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MEASURES FOR STABILITY OF SLOPE ESTIMATION ON THE SECOND ORDER RESPONSE SURFACE AND EQUALLY-STABLE SLOPE ROTATABILITY†

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ABSTRACT

This paper introduces new measures for the stability of slope estimation on the second order response surface at a point and on a sphere. As a measure of point stability of slope estimation, we suggest a point dispersion measure of slope variances over all directions at a point. A spherical dispersion measure is also proposed as a measure of spherical stability of slope estimation on each sphere. Some designs are studied to explore the usefulness of the proposed measures. Using the point dispersion measure, another concept of slope rotatability called equally-stable slope rotatability is proposed as a useful property of response surface designs. We provide a set of conditions for a design to have equally-stable slope rotatability.

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1. INTRODUCTION

In response surface designs, it has been recognized in recent years that the difference between estimated responses at two points may be of greater interest than the absolute response at each point in experimental regions. Herzberg (1967), Box and Draper (1980) and Huda and Mukerjee (1984) have considered the problem related to estimation of differences in response. If differences at two

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close points in experimental regions are involved, estimation of local slopes (first derivatives) of a response surface becomes more important. For example, it might be of interest to estimate the reaction rates in chemical experiments, the rates of change in the yield of a crop, or the rates of disintegration of a radioactive material in an animal.

Since the first work of Atkinson (1970) in this area, many research papers have been subsequently published such as Ott and Mendenhall (1972), Murty and Studden (1972), Myers and Lahoda (1975), Hader and Park (1978), Mukerjee and Huda (1985), Park (1987), Park and Kim (1992), Huda and Shafiq (1992), Draper and Ying (1994), Ying *et al.* (1995a, b), Kim *et al.* (1996), Park and Kwon (1998) and so on. Especially, as an analogue to variance dispersion graph (VDG) in Giovannitti-Jesen and Myers (1989), Jang and Park (1993) proposed the slope variance dispersion graph (SVDG) for estimating the slope of a response surface.

In estimating the slope of a response surface, analogous concepts of rotatability in Box and Hunter (1957), which are important and desirable properties of response surface designs, have been developed. Hader and Park (1978) firstly introduced the concept of slope rotatability over axial directions and studied slope rotatable central composite designs. Park (1987) extended this concept to the case which considers the variances of estimated slope over all directions. Recently, Park and Kwon (1998) introduced another concept of slope rotatability which is based on the maximum value of the variances of estimated slope over all directions.

The concepts of slope rotatability in Park (1987) and Park and Kwon (1998) are based on reliability measures of slope estimation over all possible directions at a point in experimental regions. As reliability measures of slope estimation at a point, they used the average and maximum of the variances of the estimated slope over all possible directions. The graphical methods of design assessment for slope estimation such as SVDG in Jang and Park (1993) and quantile plots in Kim *et al.* (1996) are also based on the average of the variances of the estimated slope over all possible directions at a point. These graphical methods focus on the spherical dispersion and distribution of the average of the variances of the estimated slope over all possible directions at each point on the sphere.

If we are interested in slope estimation over all possible directions at a point, stability of the variances of the estimated slope over all possible directions at each point may be as important as their reliabilities such as the average and maximum of the variances. Hence, the aim of this paper is to propose some dispersion measures for stability of slope estimation at a point and on a sphere. Also, we introduce a new concept of slope rotatability based on stability of the variances of estimated slope over all possible directions at a point. From now on, slope variance means the variance of estimated slope at a point. Note that the concept of stability of slope estimation implies the stability of slope variances over all possible directions at a point or on a sphere.

The remainder of this paper is organized as follows. Section 2 introduces concepts of slope rotatability. Pointwise and spherical dispersion measures for stability of slope estimation are proposed in Section 3. A new type of slope rotatability based on these measures is stated in Section 4. Section 5 draws some concluding remarks.

2. CONCEPTS OF SLOPE ROTATABILITY

Following the usual response surface model, we assume that the dependent variable is adequately approximated by a low order polynomial of k independent variables, $x_1, x_2, \ldots, x_k$. Independent variables are coded so that their origin is the center of a region of interest over which the polynomial approximation is used. It is also assumed that the mean of a response is adequately represented by the second order model in k independent variables, $\mathbf{x}' = (x_1, x_2, \ldots, x_k)$, of the form

$$\eta(\mathbf{x}) = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{i<j}^{k} \beta_{ij} x_i x_j, \tag{2.1}$$

which can be written in matrix notation as

$$\eta(\mathbf{x}) = \mathbf{x_s}'\boldsymbol{\beta}. \tag{2.2}$$

Here, $\mathbf{x_s} = (1, x_1, x_2, \ldots, x_k, x_1^2, x_2^2, \ldots, x_k^2, x_1x_2, x_1x_3, \ldots, x_{k-1}x_k)'$ and $\boldsymbol{\beta}$ is $p \times 1$ column vector of the corresponding coefficients, where $p = (k+1)(k+2)/2$.

The coefficients in the polynomial are to be estimated, by the method of least squares, from observations on the response variable,

$$y_u = \eta(\mathbf{x}_u) + \epsilon_u, \; u = 1, 2, \ldots, N, \tag{2.3}$$

where $\mathbf{x}_u$ is a version of $\mathbf{x}$ and observations are taken at N selected combinations of the x variables. The ϵ_u's are assumed to be uncorrelated random errors with zero mean and constant variance σ^2. The $\boldsymbol{\beta}$ is then estimated by the method of least squares

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \tag{2.4}$$

in which $\mathbf{X}$ is the $N \times p$ matrix (rank p) of values of the p elements of $\mathbf{x_s}$ taken at the design points, and $\mathbf{y}$ is the $N \times 1$ vector of y observations. The variance-covariance matrix of $\mathbf{b}$ is given by $\mathrm{Var}(\mathbf{b}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$. Then, $\widehat{y}(\mathbf{x})$, the estimate of $\eta(\mathbf{x})$, and its variance are given by

$$\widehat{y}(\mathbf{x}) = \mathbf{x_s}'\mathbf{b} \quad \text{and} \quad \mathrm{Var}\{\widehat{y}(\mathbf{x})\} = \sigma^2\mathbf{x_s}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x_s}. \tag{2.5}$$

In order to explore the variations of mean response, estimation of the first derivatives (or slopes) of a response surface is of interest. For the second order model, the first derivative of $\widehat{y}(\mathbf{x})$ with respect to x_i (i^{th} axial direction) is given by

$$\widehat{s}_i(\mathbf{x}) = \frac{\partial\widehat{y}(\mathbf{x})}{\partial x_i} = b_i + 2b_{ii}x_i + \sum_{j\neq i}^{k} b_{ij}x_j, \tag{2.6}$$

which can be written in matrix notation as

$$\frac{\partial\widehat{y}(\mathbf{x})}{\partial x_i} = d_i(\mathbf{x})\mathbf{b}. \tag{2.7}$$

Here, $\mathbf{b}$ is given in (2.4) and $d_i(\mathbf{x})$ is $1 \times p$ row vector which equals to $\partial\mathbf{x_s}'/\partial x_i$. For example, when $k = 2$, $d_1(\mathbf{x}) = (0, 1, 0, 2x_1, 0, x_2)$ and $d_2(\mathbf{x}) = (0, 0, 1, 0, 2x_2, x_1)$.

