Journal of the Korean Statistical Society (2003), **32**: *4, pp 359–371*

NEW BOUNDS ON THE OVERFLOW PROBABILITY IN JACKSON NETWORKS

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ABSTRACT

We consider the probability that the total population of a stable Jackson network reaches a given large value. By using the fluid limit of the reversed network, we derive new upper and lower bounds on this probability, which are sharper than those in Glasserman and Kou (1995). In particular, the improved lower bound is useful for analyzing the performance of an importance sampling estimator for the overflow probability in Jackson tandem networks. Bounds on the expected time to overflow are also obtained.

AMS 2000 subject classifications. Primary 60K25; Secondary 60K20.
Keywords. Jackson network, overflow probability, importance sampling estimator, asymptotics.

1. INTRODUCTION

We analyze a rare-event probability in queueing networks such as

$$p_K := P\{\text{network population reaches } K \text{ before returning to } 0, \text{ after leaving } 0\}.$$

If we think of K as an upper limit on the network population, this probability becomes a type of *overflow probability.* The networks we consider are Jackson networks, namely networks of exponential servers with Bernoulli routing and Poisson exogenous arrivals. It is generally accepted that simulations have been used to estimate the overflow probability in Jackson networks because the overflow probability in these networks is analytically intractable. Based on a heuristic application of large-deviations techniques, Parekh and Walrand (1989) proposed

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importance sampling estimators for simulation of the overflow probability in various Jackson networks. For example, in Jackson tandem networks their estimator corresponds to interchanging the arrival rate and the slowest service rate. Glasserman and Kou (1993) proved the asymptotic logarithmic limit of p_K:

$$\lim_{K\to\infty} \frac{1}{K} \log p_K = \log \rho_*, \tag{1.1}$$

where ρ_* is the load of the most highly loaded node. Glasserman and Kou (1995) also investigated the performance of Parekh and Walrand's estimators by showing the bounds on p_K as follows:

$$a_1 \rho_*^K K^{-1} \le p_K \le a_2 \rho_*^K (K+1)^n, \tag{1.2}$$

where a_1, a_2 are constants and n is the number of nodes in the network. (1.1) is the immediate consequence of bounds in (1.2).

In this paper we use the time reversal and the fluid limit to get new bounds on p_K which are sharper than (1.2). With the improved lower bound we can show that Parekh and Walrand's estimators have the bounded relative error without the additional assumption proposed by Glasserman and Kou (1995). The bounds on the expected time to overflow are also considered.

Anantharam and Ganesh (1994) established the bounds on the overflow probability based on the individual node, whereas we obtain the bounds based on the total population.

2. New Bounds on the Overflow Probability

A Jackson network consists of n nodes that operate on a FIFO (First-In-First-Out) basis. Customers arrive at node i from outside the system according to a Poisson process with rate $\overline{\lambda}_i$ and, if necessary, wait in a queue until the server gets free to serve. Service time is exponentially distributed with mean $1/\mu_i$. Once service is completed, the customer is routed to node j with probability r_{ij} or leaves the system with probability $r_{i\cdot} := 1 - \sum_{j=1}^n r_{ij}$.

We say that node i feeds node j if there is a sequence $k_1, k_2, \ldots, k_q$ such that $r_{i\,k_1} r_{k_1\,k_2} \cdots r_{k_q\,j} > 0$. A network is *exogenously supplied* if each node i has an exogenous arrival rate $\overline{\lambda}_i \neq 0$ or can be fed by another node j for which $\overline{\lambda}_j \neq 0$. The network is *open* if every node i has an exit probability $r_{i\cdot} \neq 0$ or feeds a node j for which $r_{j\cdot} \neq 0$. We assume that the network is both exogenously supplied and open.

A Jackson network can be described as a Markov jump process $\{X(t); t \geq 0\}$ on $\mathcal{S} \equiv \mathbf{N}^n$, where the state $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{S}$ depicts the system when there are x_i customers waiting or being served at node i.

Jackson (1957) gave an expression for the invariant measure of a Jackson network. His results are restated in the following theorem (see Brémaud, 1981).

