$\inf_{\lambda\in\beta^*} cl(int(\lambda))>1-\alpha+\varepsilon$$

for each finite subfamily $\beta^*\subset\beta$, also has the property $\inf_{\lambda\in\beta}\lambda>1-\alpha$

Proof ((a) ⇒ (b)) Let $\beta\subset I^X$ with $F_\tau(\lambda)>\alpha$ for all $\lambda\in\beta$and $\alpha\in I_1$ with the given property, i.e., there exists $\varepsilon\in(0,\alpha)$ such that, for finite subfamily $\beta^*\subset\beta$.

$$\inf_{\lambda\in\beta^*} cl(int(\lambda))>1-\alpha+\varepsilon.$$

Suppose $\inf_{\lambda\in\beta}\lambda\le 1-\alpha$. Then, $\sup_{\lambda\in\beta}(1-\lambda)\ge\alpha$.

By (a), for every $\varepsilon>0$, there exists a finite family $\beta^*\subset\beta$ such that

$$\sup_{\lambda\in\beta^*} int(cl(1-\lambda))\ge\alpha-\varepsilon,$$

which implies that $\inf_{\lambda\in\beta^*} cl(int(\lambda))\le 1-\alpha+\varepsilon$.

But this contradicts the given property of β.

Thus, $\inf_{\lambda\in\beta}\lambda>1-\alpha$.

((b) ⇒ (a)) Let $\beta\subset I^X$ with $\tau(\lambda)>\alpha$ for all $\lambda\in\beta$ with $\sup_{\lambda\in\beta}\lambda\ge\alpha$. Suppose that (X, T) is not fuzzy nearly compact. Then there exists $\varepsilon\in(0,\alpha)$ such that

$$\sup_{\lambda\in\beta^*} int(cl(\lambda))<\alpha-\varepsilon$$

for every finite subfamily $\beta^*\subset\beta$. Therefore,

$$1-\sup_{\lambda\in\beta^*} int(cl(\lambda))=\inf_{\lambda\in\beta^*}(1-int(cl(\lambda))) >1-\alpha+\varepsilon.$$

Since, $1-int(cl(\lambda))=cl(int(1-\lambda))$, we obtain

$$\inf_{\lambda\in\beta^*} cl(int(1-\lambda))>1-\alpha+\varepsilon$$

for the above $\alpha\in I_1$, $\varepsilon\in(0,\alpha)$ and for every finite subfamily $\beta^*\subset\beta$.

Therefore $\{1-\lambda \mid \lambda\in\beta\}\subset I^X$with $F_\tau(1-\lambda)>\alpha$ having the property (b). So, by (b) it follows that $\inf_{\lambda\in\beta}(1-\lambda)>1-\alpha$ Therefore we obtain $1-\inf_{\lambda\in\beta}(1-\lambda)<\alpha$ i.e., $\sup_{\lambda\in\beta}\lambda<\alpha$ which contradicts the given property of β.

Thus (X,τ) must be fuzzy nearly compact.

Theorem 4.2 In a fts (X,τ), the following are equivalent:

(a) (X,τ) is fuzzy almost compact.

(b) Each family $\beta\subset I^X$ with $F_\tau(\lambda)>\alpha$ for all $\lambda\in\beta$, having the property that there exists $\varepsilon\in(0,\alpha)$ such that $\inf_{\lambda\in\beta^*} int(\lambda)>1-\alpha+\varepsilon$,for each finite subfamily $\beta^*\subset\beta$, also has the property $\inf_{\lambda\in\beta}\lambda>1-\alpha$.

Proof Similar to the proof of Theorem 4.1.

Definition 4.3 Let (X,τ) be a fts. A fuzzy set $\nu\in I^X$ is said to be nearly compact fuzzy set iff for each family $\beta\subset I^X$ with $\tau(\lambda)>\alpha$ for all $\lambda\in\beta$ such that $\sup_{\lambda\in\beta}\lambda\ge\nu$ and for each $\varepsilon\in(0,\alpha)$ there exists a finite family $\beta^*\subset\beta$ such that

$$\sup_{\lambda\in\beta^*} int(cl(\lambda))\ge\nu-\varepsilon.$$

In the crisp case of τ, the nearly compact fuzzy set coincides with the Lo-fuzzy nearly compact fuzzy set given by Bulbul [2].

Remark 4.4 A fts (X,τ) is fuzzy nearly compact iff each constant fuzzy set α $(0<\alpha\le 1)$ is a nearly compact fuzzy set.