Then the variance of the estimated derivative over the i^{th} axial direction is given by

$$\mathrm{Var}\{\widehat{s}_i(\mathbf{x})\} = \sigma^2 d_i(\mathbf{x})(\mathbf{X}'\mathbf{X})^{-1}d_i'(\mathbf{x}). \tag{2.8}$$

The analogue of the Box-Hunter rotatability criterion is a requirement that $\mathrm{Var}\{\widehat{s}_i(\mathbf{x})\}$ be constant on circles ($k = 2$), spheres ($k = 3$) or hyperspheres ($k > 3$) centered at the design origin. Estimates of the slope would then be equally reliable for all points $\mathbf{x}$ that are equidistant from the design center. Hader and Park (1978) referred to this property as slope rotatability over axial directions.

In practice, however, it is often of interest to estimate the slope of a response surface at a point $\mathbf{x}$, not only over axial directions, but also over any specified direction. Let the estimated slope vector be

$$\widehat{\mathbf{s}}(\mathbf{x}) = \begin{pmatrix} \partial\widehat{y}(\mathbf{x})/\partial x_1 \\ \partial\widehat{y}(\mathbf{x})/\partial x_2 \\ \vdots \\ \partial\widehat{y}(\mathbf{x})/\partial x_k \end{pmatrix} = \begin{pmatrix} d_1(\mathbf{x})\mathbf{b} \\ d_2(\mathbf{x})\mathbf{b} \\ \vdots \\ d_k(\mathbf{x})\mathbf{b} \end{pmatrix} = D(\mathbf{x})\mathbf{b} \tag{2.9}$$

where $D(\mathbf{x})$ is the matrix arising from the differentiation of $\widehat{y}(\mathbf{x})$ with respect to each of k independent variables, that is, $D'(\mathbf{x}) = (d_1'(\mathbf{x}), d_2'(\mathbf{x}), \ldots, d_k'(\mathbf{x}))$ and $d_i(\mathbf{x})$ are defined in (2.7).

The estimated derivative at any point $\mathbf{x}$ in the direction specified by $k \times 1$ vector of a direction cosine $\mathbf{c} = (c_1, c_2, \dots, c_k)'$ is given by $\mathbf{c}'\widehat{\mathbf{s}}(\mathbf{x})$ where $\sum_{i=1}^{k} c_i^2 = 1$. Then, the variance of this specified slope can be written as

$$\begin{aligned} V_{\mathbf{c}}(\mathbf{x}) &= \mathrm{Var}\{\mathbf{c}'\widehat{\mathbf{s}}(\mathbf{x})\} \\ &= \mathbf{c}'\mathrm{Var}\{\widehat{\mathbf{s}}(\mathbf{x})\}\mathbf{c} \\ &= \sigma^2 \mathbf{c}' D(\mathbf{x})(\mathbf{X}'\mathbf{X})^{-1} D'(\mathbf{x})\mathbf{c} \\ &= \sigma^2 \mathbf{c}' M(\mathbf{x})\mathbf{c}, \end{aligned} \tag{2.10}$$

where $M(\mathbf{x}) = D(\mathbf{x})(\mathbf{X}'\mathbf{X})^{-1}D'(\mathbf{x})$. For example, when $\mathbf{c}$ is $(1, 0, \dots, 0)'$, $V_{\mathbf{c}}(\mathbf{x})$ is the variance of the estimated slope over the first axial direction.

Regarding the slope estimation over all possible directions at a point $\mathbf{x}$, Atkison (1970) considered $\overline{V}(\mathbf{x})$, which corresponds to the average of $V_{\mathbf{c}}(\mathbf{x})$ over all possible directions. It is given by

$$\overline{V}(\mathbf{x}) = \frac{\sigma^2}{k}\mathrm{tr}\big(M(\mathbf{x})\big), \tag{2.11}$$

where tr denotes the trace of a matrix. The expression (2.11) may also be viewed as a simple average of the variances of the estimated slope in the k-axial directions. For some designs, it is possible to make this averaged variance be constant for all points $\mathbf{x}$ that are equidistant from the design center. In that case, $\overline{V}(\mathbf{x})$ is a function of only $\rho^2 = \sum_{i=1}^{k} x_i^2$. Park (1987) called this property slope rotatability over all directions and gave the necessary and sufficient conditions for a design to be slope rotatable over all directions.

For these, let $[ii]$, $[iiii]$ and $[iijj]$ denote the pure second, fourth order moments and mixed fourth order moments, respectively. That is,

$$[ii] = \frac{1}{N}\sum_{u=1}^{N} x_{iu}^2, \quad [iiii] = \frac{1}{N}\sum_{u=1}^{N} x_{iu}^4 \quad \text{and} \quad [iijj] = \frac{1}{N}\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2$$

for all $i \neq j$. If a design satisfies the following conditions, it is slope rotatable over all directions.

[1] All odd-order moments up to order 4 are zero.

[2] $[ii]$ are equal for all i.

[3] $[iiii]$ are equal for all i.

[4] $[iijj]$ are equal for all $i \neq j$.

(2.12)

In general, the design which satisfies conditions in (2.12) is called symmetric permutation invariant designs. Following Box and Hunter (1957), by adding another condition such as $[iiii] = 3[iijj]$ for all $i \neq j$ to the above conditions, we have the necessary and sufficient conditions for a design to be rotatable. Hence, the class of rotatable designs is a subset of the class of slope rotatable designs over all directions. Park (1987) listed several examples of slope rotatability over all directions and turned out that most commonly used response surface designs belong to this class.

Recently, Park and Kwon (1998) proposed another concept of slope rotatability, which was called slope rotatability with equal maximum directional variance, for the second order response surface models. Let $V_{\max}(\mathbf{x})$ be the maximum directional variance at a point $\mathbf{x}$ in the region of interest, that is, $V_{\max}(\mathbf{x}) = \max_{\mathbf{c}:\mathbf{c}'\mathbf{c}=1} V_{\mathbf{c}}(\mathbf{x})$. A design is said to be slope rotatable design with equal maximum directional variance if $V_{\max}(\mathbf{x})$ would be equal for all points $\mathbf{x}$ that are equidistant from the design center, that is, $V_{\max}(\mathbf{x})$ is a function of only $\rho^2 = \sum_{i=1}^{k} x_i^2$. They gave the necessary and sufficient conditions for a design to be slope rotatable with equal maximum directional variance in the case of $k = 2$. It is also shown that the conditions for rotatability are sufficient for a design to be slope rotatable with equal maximum directional variance.

3. MEASURES FOR STABILITY OF SLOPE ESTIMATION AT A POINT AND ON A SPHERE

3.1. Point dispersion measure of slope variances

In order to investigate the stability of slope estimation at a point $\mathbf{x}$, a dispersion measure of slope variances at a point $\mathbf{x}$ is necessary. Consider a measure of the variation in $V_{\mathbf{c}}(\mathbf{x})$ where $\mathbf{c}$ uniformly takes values in $U_1 = \{\mathbf{c} : \sum_{i=1}^{k} c_i^2 = 1\}$. The most conventional measures of point stability of slope estimation might be based on deviation of slope variances over all possible directions at a point $\mathbf{x}$. Hence, throughout this paper, we define a point dispersion measure, denoted by $S^2(\mathbf{x})$, of slope variances at a point $\mathbf{x}$ as

$$S^2(\mathbf{x}) = \psi \int_{\mathbf{c}'\mathbf{c}=1} \left\{ V_{\mathbf{c}}(\mathbf{x}) - \overline{V}(\mathbf{x}) \right\}^2 d\mathbf{c} \tag{3.1}$$

where $\psi^{-1} = \int_{\mathbf{c}'\mathbf{c}=1} d\mathbf{c}$.

