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ANNIHILATOR CONDITIONS ON RINGS AND NEAR-RINGS

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ABSTRACT. In this paper, we initiate the study of some annihilator conditions on polynomials which were used by Kaplansky [Rings of operators. W. A. Benjamin, Inc., New York, 1968] to abstract the algebra of bounded linear operators on a Hilbert spaces with Baer condition. On the other hand, p.p.-rings were introduced by Hattori [A foundation of torsion theory for modules over general rings. *Nagoya Math. J.* **17** (1960) 147–158] to study the torsion theory. The purpose of this paper is to introduce the near-rings with Baer condition and near-rings with p.p. condition which are somewhat different from ring case, and to extend a results of Armendariz [A note on extensions of Baer and P.P.-rings. *J. Austral. Math. Soc.* **18** (1974), 470–473] and Jøndrup [p.p. rings and finitely generated flat ideals. *Proc. Amer. Math. Soc.* **28** (1971) 431–435].

1. INTRODUCTION

Kaplansky [5] introduced the Baer rings as rings in which every left (right) annihilator ideal is generated by an idempotent. On the other hand, Hattori [3] introduced the left p.p.-rings as rings in which any principal left ideal is projective. In this paper we introduce Baer near-rings and p.p.-near-rings and study some of their properties and give some examples. Let G be an additively written (but not necessarily abelian) group with zero 0 and $M_0(G) = \{f : G \to G \mid f(0) = 0\}$ the near-ring of all zero fixing mappings on G. We show that $M_0(G)$ is a Baer near-ring. As a corollary, we show that every zero-symmetric near-ring can be embedded into a Baer near-ring. Let R be a commutative ring with identity. It is well known that R is a Baer (*resp.* p.p.-) ring if and only if the polynomial ring $R[x]$ is a Baer (*resp.* p.p.-) ring (see *e. g.*, Armendariz [1] and Jøndrup [4]). Corresponding to this result, we will prove that the zero-symmetric part of $R[x]$ is a Baer (*resp.* p.p.-) near-ring if and only if

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R is a Baer (*resp.* p.p.-) ring. Finally we study the structure of a zero-symmetric reduced p. p.-near-ring with identity.

2. BAER NEAR-RINGS AND P.P.-NEAR-RINGS

A (right) near-ring is a set N with two binary operations $+$ and $\cdot$ such that $(N,+)$ is a not necessarily abelian group with identity 0, $(N,\cdot)$ is a semigroup and $(x+y)z = xz + yz$ for all $x, y, z \in N$. Some basic definitions and concepts in near-ring theory can be found in Meldrum [6] and Pilz [7].

For a subset S of a near-ring N, the set $\{n \in \mid NnS = 0\}$ is called the *annihilator* of S in N which is denoted by $\text{Ann}_N(S) = \text{Ann}(S)$.

A near-ring N is called a *Baer near-ring* if, for any subset S of N, $\text{Ann}(S) = \text{Ann}(e)$ for some idempotent $e \in N$. The following proposition is obvious.

Proposition 1. *Let N_i $(i \in I)$ be a family of near-rings. Then the direct product $\prod_{i\in I} N_i$ is a Bear near-ring if and only if N_i is a Bear near-ring for each $i \in I$.*

A near-ring N is said to be *integral* if N has no nonzero divisors of zero (*cf.* Pilz [7, 1.14, p. 11]).

Example 1.

(1) Every integral near-ring with identity is a Baer near-ring.
(2) Every constant near-ring is a Baer near-ring.
(3) A direct product of integral near-rings with identity is a Baer near-ring.

Let G be an additively written (but not necessarily abelian) group with zero 0 and $M_0(G) = \{f : G \to G \mid f(0) = 0\}$ the near-ring of all zero fixing mappings on G (see Pilz [7, 1.4, p. 8]). Beidleman [2, Theorem 1] proved that $M_0(G)$ is a regular near-ring. We shall prove that $M_0(G)$ is Baer.