THEOREM 2.1. *For an exogenously supplied and open Jackson network for which the solution* $(\lambda_1, \ldots, \lambda_n)$ *to the traffic equations*

$$\lambda_i = \overline{\lambda}_i + \sum_{j=1}^{n} \lambda_j r_{ji}, \quad i = 1, 2, \ldots, n$$

satisfies the light traffic conditions

$$\rho_i := \frac{\lambda_i}{\mu_i} < 1, \quad i = 1, 2, \ldots, n, \tag{2.1}$$

the stationary distribution $\pi(\vec{x})$ *of* $\vec{x} = (x_1, \ldots, x_n) \in \mathcal{S}$ *is given by the product*

$$\pi(\vec{x}) = \prod_{i=1}^{n} (1 - \rho_i) \rho_i^{x_i}.$$

The ratio ρ_i is called the *load* on node i. We call a Jackson network *stable* if the light traffic conditions (2.1) hold. We assume that the light traffic conditions hold and $X(0) = \vec{0}$. We further assume that node 1 has the maximal load ρ_*, that is, $\rho_* = \rho_1 > \rho_i$ for all $i = 2, \ldots, n$.

Let us define an *overflow set* by

$$C_K = \{\vec{x} \in \mathcal{S} : x_1 + x_2 + \cdots + x_n = K\},$$

the set of states in which the network population is exactly K. Then p_K can be written by

$$p_K = P\{X(t) \text{ hits } C_K \text{ before } \vec{0} \,|\, X(t) \text{ leaves } \vec{0}\}.$$

We bound p_K by first bounding the stationary probability of the set C_K, $\pi(C_K)$.

LEMMA 2.1. *For all* $K \geq 1$,

$$b_1 \rho_*^K \leq \pi(C_K) \leq b_2 \rho_*^K,$$

where b_1 *and* b_2 *are positive constants.*

PROOF. From Theorem 2.1, the stationary probability $\pi(C_K)$ is given by

$$\begin{aligned}\pi(C_K) &= \sum_{x_1+\cdots+x_n=K}\cdots\sum\prod_{i=1}^{n}(1-\rho_i)\rho_i^{x_i}\\ &= \prod_{i=1}^{n}(1-\rho_i)\rho_*^K\sum_{x_1+\cdots+x_n=K}\cdots\sum\prod_{i=1}^{n}\left(\frac{\rho_i}{\rho_*}\right)^{x_i}\\ &= \prod_{i=1}^{n}(1-\rho_i)\rho_*^K\sum_{k=0}^{K}\sum_{x_1+\cdots+x_n=k}\cdots\sum\prod_{i=2}^{n}\left(\frac{\rho_i}{\rho_*}\right)^{x_i},\end{aligned}$$

where the last equality follows from $\rho_1/\rho_* = 1$.

Since $\rho_i/\rho_* < 1$ for $i = 2, \dots, n$, we can get the upper bound on $\pi(C_K)$:

$$\pi(C_K) \le b_2\rho_*^K,$$

where $b_2 = (1-\rho_*)\rho_*^{n-1}\prod_{i=2}^{n}(1-\rho_i)/(\rho_*-\rho_i)$, independent of K.

To obtain the lower bound on $\pi(C_K)$, we denote

$$\vec{x}_K := (K, 0, \dots, 0),$$

the state in which there are K customers at the maximally loaded node 1 and no customers anywhere else. Clearly $\vec{x}_K \in C_K$. Thus we have

$$\pi(C_K) \ge b_1\rho_*^K,$$

where $b_1 = (1-\rho_*)\prod_{i=2}^{n}(1-\rho_i)$. □

REMARK. Glasserman and Kou (1995) obtained other bounds on $\pi(C_K)$ as follows:

$$\rho_*^K\prod_{i=1}^{n}(1-\rho_i) \le \pi(C_K) \le \rho_*^K\prod_{i=1}^{n}(1-\rho_i)(K+1)^{n-1}. \tag{2.2}$$

When $n = 1$, the upper bound in (2.2) meets that in Lemma 2.1, whereas for a large value K, the bound derived in Lemma 2.1 is sharper than that in (2.2).