Without loss of generality, we may assume the common variance σ^2 is equal to 1. For the second order model, the (i, j) element of the $k \times k$ matrix $M(\mathbf{x})$

can be represented by

$$\begin{aligned} m_{ij}(\mathbf{x}) &= \mathrm{Cov}(b_i, b_j) + \sum_{l=1}^{k} \Big\{\mathrm{Cov}(b_i, q_{jl}) + \mathrm{Cov}(b_j, q_{il})\Big\} x_l \\ &\quad + \sum_{l=1}^{k} \sum_{m=1}^{k} \mathrm{Cov}(q_{il}, q_{jm}) x_l x_m \end{aligned} \tag{3.2}$$

where $q_{il} = 2b_{ii}$ if $i = l$, b_{il} if $i < l$ and b_{li} if $i > l$. Combining this with (2.10), the slope variance with a direction cosine $\mathbf{c}$ at a point $\mathbf{x}$ can be written as

$$V_{\mathbf{c}}(\mathbf{x}) = \sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j m_{ij}(\mathbf{x}). \tag{3.3}$$

Now, we have the following representation for $\overline{V^2}(\mathbf{x})$, which is the average of $V_{\mathbf{c}}^2(\mathbf{x})$ over all possible directions at a point $\mathbf{x}$.

LEMMA 3.1. *The average of $V_{\mathbf{c}}^2(\mathbf{x})$ over all possible directions at a point $\mathbf{x}$ is*

$$\overline{V^2}(\mathbf{x}) = \frac{\{\mathrm{tr}(M(\mathbf{x}))\}^2 + 2\mathrm{tr}(M'(\mathbf{x})M(\mathbf{x}))}{k(k+2)}. \tag{3.4}$$

PROOF. From (3.3), $\overline{V^2}(\mathbf{x})$ can be written as

$$\overline{V^2}(\mathbf{x}) = \psi \int_{\mathbf{c}'\mathbf{c}=1} V_{\mathbf{c}}^2(\mathbf{x}) d\mathbf{c} = \psi \int_{\mathbf{c}'\mathbf{c}=1} \left\{ \sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j m_{ij}(\mathbf{x}) \right\}^2 d\mathbf{c}, \tag{3.5}$$

which can be reduced to

$$\psi \int_{\mathbf{c}'\mathbf{c}=1} \left[\sum_{i=1}^{k} m_{ii}^2(\mathbf{x}) c_i^4 + 2 \sum_{i<j}^{k} \Big\{ m_{ii}(\mathbf{x}) m_{jj}(\mathbf{x}) + 2m_{ij}^2(\mathbf{x}) \Big\} c_i^2 c_j^2 \right] d\mathbf{c}. \tag{3.6}$$

Simple algebra gives that

$$\psi \int_{\mathbf{c}'\mathbf{c}=1} c_i^4 d\mathbf{c} = \frac{3}{k(k+2)} \text{ and } \psi \int_{\mathbf{c}'\mathbf{c}=1} c_i^2 c_j^2 d\mathbf{c} = \frac{1}{k(k+2)} \text{ for all } i \neq j. \tag{3.7}$$

By substituting (3.7) into (3.6), (3.6) becomes

$$\frac{3\sum_{i=1}^{k} m_{ii}^2(\mathbf{x}) + 2\sum_{i<j}^{k} m_{ii}(\mathbf{x}) m_{jj}(\mathbf{x}) + 4\sum_{i<j}^{k} m_{ij}^2(\mathbf{x})}{k(k+2)}. \tag{3.8}$$

Result holds from the fact that $\{\mathrm{tr}(M(\mathbf{x}))\}^2$ and $\mathrm{tr}(M'(\mathbf{x})M(\mathbf{x}))$ can be expressed as

$$\sum_{i=1}^{k} m_{ii}^2(\mathbf{x}) + 2\sum_{i<j}^{k} m_{ii}(\mathbf{x})m_{jj}(\mathbf{x}) \quad \text{and} \quad \sum_{i=1}^{k} m_{ii}^2(\mathbf{x}) + 2\sum_{i<j}^{k} m_{ij}^2(\mathbf{x}),$$

respectively. □

Let $\mu_i(\mathbf{x}), i = 1, 2, \ldots, k$, be the eigenvalues of the $k \times k$ matrix $M(\mathbf{x})$. Since the matrix $M(\mathbf{x})$ is the variance-covariance matrix of the estimated slope vector at a point $\mathbf{x}$, all the eigenvalues of the matrix $M(\mathbf{x})$ are nonnegative. The averages of $V_{\mathbf{c}}(\mathbf{x})$ and $V_{\mathbf{c}}^2(\mathbf{x})$ over all possible directions at a point $\mathbf{x}$ can be written as

$$\overline{V}(\mathbf{x}) = \frac{\sum_{i=1}^{k} \mu_i(\mathbf{x})}{k} \quad \text{and} \quad \overline{V^2}(\mathbf{x}) = \frac{\{\sum_{i=1}^{k} \mu_i(\mathbf{x})\}^2 + 2\sum_{i=1}^{k} \mu_i^2(\mathbf{x})}{k(k+2)}, \tag{3.9}$$

since the traces of the matrix $M(\mathbf{x})$ and $M'(\mathbf{x})M(\mathbf{x})$ can be represented by $\sum_{i=1}^{k} \mu_i(\mathbf{x})$ and $\sum_{i=1}^{k} \mu_i^2(\mathbf{x})$, respectively.

Therefore, from Lemma 3.1 and the definition of $S^2(\mathbf{x})$ in (3.1), it is easy to show that the point dispersion measure, $S^2(\mathbf{x})$, can be rewritten as

$$\begin{aligned} S^2(\mathbf{x}) &= \frac{2}{k^2(k+2)}\Big[k\,\mathrm{tr}\big(M'(\mathbf{x})M(\mathbf{x})\big) - \big\{\mathrm{tr}\big(M(\mathbf{x})\big)\big\}^2\Big] \\ &= \frac{2}{k^2(k+2)}\sum_{i<j}^{k}\Big[\big\{m_{ii}(\mathbf{x}) - m_{jj}(\mathbf{x})\big\}^2 + 2km_{ij}^2(\mathbf{x})\Big] \\ &= \frac{2}{k^2(k+2)}\sum_{i<j}^{k}\big\{\mu_i(\mathbf{x}) - \mu_j(\mathbf{x})\big\}^2. \end{aligned} \tag{3.10}$$

Note that from (3.10), $S^2(\mathbf{x})$ becomes zero if and only if all the eigenvalues of the matrix $M(\mathbf{x})$ are equal, that is, $M(\mathbf{x}) = c\mathbf{I}_k$, where c is a positive constant and $\mathbf{I}_k$ is the $k \times k$ identity matrix. For example, when $\mathbf{x}$ is a center point or an axial point, the values of $S^2(\mathbf{x})$ equals zero for the second order model.