Theorem 1. *The near-ring $M_0(G)$ is a Baer near-ring.*

Proof. Let S be a subset of $M_0(G)$ and let $H = \{s(g) \mid s \in S,\ g \in G\}$. Let e be a mapping on G such that if $x \in H$, then $e(x) = x$ and $e(y) = 0$ for any $y \in G - H$. Then e is an idempotent of $M_0(G)$ and $\text{Ann}(S) = \text{Ann}(e)$. This implies that $M_0(G)$ is a Baer near-ring. □

Corollary 1. *Every zero-symmetric near-ring can be embedded into a Baer near-ring.*

Proof. By Pilz [7, 1.102, p. 11], every zero-symmetric near-ring can be embedded into a zero-symmetric near-ring with identity. Let N be a zero-symmetric near-ring with identity. By Theorem 1, $M_0(N)$ is a Bear near-ring. For any $r \in N$, the mapping $f_r : t \in N \to rt \in N$ is an element of $M_0(N)$. Since N contains an identity, the mapping $f : N \to M_0(N); r \mapsto f_r$ is a near-ring monomorphism. □

An associative ring R called a left p.p.-ring if every principal left ideal of R is projective. This is equivalent to the condition that, for any $a \in R$, $\mathrm{Ann}(a) = \mathrm{Ann}(e)$ for some idempotent $e \in R$. A right p.p.-ring is defined in a symmetric way.

Now we call a near-ring N a *p.p.-near-ring* if, for any $a \in N$, $\mathrm{Ann}(a) = \mathrm{Ann}(e)$ for some idempotent $e \in N$. Clearly a Baer near-ring is a p.p.-near-ring.

Following Beidleman [2], we call a near-ring N regular if, for any $x \in N$, here exists $y \in N$ such that $xyx = x$.

Example 2. Every regular near-ring is a p.p.-near-ring. In fact, for any $x \in N$, there exists $y \in N$ such that $xyx = x$. Then xy is an idempotent and $\mathrm{Ann}(x) = \mathrm{Ann}(xy)$.

Let R be a commutative ring with identity and let $R[x]$ denote the set of all polynomials in one indeterminate over R. Under usual addition $+$ and substitution $\circ$ of polynomials, $(R[x], +, \circ)$ becomes a near-ring. Following Pilz [7, 7.78, p. 221], $R_0[x]$ denotes the zero symmetric part of $R[x]$, that is

$$R_0[x] = \Big\{ \sum_{i=1}^{n} a_i x^i \mid a_i \in R, n \geq 1 \Big\}.$$

The following is a near-ring theoritic modification of Jøndrup [4, Theorem 2.1].

Theorem 2. *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

1) *$R_0[x]$ is a p.p.-near-ring.*
2) *R is a p.p.-ring.*

Proof.

1) $\Rightarrow$ 2). First we claim that R is reduced. Suppose that $a \in R$ with $a^2 = 0$. By hypothesis, there exists an idempotent $f \in R_0[x]$ such that $\mathrm{Ann}(ax) = \mathrm{Ann}(f)$. Let $f = a_1x + a_2x^2 + \cdots + a_nx^n$ with $a_i \in R$. Since f is an idempotent, we have

$a_1^2 = a_1$. Since $ax \in \mathrm{Ann}(ax)$, $ax \circ f = af = 0$. In particular, $aa_1 = 0$. Since $x - f \in \mathrm{Ann}(f)$, $0 = (x-f) \circ ax = ax^2 - f(ax)$. Hence $ax^2 = a_1 ax = 0$, that is $a = 0$. This proves that R is reduced. Since R is reduced, the set of idempotents of $R_0[x]$ is just $\{ex \mid e^2 = e \in R\}$. Now let r be an arbitrary element of R. By hypothesis, there exists an idempotent $e \in R$ such that $\mathrm{Ann}(rx) = \mathrm{Ann}(ex)$. Clearly this implies that $\{s \in R \mid sr = 0\} = R(1-e)$. Hence R is a p.p.-ring.