THEOREM 2.2. *Consider an exogenously supplied and open Jackson network which satisfies the stability condition. Then, for some positive constants c_1, c_2 that do not depend on the population size K but may depend on the network parameters, we have*

$$c_1\rho_*^K \le p_K \le c_2\rho_*^K. \tag{2.3}$$

For large enough K, we have the explicit estimates given by

$$c_1 = \frac{\sum(\overline{\lambda}_i + \mu_i)}{\sum \overline{\lambda}_i} \cdot \frac{\mu_1(1 - r_{11})}{\lambda_1' + \mu_1(1 - r_{11})} \left(1 - \frac{\lambda_1' - \mu_1 r_{11}}{\mu_1 - \mu_1 r_{11}}\right), \tag{2.4}$$

where λ_1' is defined in (3.9) and

$$c_2 = \frac{\sum(\overline{\lambda}_i + \mu_i)}{\sum \overline{\lambda}_i} \prod_{i=2}^{n} \left(1 - \frac{\rho_i}{\rho_*}\right)^{-1}. \tag{2.5}$$

PROOF. The proof is given in Section 3. □

REMARK. Glasserman and Kou (1995) pointed out one simple case of Jackson tandem network in which Theorem 2.2 holds. In an n-node Jackson tandem network with an arrival rate λ and service rates μ_i satisfying

$$\frac{1}{\lambda} > \sum_{i=1}^{n} \frac{1}{\mu_i}, \tag{2.6}$$

they showed that $p_K \geq b\rho_*^K$ with a constant b. Under the condition (2.6), they also proved that Parekh and Walrand's estimators have the bounded relative error in certain parameter regions. However, from Theorem 2.2 we do not need the additional condition in (2.6) any more to get the same result.

COROLLARY 2.1. *Let S_K be the first hitting time at C_K, starting from $\vec{0}$. Then, for sufficiently large K, we have*

$$d_1 \rho_*^{-K} \leq E(S_K) \leq d_2 \rho_*^{-K},$$

where d_1 and d_2 are constants, independent of K.

PROOF. Let δ have the distribution of the time taken to return to $\vec{0}$, starting from $\vec{0}$ and conditioned on not visiting C_K and let $\delta_1, \delta_2, \ldots$ be *iid* random variables with the distribution of δ. Let Δ denote the time to hit C_K, starting from $\vec{0}$ and conditioned on not returning to $\vec{0}$. Then, since the evolution of the process starts afresh each time it hits $\vec{0}$, it is easy to see that

$$S_K = \begin{cases} \Delta, & \text{with probability } p_K, \\ \sum_{k=1}^{\nu} \delta_k + \Delta, & \text{with probability } 1 - p_K, \end{cases} \tag{2.7}$$

where ν denotes the number of returns to $\vec{0}$ before hitting C_K, which is a geometric random variable with parameter p_K, *i.e.*,

$$P\{\nu = i\} = (1-p_K)^{i-1} p_K, \quad i = 1, 2, \ldots$$

From (2.7) it follows that

$$\begin{aligned} E(S_K) &= \frac{1-p_K}{p_K} E(\delta) + E(\Delta) \\ &= \frac{1}{p_K}\{(1-p_K)E(\delta) + p_K E(\Delta)\}. \end{aligned}$$

Here $(1-p_K)E(\delta) + p_K E(\Delta)$ is the expected time taken to either return to $\vec{0}$ or visit C_K, starting from $\vec{0}$. This expected time is clearly dominated by the expected time to return to $\vec{0}$, starting from $\vec{0}$ in the network with infinite buffers, denoted by $E(S_0)$. So, we have

$$E(S_K) \le \frac{1}{p_K} E(S_0).$$

From the renewal theorem it follows that

$$\pi(\vec{0}) = \frac{(\sum \overline{\lambda}_i)^{-1}}{E(S_0)}$$

since the expected time spent in $\vec{0}$ in each visit to it is $(\sum \overline{\lambda}_i)^{-1}$ and the expected time between visits is $E(S_0)$. Substituting for $E(S_0)$ above gives

$$E(S_K) \le d_2 \rho_*^{-K},$$

where

$$d_2 = \frac{1}{c_1 \sum \overline{\lambda}_i} \prod_{i=1}^{n} (1-\rho_i)^{-1}$$

can be explicitly computed from the estimate for c_1 in Theorem 2.2.