To be specific, let Ω be the class of symmetric permutation invariant designs with $[ii] = \lambda_2$, $[iiii] = \lambda_4$ and $[iijj] = \lambda_{22}$ for all $i \neq j$ under $\lambda_4 > \lambda_{22}$ and $\lambda_4 + (k-1)\lambda_{22} - k\lambda_2^2 > 0$. For a design in Ω, the traces of the matrix $M(\mathbf{x})$ and $M'(\mathbf{x})M(\mathbf{x})$ can be simply written as

$$\mathrm{tr}\big(M(\mathbf{x})\big) = N^{-1}(k\lambda_2^{-1} + A\rho^2) \tag{3.11}$$

and

$$\operatorname{tr}\big(M'(\mathbf{x})M(\mathbf{x})\big) = N^{-2}\Big(k\lambda_2^{-2} + 2\lambda_2^{-1}A\rho^2 + B\rho^4 + C\sum_{i=1}^{k} x_i^4\Big), \tag{3.12}$$

where A, B, C, P and Q are

$$\begin{aligned} A &= 4P + (k-1)\lambda_{22}^{-1}, \\ B &= 8P\lambda_{22}^{-1} + (k-2)\lambda_{22}^{-2} + (4Q + \lambda_{22}^{-1})^2, \\ C &= 8(P+Q)(2P - 2Q - \lambda_{22}^{-1}), \\ P &= \frac{\lambda_4 + (k-2)\lambda_{22} - (k-1)\lambda_2^2}{(\lambda_4 - \lambda_{22})\{\lambda_4 + (k-1)\lambda_{22} - k\lambda_2^2\}}, \\ Q &= \frac{\lambda_2^2 - \lambda_{22}}{(\lambda_4 - \lambda_{22})\{\lambda_4 + (k-1)\lambda_{22} - k\lambda_2^2\}}, \end{aligned} \tag{3.13}$$

respectively. Thus, from (3.10), (3.11) and (3.12), we may write

$$S^2(\mathbf{x}) = \frac{2}{N^2k^2(k+2)}\Big\{(kB - A^2)\rho^4 + kC\sum_{i=1}^{k} x_i^4\Big\}. \tag{3.14}$$

Let us show an example for the change of $S^2(\mathbf{x})$ values for central composite designs.

EXAMPLE 1. Consider the central composite designs (CCDs) which consist of 4 factorial points, 4 axial points with arbitrary α values and one center point. Then, the nonzero design moments are $\lambda_2 = (4 + 2\alpha^2)/9$, $\lambda_4 = (4 + 2\alpha^4)/9$ and $\lambda_{22} = 4/9$. For these CCDs, we are interested in comparing point stabilities of slope estimation at some points. Figure 1 depicts the change of the point dispersion measure $S^2(\mathbf{x})$ at $(1,1), (\sqrt{2},0), (1,0)$ and $(1/\sqrt{2}, 1/\sqrt{2})$, where $0 < \alpha \leq 2$. From this figure, the values of $S^2(\mathbf{x})$ at $(1,0)$ and $(\sqrt{2},0)$ are strictly decreasing as α increases, but not always at $(1,1)$ and $(1/\sqrt{2}, 1/\sqrt{2})$. In case of $\alpha = 1$, among these four points $(1/\sqrt{2}, 1/\sqrt{2})$ is the best for the point stability of slope estimation. In general, when we construct stable designs for slope estimation, the values of $S^2(\mathbf{x})$ should be made as small as possible.

3.2. Spherical dispersion measure of slope variances

When we are interested in stability of all slope variances on the sphere with a radius ρ, it may not be sufficient to use only the arithmetic mean of the point

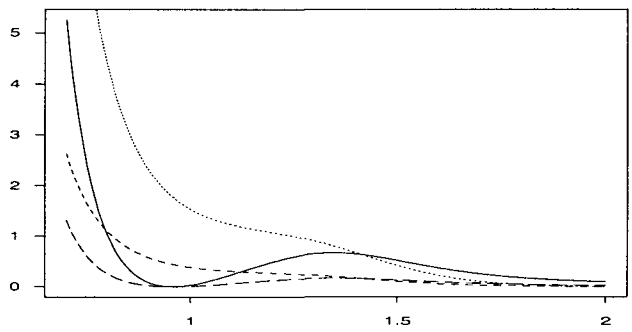

FIGURE 1 *Plots of $S^2(\mathbf{x})$ (y-axis) versus α (x-axis) for CCDs at some points. Solid, dotted, short-dashed and long-dashed lines correspond to the cases that* $\mathbf{x}$ *is* $(1,1), (\sqrt{2},0), (1,0)$ *and* $(1/\sqrt{2}, 1/\sqrt{2})$*, respectively.*

dispersion measure as a spherical stability criterion of slope estimation on the sphere. In this subsection, we propose a new spherical dispersion measure of all slope variances on a sphere as a measure of spherical stability of slope estimation. Let us define the spherical mean of all slope variances on the sphere with a radius ρ, denoted by $\overline{V}_\rho$, as

$$\overline{V}_\rho = \phi_\rho \int_{U_\rho} \left\{ \psi_1 \int_{u_1} V_{\mathbf{c}}(\mathbf{x}) d\mathbf{c} \right\} d\mathbf{x} = \phi_\rho \int_{U_\rho} \overline{V}(\mathbf{x}) d\mathbf{x} \tag{3.15}$$

where

$$U_\rho = \left\{ \mathbf{x} : \sum_{i=1}^{k} x_i^2 = \rho^2 \right\}, \quad \phi_\rho^{-1} = \int_{U_\rho} d\mathbf{x},$$
$$u_1 = \left\{ \mathbf{c} : \sum_{i=1}^{k} c_i^2 = 1 \right\}, \quad \psi_1^{-1} = \int_{u_1} d\mathbf{c}.$$

Then the proposed spherical dispersion measure, denoted by $S_t^2(\rho)$, on U_ρ is defined by

$$S_t^2(\rho) = \phi_\rho \int_{U_\rho} \left[\psi_1 \int_{u_1} \{ V_{\mathbf{c}}(\mathbf{x}) - \overline{V}_\rho \}^2 d\mathbf{c} \right] d\mathbf{x}. \tag{3.16}$$

Note that we can decompose the deviation, $V_{\mathbf{c}}(\mathbf{x}) - \overline{V}_\rho$, into two parts, as follows:

$$V_{\mathbf{c}}(\mathbf{x}) - \overline{V}_\rho = \{ V_{\mathbf{c}}(\mathbf{x}) - \overline{V}(\mathbf{x}) \} + \{ \overline{V}(\mathbf{x}) - \overline{V}_\rho \}. \tag{3.17}$$