2) $\Rightarrow$ 1). Let $f = a_1x + \cdots + a_nx^n \in R_0[x]$ and $g = b_1x + \cdots + b_mx^m \in R_0[x]$. First we claim that $f \circ g = 0$ if and only if $a_ib_j = 0$ for all i, j. It suffices to prove the 'only if' part. Let P be an arbitrary prime ideal of R and let $\bar{f}$ and $\bar{g}$ denote the image of f and g in $(R/P)[x]$ respectively. Since R/P is an integral domain and since $\bar{f} \circ \bar{g} = 0$, we can easily see that either $\bar{f} = 0$ or $\bar{g} = 0$ holds. Hence $a_ib_j \in P$ for all i, j. Since P is an arbitrary prime ideal, this implies that $a_ib_j \in \mathrm{Rad}(R)$, where $\mathrm{Rad}(R)$ denote the prime radical of R. Since R is a commutative p.p.-ring, R is reduced and hence $\mathrm{Rad}(R) = 0$. This proves our claim. Therefore $a_1, \ldots, a_n \in \mathrm{Ann}_R(b_1, \ldots, b_m)$. Since R is a p.p.-ring, for each i, there exists an idempotent $e_i \in R$ such that $\mathrm{Ann}(b_i) = \mathrm{Ann}(e_i)$. If $n = 2$, then $f = e_1 + e_2 - e_1e_2$ is an idempotent and $\mathrm{Ann}_R(b_1, b_2) = \mathrm{Ann}(f)$. Using induction on n, we can find an idempotent e of R such that $\mathrm{Ann}_R(b_1, \ldots, b_m) = \mathrm{Ann}(e)$. Then ex is an idempotent of $R_0[x]$ and $\mathrm{Ann}(g) = \mathrm{Ann}(ex)$. Therefore $R_0[x]$ is a p.p.-near-ring. □

The next theorem gives more examples of Baer near-rings.

Theorem 3. *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

1) *$R_0[x]$ is a Baer near-ring.*
2) *R is a Baer ring.*

Proof.

1) $\Rightarrow$ 2). Let T be a subset of R and consider the suset $S = \{tx \mid t \in T\}$ of $R_0[x]$. As saw in the proof of 1) $\Rightarrow$ 2) of Theorem 2, the set of idempotents of $R_0[x]$ is just $\{ex \mid e^2 = e \in R\}$. Since $R_0[x]$ is Baer, $\mathrm{Ann}(S) = \mathrm{Ann}(ex)$ for some idempotent $e \in R$. Then we can easily see that $\mathrm{Ann}_R(T) = \mathrm{Ann}_R(e)$. Hence R is a Baer ring.

2) $\Rightarrow$ 1). Let S be a subset of $R_0[x]$ and consider the set T of all coefficients of $g(x) \in S$. Let $f = a_1x + \cdots + a_nx^n \in \mathrm{Ann}(S)$. As saw in the proof of 2) $\Rightarrow$ 1) of Theorem 2, $a_i \in \mathrm{Ann}_R(T)$ for all i. Since R is a Baer ring, there exists an idempotent

e such that $\mathrm{Ann}_R(T) = \mathrm{Ann}_R(e)$. Now we can easily see that $\mathrm{Ann}(S) = \mathrm{Ann}(ex)$. This proves that $R_0[x]$ is a Baer near-ring. □

Corollary 2. *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

1) *R is a von Neumann regular ring.*
2) *$(R/I)_0[x]$ is a p.p.-near-ring for all ideals I of R.*

Proof.

1) $\Rightarrow$ 2). If R is regular, then R/I is regular for every ideal I of R, so that R/I is a p.p.-ring. Hence this follows from Theorem 1.

2) $\Rightarrow$ 1). As saw in the proof of 1) $\Rightarrow$ 2) of Theorem 2, R/I is reduced for every ideal I of R. Let $a \in R$ and consider the ideal Ra^2 of R. Since R/Ra^2 is reduced and since $a + Ra^2 \in R/Ra^2$ is nilpotent, we have $a \in Ra^2$. This implies that R is von Neumann regular. □

Let R be an associative ring with identity and let M be a unital left R-module. If we define a multiplication on the additive group $R \oplus M$ by $(a, b) \circ (c, d) = (ac, ad + b)$ for (a, b), $(c, d) \in R \oplus M$, $R \oplus M$ becomes a near-ring with identity $(1, 0)$.

Theorem 4. *Let R be an associative ring with identity and let M be a unital left R-module. Then the following conditions are equivalent:*

1) *$R \oplus M$ is a p.p.-near-ring.*
2) *R is a left p.p.-ring.*

Proof.

2) $\Rightarrow$ 1). We can easily see that, for $(c, d) \in R \oplus M$,

$$\mathrm{Ann}(c, d) = \{(a, -ad) \mid a \in \mathrm{Ann}(s)\}.$$

Since R is a left p.p.-ring, there is an idempotent $e \in R$ such that $\mathrm{Ann}_R(c) = \mathrm{Ann}(e)$. Then $(e, (1 - e)d)$ is an idempotent of $R \oplus M$ and $\mathrm{Ann}(c, d) = \mathrm{Ann}(e, (1 - e)d)$.