Now we observe that δ stochastically dominates the time spent in $\vec{0}$, which is exponentially distributed random variable of rate $\sum \overline{\lambda}_i$. Since an independent geometric sum of independent exponential random variables is also exponential, it follows that $\sum_{i=1}^{\nu} \delta_i$ stochastically dominates an exponential random variable of rate $p_K \sum \overline{\lambda}_i$. Using (2.7), we obtain

$$E(S_K) \ge \frac{1-p_K}{p_K \sum \overline{\lambda}_i}.$$

Hence, the lower bound on $E(S_K)$ for large enough K is given by

$$E(S_K) \ge d_1 \rho_*^{-K},$$

where $d_1 = 1/(2c_2 \sum \overline{\lambda}_i)$. □

3. PROOF OF THEOREM 2.2

In this section, we discuss the proof of Theorem 2.2. The upper bound is given in Section 3.1 and the lower bound is derived in Section 3.2.

3.1. The upper bound

Let $\widehat{X}(n)$ be a discrete-time Markov chain obtained by embedding at the virtual jump times of the original process $X(t)$. The virtual jump process is the sum of the exogenous arrival process and the virtual departure processes of the individual nodes. These are independent Poisson processes, with the future independent of the current state; hence the virtual jump process is Poisson of rate $\sum(\bar{\lambda}_i + \mu_i)$ with the future independent of the current state. Then, this uniformized Markov chain $\widehat{X}(n)$ has the same stationary distribution π as the original process $X(t)$.

Next, let $\widehat{Y}(n)$ be obtained from $\widehat{X}(n)$ by watching it in the set $\{\vec{0}\} \cup C_K$. Then, $\widehat{Y}(n)$ is also a discrete-time Markov chain with the stationary distribution $\widehat{\pi}$ given by

$$\widehat{\pi}(\vec{x}) = \left\{ \sum_{\vec{y} \in \{\vec{0}\} \cup C_K} \pi(\vec{y}) \right\}^{-1} \pi(\vec{x})$$

and the transition matrix $\widehat{P}$ defined by $\widehat{P}(\vec{x}, \vec{y}) = P\{\widehat{Y}(n+1) = \vec{y} \,|\, \widehat{Y}(n) = \vec{x}\}$ for all $\vec{x}, \vec{y} \in \{\vec{0}\} \cup C_K$. Specifically, we have

$$\begin{aligned}
\widehat{P}(\vec{0},\vec{0}) &= P\{\widehat{X}(n) = \vec{0} \text{ before } \widehat{X}(n) \in C_K | \widehat{X}(1) \neq \vec{0}, \widehat{X}(0) = \vec{0}\} \\
&\quad \times P\{\widehat{X}(1) \neq \vec{0} | \widehat{X}(0) = \vec{0}\} + P\{\widehat{X}(1) = \vec{0} | \widehat{X}(0) = \vec{0}\} \\
&= \frac{\sum \bar{\lambda}_i}{\sum(\bar{\lambda}_i + \mu_i)} P\{\text{return to } \vec{0} \text{ before hitting } C_K | \widehat{X}(1) \neq \vec{0}, \widehat{X}(0) = \vec{0}\} \\
&\quad + \frac{\sum \mu_i}{\sum(\bar{\lambda}_i + \mu_i)} \\
&= \frac{\sum \bar{\lambda}_i}{\sum(\bar{\lambda}_i + \mu_i)}(1 - p_K) + \frac{\sum \mu_i}{\sum(\bar{\lambda}_i + \mu_i)} \\
&= 1 - \frac{p_K \sum \bar{\lambda}_i}{\sum(\bar{\lambda}_i + \mu_i)}. \quad \text{(Anantharam and Ganesh, 1994)} \qquad (3.1)
\end{aligned}$$

Therefore,

$$p_K = \sum_{\vec{x}\in C_K} \widehat{P}(\vec{0},\vec{x}) \frac{\sum(\overline{\lambda}_i + \mu_i)}{\sum \overline{\lambda}_i}$$

using $1 - \widehat{P}(\vec{0},\vec{0}) = \sum_{\vec{x}\in C_K} \widehat{P}(\vec{0},\vec{x})$.