The first term in (3.17) is the the deviation of slope variance with a direction cosine $\mathbf{c}$ at a point $\mathbf{x}$ from the average of slope variances over all directions at a point $\mathbf{x}$. The second term in (3.17) is the deviation of the average of slope variances over all directions at a point $\mathbf{x}$ from the spherical mean of all slope variances on the sphere. From the definition of $\overline{V}(\mathbf{x})$ integration of the cross-product term in (3.17) with respect to $\mathbf{c}$ equals zero. And hence, the spherical dispersion measure, $S_t^2(\rho)$, can be decomposed into two spherical dispersion measures, which are called "spherical point dispersion measure", $S_p^2(\rho)$ and "spherical rotation dispersion measure", $S_r^2(\rho)$, respectively. They are defined by

$$S_p^2(\rho) = \phi_\rho \int_{U_\rho} \left[\psi_1 \int_{u_1} \left\{V_\mathbf{c}(\mathbf{x}) - \overline{V}(\mathbf{x})\right\}^2 d\mathbf{c}\right] d\mathbf{x} = \phi_\rho \int_{U_\rho} S^2(\mathbf{x}) d\mathbf{x} \tag{3.18}$$

and

$$S_r^2(\rho) = \phi_\rho \int_{U_\rho} \left[\psi_1 \int_{u_1} \left\{\overline{V}(\mathbf{x}) - \overline{V}_\rho\right\}^2 d\mathbf{c}\right] d\mathbf{x} = \phi_\rho \int_{U_\rho} \left\{\overline{V}(\mathbf{x}) - \overline{V}_\rho\right\}^2 d\mathbf{x}. \tag{3.19}$$

The latter one measures the nearness to slope rotatability over all directions of the design as a whole.

When a design is slope rotatable over all directions, the values of $\overline{V}(\mathbf{x})$ are equal to $\overline{V}_\rho$ for all points $\mathbf{x}$ on U_ρ. In this case, the values of $S_r^2(\rho)$ are always zero regardless of the values of ρ, that is, $S_t^2(\rho)$ is equal to $S_p^2(\rho)$. But, for other designs, this is not the case. Hence, if we are interested in spherical stability of slope estimation on U_ρ, it is recommendable to investigate each value of $S_t^2(\rho)$, $S_p^2(\rho)$ and $S_r^2(\rho)$.

Now, let us consider the general forms of $S_p^2(\rho)$ and $S_r^2(\rho)$. For simplification of notation, denote some covariances as follows:

$$\begin{aligned} c_{i,j} &= \mathrm{Cov}(b_i, b_j), \\ c_{ij,l} &= \mathrm{Cov}(b_i, q_{jl}) + \mathrm{Cov}(b_j, q_{il}), \\ c_{il,jm} &= \mathrm{Cov}(q_{il}, q_{jm}). \end{aligned} \tag{3.20}$$

Then, $S^2(\mathbf{x})$ in (3.10) can be written as

$$\begin{aligned} S^2(\mathbf{x}) = \frac{2}{k^2(k+2)} \Big\{ & A_k + \sum_{l=1}^{k} B_{k,l} x_l^2 + \sum_{l=1}^{k} C_{k,l} x_l^4 + \sum_{l<m}^{k} D_{k,lm} x_l^2 x_m^2 \\ & + (\text{odd-terms of } x\text{'s}) \Big\}, \end{aligned} \tag{3.21}$$

where

$$A_k = k\sum_{i=1}^{k}\sum_{j=1}^{k} c_{i,j}^2 - \left(\sum_{i=1}^{k} c_{i,i}\right)^2,$$

$$B_{k,l} = k\sum_{i=1}^{k}\sum_{j=1}^{k}\left(c_{ij,l}^2 + 2c_{i,j}c_{il,jl}\right) - \left\{\left(\sum_{i=1}^{k} c_{ii,l}\right)^2 + 2\sum_{i=1}^{k} c_{i,i}\sum_{i=1}^{k} c_{il,il}\right\},$$

$$C_{k,l} = k\sum_{i=1}^{k}\sum_{j=1}^{k} c_{il,jl}^2 - \left(\sum_{i=1}^{k} c_{il,il}\right)^2,$$

$$D_{k,lm} = k\sum_{i=1}^{k}\sum_{j=1}^{k}\left\{2c_{il,jl}c_{im,jm} + \left(c_{il,jm} + c_{im,jl}\right)^2\right\}$$
$$-2\left\{\sum_{i=1}^{k} c_{il,il}\sum_{i=1}^{k} c_{im,im} + 2\left(\sum_{i=1}^{k} c_{il,im}\right)^2\right\}.$$

Through some calculations, $S_p^2(\rho)$ can be rewritten as

$$S_p^2(\rho) = \frac{2}{k^3(k+2)^2}\left\{k(k+2)A_k + (k+2)B_k\rho^2 + (3C_k + D_k)\rho^4\right\}, \qquad (3.22)$$

where

$$B_k = \sum_{l=1}^{k} B_{k,l},\ C_k = \sum_{l=1}^{k} C_{k,l} \text{ and } D_k = \sum_{l<m}^{k} D_{k,lm}.$$

It is also shown that $S_r^2(\rho)$ has the following expression.

$$S_r^2(\rho) = \frac{\rho^2}{k^4(k+2)}\left\{k(k+2)E_k + 2\rho^2(F_k + 2kG_k)\right\}, \qquad (3.23)$$

where

$$E_k = \sum_{l=1}^{k}\left(\sum_{i=1}^{k} c_{ii,l}\right)^2,$$

$$F_k = (k-1)\sum_{l=1}^{k}\left(\sum_{i=1}^{k} c_{il,il}\right)^2 - 2\sum_{l<m}^{k}\left(\sum_{i=1}^{k} c_{il,il}\right)\left(\sum_{i=1}^{k} c_{im,im}\right),$$

$$G_k = \sum_{l<m}^{k}\left(\sum_{i=1}^{k} c_{il,im}\right)^2.$$

In order to evaluate the spherical stability of slope estimation, some symmetric designs are investigated in the following example.

EXAMPLE 2. For symmetric permutation invariant designs, the spherical dispersion measures of slope estimation on U_ρ is given by

$$S_t^2(\rho) = S_p^2(\rho) = \frac{2\rho^4}{N^2k^2(k+2)^2}\Big\{k(k+2)(4Q+\lambda_{22}^{-1})^2 - (k+2)(4P-\lambda_{22}^{-1})^2 + 24k(P+Q)(2P-2Q-\lambda_{22}^{-1})\Big\}.$$

For these designs, $S_r^2(\rho)$ is always zero and $S_p^2(\rho)$ is equivalent to the arithmetic mean of the point dispersion measure of slope estimation at all points $\mathbf{x}$ on U_ρ. Hence, the spherical stability of slope estimation on U_ρ can be measured or investigated by only through $S_p^2(\rho)$. Figure 2 depicts values of $S_p^2(\rho)$ for CCDs which consist of 4 factorial points, 4 axial points with arbitrary α values and n_0 center points. From this figure, we get the impression that larger value of n_0 results in more stable design for each fixed α. At the same time, as α increases, $S_p^2(\rho)$ gets smaller for each ρ.