1) $\Rightarrow$ 2). We first note that the set of all idempotents of $R \oplus M$ is equal to $\{(e, (1 - e)x) \mid e = e^2 \in R,\ x \in M\}$. Hence, for any $c \in R$, there exists an idempotent $e \in R$ and an $x \in M$ such that $\mathrm{Ann}(c, 0) = \mathrm{Ann}(e, (1 - e)x)$. By the way, $\mathrm{Ann}(c, 0) = \{(a, 0) \mid a \in \mathrm{Ann}(c)\}$. On the other hand, $(1 - e, -(1 - e)x) \in \mathrm{Ann}(e, (1 - e)x)$. Hence $(1 - e)x = 0$, and so $\mathrm{Ann}(c, 0) = \mathrm{Ann}(e, 0)$. This implies $\mathrm{Ann}(c) = \mathrm{Ann}(e)$. Therefore R is a left p.p.-ring. □

A near-ring with no non-zero nilpotent elements is said to be *reduced.* For the rest of this paper, we shall study the structure of zero-symmetric reduced p.p.-near-rings with identity.

Proposition 2. *Let N be a zero-symmetric reduced p.p.-near-ring with identity. Then, for any finitely many elements $a_1, \ldots, a_n \in N$, there exists an idempotent $e \in N$ such that* $\text{Ann}(a_1, \ldots, a_n) = \text{Ann}(e)$.

Proof. Since N is a p.p.-near-ring, there exist idempotents $e_1, \ldots, e_n \in N$ such that $\text{Ann}(a_i) = \text{Ann}(e_i)$ for each i. By Ramakotaiah & Sambasiva Rao [8, Lemma 0.2] (or Pilz [7, Proposition 9.43(b), p. 304]), all idempotents of N is central. Then, by the same method as in the proof of 2) $\Rightarrow$ 1) of Theorem 2, we can find an idempotent $e \in N$ such that $\text{Ann}(a_1, \ldots a_n) = \text{Ann}(e)$. □

Proposition 3. *Let N be a zero-symmetric reduced p.p.-near-ring with identity. If N has no infinitely many nonzero orthogonal idempotents, then N is a direct sum of finitely many integral near-rings.*

Proof. Let $\text{Ann}(a)$ be a minimal element in $\{\text{Ann}(t) \neq 0 \mid t \in N\}$. By hypothesis, there exists an idempotent $e_1 \in N$ such that $\text{Ann}(a) = \text{Ann}(e_1)$. We claim that $N(1-e_1)$ is an integral near-ring. Let b, $c \in N(1-e_1)$ such that $bc = 0$ and $c \neq 0$. Then $\text{Ann}(c+e_1) \subsetneq \text{Ann}(e_1)$. By minimality of $\text{Ann}(e_1)$, we conclude that $\text{Ann}(c+e_1) = 0$. Clearly $b \in \text{Ann}(c+e_1)$, whence $b = 0$. This proves our claim.

Next we choose a minimal element $\text{Ann}(e_2)$ with $e_2 = e_2^2$ in $\{\text{Ann}(t) \neq 0 \mid t \in Ne_1\}$. Then we can also show that $N(e_1 - e_2)$ is an integral near-ring. Continuing this process, we obtain othogonal idempotents $e_0 = 1, e_1, e_2, \ldots$ of N such that $N(e_i - e_{i+1})$ is integral near-ring for each $i = 0, 1, \ldots$. Since

$$1 - e_1, e_1 - e_2, \ldots, e_{n-1} - e_n, \ldots$$

are orthogonal idempotents, by hypothesis there exists a natural number n such that $e_n = 0$. Then $N = N(1-e_1) \oplus \cdots \oplus N(e_{n-2} - e_{n-1}) \oplus Ne_{n-1}$ and $N(1-e_1), \ldots, N(e_{n-2} - e_{n-1})$, Ne_{n-1} are all integral near-rings. □

Proposition 4. *Let N be a zero-symmetric reduced p.p.-near-ring with identity. Then, for any $a \in N$, there exists a non zero-divisor $d \in N$ and an idempotent $e \in N$ such that $a = ed$.*

Proof. By hypothesis, there exists an idempotent $e \in N$ such that $\text{Ann}(a) = \text{Ann}(e)$. Since every idempotent of N is central, we have $a = ea = e(a + (1-e))$. By

Ramakotaiah & Sambasiva Rao [8, Lemma 0.1] $xy = 0$, whence $x, y \in N$ implies $yx = 0$. Using this property we can easily see that $a+(1-e)$ is a non zero-divisor. □

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