Now, let $\widetilde{Y}(n)$ be the time reversal of $\widehat{Y}(n)$, so $\widetilde{Y}(n)$ is a Markov chain with the same stationary distribution $\widehat{\pi}$ and its transition matrix $\widetilde{P}$ given by

$$\widetilde{P}(\vec{x},\vec{y}) = \frac{\widehat{\pi}(\vec{y})\widehat{P}(\vec{y},\vec{x})}{\widehat{\pi}(\vec{x})}, \quad \vec{x},\vec{y} \in \{\vec{0}\} \cup C_K.$$

Thus, p_K can be rewritten as

$$p_K = \frac{1}{\pi(\vec{0})} \sum_{\vec{x}\in C_K} \pi(\vec{x}) \widetilde{P}(\vec{x},\vec{0}) \frac{\sum(\overline{\lambda}_i + \mu_i)}{\sum \overline{\lambda}_i}.$$

Since $\widetilde{P}(\vec{x},\vec{0}) \leq 1$ for all $\vec{x} \in C_K$, the upper bound in (2.3) and the estimate for c_2 in (2.5) are obtained from Lemma 2.1.

3.2. The lower bound

Let $\widehat{Z}(n)$ be a discrete-time Markov chain obtained from $\widehat{X}(n)$ by watching it in the set $\{\vec{0}, \vec{x}_K\}$, and let $\widehat{Q}$ be its transition matrix. Then, since $\vec{x}_K \in C_K$, it is easy to see that

$$\widehat{P}(\vec{0},\vec{0}) \leq \widehat{Q}(\vec{0},\vec{0}).$$

Therefore, from (3.1) and the fact that $\widehat{Q}(\vec{0},\vec{0}) + \widehat{Q}(\vec{0},\vec{x}_K) = 1$ it follows that

$$p_K \geq \widehat{Q}(\vec{0},\vec{x}_K) \frac{\sum(\overline{\lambda}_i + \mu_i)}{\sum \overline{\lambda}_i}. \tag{3.2}$$

Let $\widetilde{Q}$ denote the transition matrix of the time reversal of $\widehat{Z}(n)$. Substituting for $\widehat{Q}(\vec{0},\vec{x}_K)$ in (3.2) gives

$$p_K \geq \frac{\pi(\vec{x}_K)}{\pi(\vec{0})} \widetilde{Q}(\vec{x}_K,\vec{0}) \frac{\sum(\overline{\lambda}_i + \mu_i)}{\sum \overline{\lambda}_i}. \tag{3.3}$$

If we can show that $\widetilde{Q}(\vec{x}_K,\vec{0})$ is bounded below by a positive constant, which is independent of K, then it is easy to see, from (3.3) and Theorem 2.1, that

$$p_K \geq c_1 \rho_*^K.$$

Now, to complete the proof of Theorem 2.2 we have to show that $\widetilde{Q}(\vec{x}_K, \vec{0}) > 0$, uniformly in K and estimate the constant c_1. This consists of two parts. In the first part we use a fluid limit of the time reversal to show that with sufficiently high probability, the queue length process $\widetilde{X}_1(t)$ at node 1, starting from K, stays below K after its fluid limit hits 0 until the reversed network $\widetilde{X}(t)$, starting from $\vec{x}_K$, becomes empty. In the second part we construct a new stable Jackson network. A queue length at each node of this new Jackson network dominates that of the original reversed network. Then we prove that the process $\widetilde{X}_1(t)$, starting from K, does not hit K until its fluid limit reaches 0, with probability bounded away from 0.