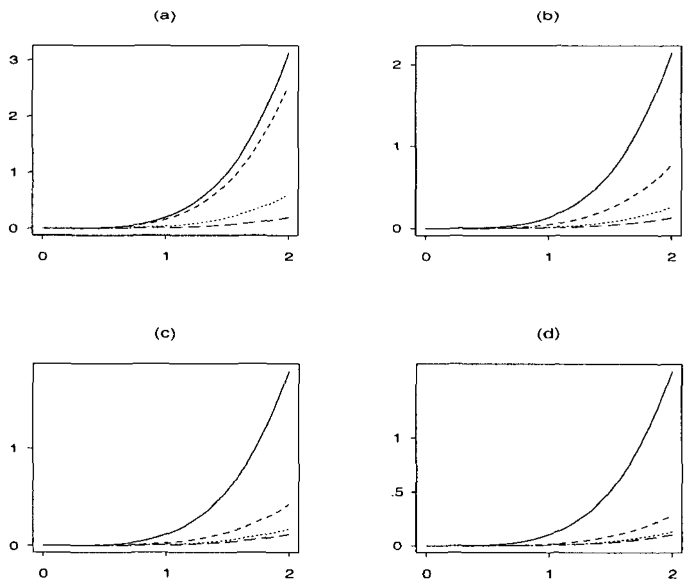

FIGURE 2 *Plots of $S_p^2(\rho)$ (y-axis) versus ρ (x-axis) for CCDs with some α values and n_0 center points. Solid, short-dashed, dotted and long-dashed lines correspond to $n_0 = 1, 2, 3$ and 4, respectively. (a) $\alpha = 1$ (b) $\alpha = \sqrt{2}$ (c) $\alpha = \sqrt{3}$ (d) $\alpha = 2$.*

For response surface designs, simultaneous use of $S_p^2(\rho)$, $S_r^2(\rho)$ and $S_t^2(\rho)$ gives more effective comparison of spherical stability of slope estimation on the sphere with a radius ρ. For instance, it can be clarified whether the main source of spherical dispersion of all slope variances on a sphere is generated by the point or the rotation. In addition, these measures will play an important role in comparing or investigating the stability of slope estimation on U_ρ for general designs which are not slope rotatable. The following example illustrates the usefulness of the spherical dispersion measure for investigating the spherical stability of slope estimation for non-symmetric designs.

EXAMPLE 3. Let us consider designs that consist of (a, b), $(-a, -b)$, $(\pm\alpha, 0)$, $(0, \pm\alpha)$ and n_0 center points. In general, these designs are not slope rotatable over all directions. In case of $\alpha^2 = (n_0 + 4)(a^2 + b^2)/4$, it is shown that these designs are also slope rotatable over all directions (Ying *et al.*, 1995a). To investigate the behavior of the proposed spherical dispersion measure for these designs, two designs are examined, which are Design 1 with $(a, b, \alpha, n_0) = (1, 1, \sqrt{2}, 1)$ and Design 2 with $(a, b, \alpha, n_0) = (\sqrt{2}, \sqrt{2}, 1, 1)$.

For these designs, spherical dispersion measures are illustrated in Figure 3. From these figures, we can observe the following facts:

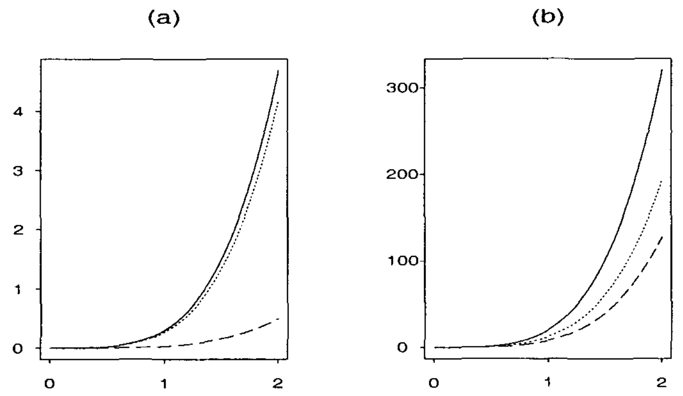

FIGURE 3 *Plots of $S_t^2(\rho)$, $S_p^2(\rho)$ and $S_r^2(\rho)$ (y-axis) versus ρ (x-axis). Solid, dotted and dashed lines correspond to $S_t^2(\rho)$, $S_p^2(\rho)$ and $S_r^2(\rho)$, respectively. (a) Design 1 (b) Design 2.*

[i] Values of $S_t^2(\rho)$, $S_p^2(\rho)$ and $S_r^2(\rho)$ on U_ρ for Design 1 are much smaller than those for Design 2, respectively. Hence, Design 1 is better than Design 2 in the sense of spherical stability of slope estimation.

[ii] Especially, Design 1 rather than Design 2 is near to slope rotatable design

over all directions since the value of $S_r^2(\rho)$ for Design 1 is much smaller than that for Design 2 on each sphere with any radius ρ.

[iii] For Design 1, it is obvious that the spherical dispersion of all slope variances on U_ρ mainly depends on the spherical point dispersion. But, for Design 2, the spherical dispersion of all slope variances on U_ρ depends on both the spherical point and rotation dispersions.

4. EQUALLY-STABLE SLOPE ROTATABILITY

For a design which is analogous to Park (1987) or Park and Kwon (1998), we can consider slope rotatability with respect to the stability of slope estimation over all possible directions at a point $\mathbf{x}$. If the values of $S^2(\mathbf{x})$ are constant on the circles ($k = 2$), spheres ($k = 3$) or hyperspheres ($k > 3$) centered at the design origin, slope variances then would be equally stable for all points $\mathbf{x}$ that are equidistant from the design center. In such case, $S^2(\mathbf{x})$ is a function of only $\rho^2 = \sum_{i=1}^k x_i^2$. Let us call this desirable property as "equally-stable slope rotatability" over all directions.

In order to explore general conditions for a design to have equally-stable slope rotatability over all directions, note from (3.2) and (3.20), the (i, j) element of the matrix $M(\mathbf{x})$ can be rewritten as

$$m_{ij}(\mathbf{x}) = c_{i,j} + \sum_{l=1}^k c_{ij,l}x_l + \sum_{l=1}^k \sum_{m=1}^k c_{il,jm}x_l x_m. \tag{4.1}$$

Then, the following theorem holds.

THEOREM 4.1. *The necessary and sufficient conditions for a design to have equally-stable slope rotatability over all directions are as follows:*

[1] $k \sum_{i=1}^k \sum_{j=1}^k c_{i,j}c_{ij,l} - \sum_{i=1}^k c_{i,i} \sum_{i=1}^k c_{ii,l} = 0$ *for all* l.

[2] $k \sum_{i=1}^k \sum_{j=1}^k c_{ij,l}c_{il,jl} - \sum_{i=1}^k c_{ii,l} \sum_{i=1}^k c_{il,il} = 0$ *for all* l.

[3] $k \sum_{i=1}^k \sum_{j=1}^k c_{ij,l}(c_{il,jm} + c_{im,jl}) - 2\sum_{i=1}^k c_{ii,l} \sum_{i=1}^k c_{il,im} = 0$ *for all* $l \neq m$.

[4] $k\sum_{i=1}^{k}\sum_{j=1}^{k} c_{il,jl}(c_{il,jm}+c_{im,jl}) - 2\sum_{i=1}^{k} c_{il,il}\sum_{i=1}^{k} c_{il,im} = 0$ *for all* $l \neq m$.