Theorem 4.5 Let (X,τ) be a fts and $\nu\in I^X$. Then the following are equivalent:

(a) ν is nearly compact fuzzy set.

(b) Each family $\beta\subset I^X$with $F_\tau(\lambda)>\alpha$ for all $\lambda\in\beta$, having the property that there exists $\varepsilon\in(0,\alpha)$ such that $\inf_{\lambda\in\beta^*} cl(int(\lambda))>1-\nu+\varepsilon$ for each finite subfamily $\beta^*\subset\beta$, also has the property $\inf_{\lambda\in\beta}\lambda>1-\nu$.

Proof Similar to the proof of Theorem 4.1.

Definition4.6 Let (X,τ) be a fts. A fuzzy set $\nu\in I^X$ is said to be almost compact fuzzy set iff for each family $\beta\subset I^X$ with $\tau(\lambda)>\alpha$ for all $\lambda\in\beta$ such that $\sup_{\lambda\in\beta}\lambda\ge\nu$ and for each $\varepsilon\in(0,\alpha)$ there exists a finite family $\beta^*\subset\beta$ such that

$$\sup_{\lambda\in\beta^*} cl(\lambda)\ge\nu-\varepsilon.$$

In the crisp case of τ, the almost compact fuzzy set coincides with the Lo-fuzzy almost compact fuzzy set given by Bulbul [2].

Remark 4.7 A fts (X,τ) is fuzzy almost compact iff each constant fuzzy set α $(0<\alpha\le 1)$ is a almost compact fuzzy set.

Theorem 4.8 Let (X,τ) be a fts and $\nu\in I^X$. Then the following are equivalent:

(a) ν is almost compact fuzzy set.

(b) Each family $\beta\subset I^X$with $F_\tau(\lambda)>\alpha$ for all $\lambda\in\beta$, having the property that there exists $\varepsilon \in (0, \alpha)$ such that $\inf_{\lambda \in \beta^*} \mathrm{int}(\lambda) > 1 - \nu + \varepsilon$ for each finite subfamily $\beta^* \subset \beta$, also has the property $\inf_{\lambda \in \beta} \lambda > 1 - \nu$.

Proof Similar to the proof of Theorem 4.1.

Theorem 4.9 If $f: X \to Y$ is a fuzzy continuous mapping, then the image of a nearly compact fuzzy set is an almost compact fuzzy set.

Proof Let ν be any nearly compact fuzzy set in X and $\beta \subset I^Y$ with $\tau_2(\mu) > \alpha$ for all $\mu \in \beta$ such that $\sup_{\mu \in \beta} \mu \geq f(\nu)$. Hence we get $\sup_{\mu \in \beta}(f^{-1}(\mu)) \geq \nu$

By fuzzy continuity of f, $\tau_1(f^{-1}(\mu)) \geq \tau_2(\mu) > \alpha$

for all $\mu \in \beta$. Since ν is nearly compact fuzzy set in X, it follows that for each $\varepsilon \in (0, \alpha)$,
there exists $\beta^* \subset \beta$ such that

$$\sup_{\mu \in \beta^*} \mathrm{int}(\mathrm{cl}(f^{-1}(\mu))) \geq \nu - \varepsilon$$

Since

$$f(\nu - \varepsilon)(y) = \sup_{x \in f^{-1}(\{y\})} (\nu - \varepsilon)(x) \geq \sup_{x \in f^{-1}(\{y\})} (\nu)(x) - \varepsilon = f(\nu)(y) - \varepsilon,$$

then

$$f(\sup_{\mu \in \beta^*} \mathrm{int}(\mathrm{cl}(f^{-1}(\mu)))) \geq f(\nu - \varepsilon) \geq f(\nu) - \varepsilon.$$

Hence we get

$$\sup_{\mu \in \beta^*} f(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu)))) \geq f(\nu) - \varepsilon.$$

But $\mathrm{int}(\mathrm{cl}(f^{-1}(\mu))) \leq \mathrm{cl}(f^{-1}(\mu)) \leq f^{-1}(\mathrm{cl}(\mu))$ (Corollary 3.1[8]) Taking direct image of both sides gives:

$$f(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu)))) \leq ff^{-1}(\mathrm{cl}(\mu)) \leq \mathrm{cl}(\mu),$$

and hence $\sup_{\mu \in \beta^*} \mathrm{cl}(\mu) \geq f(\nu) - \varepsilon$. So $f(\nu)$ is almost compact fuzzy set of Y.

Theorem 4.10 The fuzzy continuous image of an almost compact fuzzy set is an almost compact fuzzy set.

Proof Similar to the proof of Theorem 4.9.

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