1. The fluid limit of the time reversal. Let $\widetilde{X}(t)$ be the time reversal of the original process $X(t)$. Then, it is known that the time reversal $\widetilde{X}(t)$ is a Markov jump process for a different Jackson network, with the same number of nodes but different parameters as follows (Walrand, 1988):

$$\begin{aligned}
\widetilde{\overline{\lambda}}_i &= \lambda_i r_{i\cdot}, & i &= 1, 2, \dots, n, \\
\widetilde{\mu}_i &= \mu_i, & i &= 1, 2, \dots, n, \\
\widetilde{r}_{ij} &= \frac{\lambda_j}{\lambda_i} r_{ji}, & i, j &= 1, 2, \dots, n, \\
\widetilde{r}_{i\cdot} &= \frac{\overline{\lambda}_i}{\lambda_i}, & i &= 1, 2, \dots, n,
\end{aligned}$$

where tildes refer to the corresponding quantities in the reversed Jackson network. Also, if the original network is exogenously supplied and open, then so is its time reversal. Moreover, the solutions to the traffic equations of the time-reversed network are also the same, *i.e.*,

$$\widetilde{\lambda}_i = \lambda_i, \quad i = 1, 2, \dots, n.$$

Let $\widetilde{X}^K(t)$ denote the Markov jump process when the time reversal $\widetilde{X}(t)$ is started with $\vec{x}_K$. It was shown in Anantharam *et al.* (1990) that the process $\widetilde{X}^K(t)$ converges to a fluid limit $X^f(t)$ in the sense that, for any $\epsilon_0 > 0$ and all $\epsilon > \epsilon_0$,

$$\lim_{K \to \infty} P\left\{ \sup_{0 \le t \le T} \left\| \frac{1}{K} \widetilde{X}^K(Kt) - X^f(t) \right\| \ge \epsilon \;\middle|\; \left\| \frac{1}{K} \widetilde{X}^K(0) - X^f(0) \right\| < \epsilon_0 \right\} = 0, \tag{3.4}$$

where $\|X\| = \max|X_i|$ and $T = \inf\{t > 0 : X_i^f(t) = 0 \text{ for all } i = 1, \dots, n\}$. It was also proved in Anantharam and Ganesh (1994) that $\sum_{i=1}^n X_i^f(t)$, the total quantity of fluid in the network, is strictly decreasing at a positive rate as long as the amount of fluid is not zero. Furthermore, the fluid limit $X_i^f(t)$ at node i stays at zero after it reaches zero until the total amount of fluid becomes empty. In other words, if we let $T_i := \inf\{t > 0 : X_i^f(t) = 0\}$ for $i = 1, 2, \dots, n$, then

$$X_i^f(t) = 0 \quad \text{for all} \quad T_i \le t \le T, \tag{3.5}$$

where T is the time at which the total amount of fluid $\sum_{i=1}^n X_i^f(t)$ hits zero. By using the relation to the fluid limit we can obtain bounds on the process $\widetilde{X}_1^K(t)$, the number of customers in the maximally loaded node 1.

Observe that

$$\lim_{K\to\infty} \frac{1}{K}\widetilde{X}_i^K(0) = X_i^f(0)$$

exists for $i = 1, 2, \dots, n$. Then, from (3.4) we can determine that the following statements are true with probability going to one as K goes to infinity;

$$\widetilde{X}_1^K(t) < \epsilon K \quad \text{for all} \quad KT_1 \le t \le KT \tag{3.6}$$

and

$$\sum_{i=1}^n \widetilde{X}_i^K(KT) < \epsilon K \tag{3.7}$$

for all $\epsilon > 0$.

Let T_K denote the first time that the actual queue length process $\widetilde{X}^K(t)$ hits the state $\vec{0}$. Then, by applying Corollary 1 in Anantharam (1989) it follows from (3.7) that $T_K - KT$ is stochastically dominated by the sum of ϵK *iid* random variables of finite mean and variance. Since the exogenous arrival process is Poisson of rate $\sum \overline{\lambda}_i$, the total number of exogenous arrivals in the period $[KT, T_K]$(taken to be empty if $KT > T_K$) is less than a constant times ϵK, with probability going to one as $K \to \infty$. This implies that with asymptotic probability one,

$$\sum_{i=1}^n \widetilde{X}_i^K(t) < Const \cdot \epsilon K \quad \text{for all} \quad KT \le t \le T_K,$$

where *Const* denotes a constant which is independent of K and $\epsilon > 0$ is arbitrary. This allows us to extend the validity of (3.6) through the period $[KT_1, T_K]$, that is,

$$\widetilde{X}_1^K(t) < Const \cdot \epsilon K \quad \text{for all} \quad KT_1 \le t \le T_K, \tag{3.8}$$

with asymptotic probability one.