[5] $$k\sum_{i=1}^{k}\sum_{j=1}^{k}\left\{c_{ij,l}c_{ij,m}+c_{i,j}(c_{il,jm}+c_{im,jl})\right\} - \left(\sum_{i=1}^{k} c_{ii,l}\sum_{i=1}^{k} c_{ii,m} + 2\sum_{i=1}^{k} c_{i,i}\sum_{i=1}^{k} c_{il,im}\right) = 0 \text{ for all } l \neq m.$$

[6] $k\sum_{i=1}^{k}\sum_{j=1}^{k} c_{ij,l}(c_{im,jn}+c_{in,jm}) - 2\sum_{i=1}^{k} c_{ii,l}\sum_{i=1}^{k} c_{im,in} = 0$ *for all* $l \neq m \neq n$.

[7] $k\sum_{i=1}^{k}\sum_{j=1}^{k} c_{il,jl}(c_{im,jn}+c_{in,jm}) - 2\sum_{i=1}^{k} c_{il,il}\sum_{i=1}^{k} c_{im,in} = 0$ *for all* $l \neq m \neq n$.

[8] $$k\sum_{i=1}^{k}\sum_{j=1}^{k}(c_{il,jm}+c_{im,jl})(c_{in,jr}+c_{ir,jn}) - 4\sum_{i=1}^{k} c_{il,im}\sum_{i=1}^{k} c_{in,ir} = 0 \text{ for all } l \neq m \neq n \neq r.$$

[9] $$k\sum_{i=1}^{k}\sum_{j=1}^{k}(c_{ij,l}^2+2c_{i,j}c_{il,jl}) - \left\{\left(\sum_{i=1}^{k} c_{ii,l}\right)^2 + 2\sum_{i=1}^{k} c_{i,i}\sum_{i=1}^{k} c_{il,il}\right\} \text{ are equal for all } l.$$

[10] $k\sum_{i=1}^{k}\sum_{j=1}^{k} c_{il,jl}^2 - \left(\sum_{i=1}^{k} c_{il,il}\right)^2$ *are equal for all* l.

[11] $$2\left\{k\sum_{i=1}^{k}\sum_{j=1}^{k} c_{il,jl}^2 - \left(\sum_{i=1}^{k} c_{il,il}\right)^2\right\} = k\sum_{i=1}^{k}\sum_{j=1}^{k}\left\{2c_{il,jl}c_{im,jm}+(c_{il,jm}+c_{im,jl})^2\right\} -2\left\{\sum_{i=1}^{k} c_{il,il}\sum_{i=1}^{k} c_{im,im} + 2\left(\sum_{i=1}^{k} c_{il,im}\right)^2\right\} \text{ for all } l \neq m.$$

PROOF. From (3.10), it is sufficient to consider only the right hand side of $S^2(\mathbf{x})$, which is given by

$$k\,\mathrm{tr}\big(M'(\mathbf{x})M(\mathbf{x})\big) - \big\{\mathrm{tr}\big(M(\mathbf{x})\big)\big\}^2. \tag{4.2}$$

By some tedious calculations with the equation (4.1), we have each coefficient of x's polynomial terms in (4.2). From these, all the coefficients of each odd-term of x's are given in the above conditions from [1] through [8], respectively. For example, the coefficient of x_l is given by

$$2\left(k\sum_{i=1}^{k}\sum_{j=1}^{k} c_{i,j}c_{ij,l} - \sum_{i=1}^{k} c_{i,i}\sum_{i=1}^{k} c_{ii,l}\right)$$

and the coefficient of $x_l x_m x_n$ is given by

$$2\left\{k\sum_{i=1}^{k}\sum_{j=1}^{k} c_{ij,l}(c_{im,jn} + c_{in,jm}) - 2\sum_{i=1}^{k} c_{ii,l}\sum_{i=1}^{k} c_{im,in}\right\}$$

for all $l < m < n$.

Therefore, for $S^2(\mathbf{x})$ to be a function of ρ^2, first, the conditions from [1] through [8] should be satisfied. Second, the coefficients of x_l^4 and x_l^2 are equal for all l, respectively. Finally, the coefficient of $x_l^2 x_m^2$ is two times as large as that of x_l^4 for all $l < m$. These conditions for each even-term of x's are given in the conditions [9], [10] and [11], respectively. From the property of an identical equation of x's, we obtain the necessary and sufficient conditions as stated above and this completes the proof. □

Although the conditions in Theorem 4.1 seem to be extremely complicated, conditions from [1] through [8] are always satisfied for commonly used response surface designs such as central composite designs and 3^k factorial designs. For a design satisfying the conditions in (2.12), $\mathrm{tr}\big(M(\mathbf{x})\big)$ is already a function of only ρ^2. Furthermore, all the conditions except [11] in Theorem 4.1 are also satisfied. Hence, we will restrict our consideration to the class of symmetric permutation invariant designs defined in Subsection 3.1.

For symmetric permutation invariant designs, the variance-covariance elements in (3.20) can be simply written as follows:

$$
\begin{aligned}
c_{i,j} &= \begin{cases} \dfrac{\lambda_2^{-1}}{N} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \\
c_{ij,l} &= \quad 0 \qquad \text{for all } i, j \text{ and } l, \\
c_{il,jm} &= \begin{cases} \dfrac{4P}{N} & \text{if } i = l = j = m, \\ \dfrac{4Q}{N} & \text{if } i = l \neq j = m, \\ \dfrac{\lambda_{22}^{-1}}{N} & \text{if } i \neq l, j \neq m \text{ and } \{i,l\} = \{j,m\}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
\tag{4.3}
$$

Here, N is the number of experimental runs, and P and Q are given in (3.13). The following theorem states sufficient conditions for a symmetric permutation invariant design to have equally-stable slope rotatability.

THEOREM 4.2. *The conditions for rotatability are sufficient for a design to have equally-stable slope rotatability.*

PROOF. If a design is rotatable, all odd-moments up to 4 are zero and the equation (4.3) is satisfied since the class of rotatable designs is a subset of the class of symmetric permutation invariant designs. Regarding the conditions in Theorem 4.1, conditions from [1] through [8] are holds trivially. Hence, it is sufficient to check only the other conditions [9], [10] and [11]. Note that [9] can be reduced to zero and [10] can be reduced to

$$N^{-2}k\{16P^2 + (k-1)\lambda_{22}^{-2}\} - N^{-2}\{4P + (k-1)\lambda_{22}^{-1}\}^2.$$

These imply that [9] and [10] are always constants for all $l = 1, 2, \ldots, k$, respectively.

In addition, the condition [11] can be simply written as

$$2P^2 = 2Q^2 + \lambda_{22}^{-1}(P+Q). \tag{4.4}$$

By using P and Q in (3.13), it follows that (4.4) is equivalent to

$$\{\lambda_4 + (k-3)\lambda_{22} - (k-2)\lambda_2^2\}(\lambda_4 - 3\lambda_{22}) = 0. \tag{4.5}$$

Then, $\lambda_4 = 3\lambda_{22}$ is sufficient for the condition [11] to be satisfied and this completes the proof. □

Theorem 4.2 implies that the class of rotatable designs is a subset of the class of equally-stable slope rotatable designs. The following corollary states the relation between symmetric permutation invariant designs and equally-stable slope rotatable designs.

COROLLARY 4.1. *Among symmetric permutation invariant designs, only rotatable designs have equally-stable slope rotatability.*

PROOF. When $k=2$, $\lambda_4+(k-3)\lambda_{22}-(k-2)\lambda_2^2$ in (4.5) is equal to $\lambda_4-\lambda_{22}$ which is always positive since $\lambda_4>\lambda_{22}$. When $k\geq 3$, if $\lambda_4+(k-3)\lambda_{22}-(k-2)\lambda_2^2$ is zero, then we have $\lambda_4+(k-1)\lambda_{22}-k\lambda_2^2=2(\lambda_{22}-\lambda_4)/(k-2)$. In this case, the value of $\lambda_4+(k-1)\lambda_{22}-k\lambda_2^2$ is always negative. And this contradicts to the conditions of symmetric permutation invariant designs. Therefore, the value of $\lambda_4+(k-3)\lambda_{22}-(k-2)\lambda_2^2$ can not be zero for all k. This implies that the unique solution of the equation (4.5) is rotatability in the class of symmetric permutation invariant designs. □

REMARK 4.1. For rotatable designs, we can note that $\overline{V}(\mathbf{x})$ and $S^2(\mathbf{x})$ of slope variances at any point $\mathbf{x}$ are always constants on the sphere with a radius ρ, respectively. Furthermore, for a rotatable design, it can be shown that the eigenvalues of the matrix $M(\mathbf{x})$ with size $k\times k$ are

$$N^{-1}\left(\lambda_2^{-1}+\lambda_{22}^{-1}\rho^2\right) \quad\text{and}\quad N^{-1}\left[\lambda_2^{-1}+2\frac{(k+1)\lambda_{22}-(k-1)\lambda_2^2}{\lambda_{22}\{(k+2)\lambda_{22}-k\lambda_2^2\}}\rho^2\right]$$

with multiplicity $k-1$ where $\rho^2=\sum_{i=1}^k x_i^2$. Hence, all measures, which can be represented by only eigenvalues of the matrix $M(\mathbf{x})$, at a point $\mathbf{x}$ are always functions of only ρ^2.

REMARK 4.2. When we consider slope variances over only axial directions at a point $\mathbf{x}$, the average of slope variances over axial directions at a point $\mathbf{x}$ is equal to that of slope variances over all possible directions. But the point dispersion measure, $S^2(\mathbf{x})$, of slope variances at a point $\mathbf{x}$ must be redefined as

$$S^2(\mathbf{x})=\frac{1}{k}\sum_{i=1}^k V_{\mathbf{c}_i}^2(\mathbf{x})-\{\overline{V}(\mathbf{x})\}^2 \tag{4.6}$$

where $\mathbf{c}_i$ is the $k\times 1$ vector which has only nonzero value at the i^{th} row element. For example, $\mathbf{c}_1=(1,0,\ldots,0)'$. In this case, $S^2(\mathbf{x})$ can be written as

$$\frac{1}{k^2}\sum_{i<j}^{k}\{m_{ii}(\mathbf{x})-m_{jj}(\mathbf{x})\}^2$$

where $m_{ii}(\mathbf{x})$ is the variance of estimated slope with respect to the i^{th} axial direction. Hence, if a design is slope rotatable over axial directions, then the values of $S^2(\mathbf{x})$ are always zero for all points $\mathbf{x}$.

5. CONCLUDING REMARKS

In this paper, we are interested in evaluating point and spherical stabilities of slope estimation over all directions. For these, we propose the point and spherical dispersion measures of slope estimation on the second order response surface model. For measuring stability of slope estimation at a point or on a sphere, we recommend these measures as new criteria of response surface designs. Furthermore, equally-stable slope rotatability is also proposed as a useful property of response surface designs. In estimating the slope on the second order response surface, the proposed concept has desirable properties as good as slope rotatability in Park (1987) and Park and Kwon (1998).

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REFERENCES

ATKINSON, A. C. (1970). "The design of experiments to estimate the slope of a response surface", *Biometrika*, **57**, 319–328.

BOX, G. E. P. AND DRAPER, N. R. (1980). "The variance function of the difference between two estimated responses", *Journal of the Royal Statistical Society*, **B42**, 79–82.

BOX, G. E. P. AND HUNTER, J. S. (1957). "Multi-factor experimental designs for exploring response surfaces", *Annals of Mathematical Statistics*, **28**, 195–241.

DRAPER, N. R. AND YING, L. H. (1994). "A note on slope rotatability over all directions", *Journal of Statistical Planning and Inference*, **41**, 113–119.

GIOVANNITTI-JENSEN, A. AND MYERS, R. H. (1989). "Graphical assessment of the prediction capability of response surface designs", *Technometrics*, **31**, 159–171.

HADER, R. J. AND PARK, S. H. (1978). "Slope-rotatable central composite designs", *Technometrics*, **20**, 413–417.

HERZBERG, A. M. (1967). "The behavior of the variance function of the difference between two estimated responses", *Journal of the Royal Statistical Society*, **B29**, 174–179.

HUDA, S. AND MUKERJEE, R. (1984). "Minimizing the maximum variance of the difference between two estimated responses", *Biometrika*, **71**, 381–385.

HUDA, S. AND SHAFIQ, M. (1992). "Minimax designs for estimating the slope of a second-order response surface in a cubic region", *Journal of Applied Statistics*, **19**, 501–507.

JANG, D. H. AND PARK, S. H. (1993). "A measure and a graphical method for evaluating slope rotatability in response surface designs", *Communications in Statistics-Theory and Methods*, **22**, 1849–1863.

KIM, H. J., UM, Y. H. AND KHURI, A. I. (1996). "Quantile plots of the average slope variance for response surface designs", *Communications in Statistics-Simulation and Computation*, **25**, 995–1014.

MUKERJEE, R. AND HUDA, S. (1985). "Minimax second- and third-order designs to estimate the slope of a response surface", *Biometrika*, **72**, 173–178.

MURTY, V. N. AND STUDDEN, W. J. (1972). "Optimal designs for estimating the slope of a polynomial regression", *Journal of the American Statistical Association*, **67**, 869–873.

MYERS, R. H. AND LAHODA, S. J. (1975). "A generalization of the response surface mean square error criterion with a specific application to the slope", *Technometrics*, **17**, 481–486.

OTT, L. AND MENDENHALL, W. (1972). "Designs for estimating the slope of a second order linear model", *Technometrics*, **14**, 341–353.

PARK, S. H. (1987). "A class of multifactor designs for estimating the slope of response surfaces", *Technometrics*, **29**, 449–453.

PARK, S. H. AND KIM, H. J. (1992). "A measure of slope-rotatability for second order response surface experimental designs", *Journal of Applied Statistics*, **19**, 391–404.

PARK, S. H. AND KWON, H. T. (1998). "Slope-rotatable designs with equal maximum directional variance for second order response surface models", *Communications in Statistics-Theory and Methods*, **27**, 2837–2851.

YING, L. H., PUKELSHEIM, F. AND DRAPER, N. R. (1995a). "Slope rotatability over all directions designs", *Journal of Applied Statistics*, **22**, 331–341.

YING, L. H., PUKELSHEIM, F. AND DRAPER, N. R. (1995b). "Slope rotatability over all directions designs for $k \geq 4$", *Journal of Applied Statistics*, **22**, 343–354.