2. Constructing a new stable Jackson network. Now, we investigate the process $\widetilde{X}_1^K(t)$ during the time period $(0, KT_1)$. Let $\widetilde{X}'(t)$ denote the process started in the same initial condition as $\widetilde{X}^K(t)$, but with the output of the node 1 replaced by its virtual departure process of the reversed network. Then, it evolves like another Jackson network of which inflow rates are given by the solution $(\lambda_1', \lambda_2', \ldots, \lambda_n')$ to the generalized traffic equation (Goodman and Massey, 1984):

$$\lambda_i' = \tilde{\tilde{\lambda}}_i + \widetilde{\mu}_1 \widetilde{r}_{1i} + \sum_{j=2}^{n} \min(\lambda_j', \widetilde{\mu}_j) \widetilde{r}_{ji}, \quad i = 1, 2, \ldots, n. \tag{3.9}$$

In addition, we can see that for all sample paths ω, the process $\widetilde{X}'(t;\omega)$ dominates $\widetilde{X}^K(t;\omega)$, that is,

$$\widetilde{X}_i'(t;\omega) \geq \widetilde{X}_i^K(t;\omega) \quad \text{for all} \quad t \geq 0, \quad i = 1, 2, \ldots, n.$$

To prove this, we apply the coloring arguments introduced in Anantharam and Ganesh (1994). Color red the virtual departures from node 1 that are not actual departures and color blue all other departures from all nodes and exogenous arrivals. Then, red customers can arrive only when node 1 is empty. Observe that when a service occurs at a node with nonempty queue, we can decide which customer in the queue departs without affecting the process of total number of customers at the nodes. So, if we assume that blue customers always have precedence over red customers, *i.e.* when a service takes place at node i, red customer at node i does not move unless there are no blue customers at node i, it follows that $\widetilde{X}^K(t)$, the process of blue customers is dominated by $\widetilde{X}'(t)$, the process of all customers.

Now, we check the stability of new Jackson network evolved by the process $\widetilde{X}'(t)$. Let us define

$$\phi_i(\vec{\eta}) = \tilde{\tilde{\lambda}}_i + \widetilde{\mu}_1 \widetilde{r}_{1i} + \sum_{j=2}^{n} \min(\eta_j, \widetilde{\mu}_j) \widetilde{r}_{ji}, \quad i = 1, 2, \ldots, n$$

for a vector $\vec{\eta} := (\eta_1, \eta_2, \ldots, \eta_n)$. Then, the solution $(\lambda_1', \lambda_2', \ldots, \lambda_n')$ to the generalized traffic equation in (3.9) is the unique fixed point of ϕ. For $\vec{\lambda} := (\lambda_1, \lambda_2, \ldots, \lambda_n)$, the solution to the traffic equation of the original Jackson network, we have

$$\begin{aligned}\phi_i\Big(\frac{1}{\rho_*}\vec{\lambda}\Big) &= \tilde{\tilde{\lambda}}_i + \widetilde{\mu}_1\widetilde{r}_{1i} + \sum_{j=2}^{n} \min\Big(\frac{1}{\rho_*}\lambda_j, \widetilde{\mu}_j\Big)\widetilde{r}_{ji} \\ &\le \frac{1}{\rho_*}\tilde{\tilde{\lambda}}_i + \widetilde{\mu}_1\widetilde{r}_{1i} + \sum_{j=2}^{n}\frac{1}{\rho_*}\lambda_j\widetilde{r}_{ji} \\ &= \frac{1}{\rho_*}\lambda_i,\end{aligned}$$

since $\rho_* = \rho_1 < 1$. From the fact that ϕ_i is increasing it follows that $\lambda'_i \le \lambda_i/\rho_*$ for all $i = 1, 2, \ldots, n$. Therefore we obtain $\lambda'_1 \le \mu_1$ and $\lambda'_i < \mu_i$ from $\rho_i < \rho_*$ for $i = 2, \ldots, n$. Suppose that $\lambda'_1 = \mu_1$. Then $(\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ can be a solution to the traffic equation of the original Jackson network. It implies $\mu_1 = \lambda_1$, which contradicts to the stability of the original network. Hence we can conclude that this new Jackson network is still stable.

Now, if we consider an initial condition where queues outside node 1 are in their stationary distributions, then the external arrival process into node 1 is Poisson of rate $\lambda'_1 - \mu_1 r_{11}$ because the departure process at each node is Poisson of rate λ'_i and $\widetilde{\mu}_1\widetilde{r}_{11} = \mu_1 r_{11}$. This external arrival process dominates the external arrival process into node 1 in the process $\widetilde{X}'(t)$, wherein the queues outside node 1 were initially empty. This can also be shown using the coloring technique employed above. Color red all customers who are in nodes $2, \ldots, n$ initially and color blue all other customers who arrive from outside the network. If we let blue customers have preemptive service priority at each node $2, \ldots, n$, then blue customers who get into the node 1 constitute the external arrival process while all customers including red customers who arrive at node 1 form a Poisson process. We thus see that $\widetilde{X}'_1(t)$ is dominated by a Markov jump process $M(t)$ of arrival rate $\lambda'_1 - \mu_1 r_{11}$, service rate μ_1, the transition probability to itself after the service r_{11}, and started at $M(0) = K$. It can be seen that for the process $M(t)$,

$$P\{M(t) = 0 \text{ before } M(t) = K\} > \frac{\mu_1(1 - r_{11})}{\lambda'_1 + \mu_1(1 - r_{11})}\left(1 - \frac{\lambda'_1 - \mu_1 r_{11}}{\mu_1 - \mu_1 r_{11}}\right) \quad (3.10)$$

for all K. Notice that the first term on the right-hand side of (3.10) is the probability that $M(t)$ is decreased by one before it is increased by one or jumps to itself and the second term is less than the probability that $M(t)$ hits 0 before K, starting from $K - 1$ given by

$$\left\{1 - \left(\frac{\lambda'_1 - \mu_1 r_{11}}{\mu_1 - \mu_1 r_{11}}\right)^K\right\}^{-1}\left(1 - \frac{\lambda'_1 - \mu_1 r_{11}}{\mu_1 - \mu_1 r_{11}}\right)$$

for all $K \geq 1$.

Let $T_0 := \inf\{t > 0 : M(t) = 0\}$. Since the stable Markov jump process $M(t)$ does not grow by K in time linear in K, with probability one, we can have

$$\lim_{K\to\infty} P\{M(t) < K \text{ for all } T_0 \leq t \leq KT_1\} = 1. \tag{3.11}$$

Hence $\widetilde{X}_1^K(t)$, which is dominated by $M(t)$, does not hit K before the time KT_1 with probability bounded uniformly away from zero because $\lambda_1' < \mu_1$ and $r_{11} < 1$.

Combining this with (3.8) gives $\widetilde{X}_1^K(t) < K$ for all $0 \leq t \leq T_K$ with a positive probability, independent of K. Thus $\widetilde{X}(t)$ with initial state $\vec{x}_K$ satisfies

$$\liminf_{K\to\infty} P\{\widetilde{X}(t) = \vec{0} \text{ before } \widetilde{X}(t) \text{ hits } \vec{x}_K\} > 0.$$

Then, since the time reversal of the watching of the embedding is the same as the watching of the embedding the time reversal, we have that $\widetilde{Q}(\vec{x}_K, \vec{0}) > 0$. Further, for large enough K, it follows from (3.8), (3.10), and (3.11) that

$$\widetilde{Q}(\vec{x}_K, \vec{0}) \geq \frac{\mu_1(1-r_{11})}{\lambda_1' + \mu_1(1-r_{11})}\left(1 - \frac{\lambda_1' - \mu_1 r_{11}}{\mu_1 - \mu_1 r_{11}}\right)$$

which implies the estimate for c_1 in (2.4).

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