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WEAKER FORMS OF COMMUTING MAPS AND EXISTENCE OF FIXED POINTS

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ABSTRACT. Weak commutativity of a pair of maps was introduced by Sessa [On a weak commutativity condition of mappings in fixed point considerations. *Publ. Inst. Math. (Beograd) (N.S.)* **32(46)** (1982), 149–153] in fixed point considerations. Thereafter a number of generalizations of this notion has been obtained. The purpose of this paper is to present a brief development of weaker forms of commuting maps, and to obtain two fixed point theorems for noncommuting and noncontinuous maps on noncomplete metric spaces.

1. INTRODUCTION

Let A be an arbitrary nonempty set and (X, d) a metric space and $f, g : A \to X$ such that

$$d(fx, fy) \leq kd(gx, gy), \text{ for all } x, y \in A \text{ and some } k, \ 0 \leq k < 1. \tag{1}$$

In 1968, using the Banach contraction principle, Goebel [13] obtained a coincidence theorem for f and g on an arbitrary set A with values in a complete metric space X satisfying $f(A) \subseteq g(A)$ and (1). The condition (1) seems to have been studied first by Machuca [25] in 1967 under heavy topological conditions. However, considering (1) when $A = X$ and introducing a new sequence of iterates, Jungck [16] used commutativity of f and g to ensure the existence of a common fixed point. Elegancy of this result and its proof fascinated several scholars and subsequently a multitude of coincidence and fixed point theorems for maps having the full force of commutativity or restricted commutativity were obtained. Several new applications have been suggested (see, for instance, [3, 9, 11, 17, 24, 29, 40, 45, 48, 49, 63, 64], and references thereof).

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In 1982, Sessa [44], relaxed the condition of commutativity in the theorem of Jungck [16] and its various generalizations by introducing the weak commutativity. Subsequently, the notion of weak commutativity of a pair of maps in metric fixed point theory was further weakened and used by Singh [50], Jungck [18], Pathak [33], Mishra [26], Gairola, Singh & Whitfield [12], Pant [29], Tivari & Singh [63], Hadžić [14] and others (see, for instance, [2, 4, 8, 10, 13, 15, 18, 19, 20, 21, 22, 23, 26, 27, 28, 29, 30, 31, 35, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 51, 52, 53, 55, 56, 60, 63, 64, 65, 66]).

Several other weaker forms of commuting maps have been studied under the following names:

Table 1. Some weaker forms of commuting maps

Forms	References
preorbitally commuting	Singh & Mishra [55] Tivari & Singh [63]
asymptotically commuting	Gairola, Singh & Whitfield [12]
compatible	Jungck [18]
weak* commuting	Pathak [33]
weak** commuting	Pathak [35]
compatible of various types (such as compatible of types (A), (B), (P), (α), (β))	Cho, Pathak, Kang & Jung [7], Jungck, Murthy & Cho [21] Pathak, Chang & Cho [36], Pathak, Chang, Kang & Lee [38]
biased	Jungck & Pathak [22]
R-weakly commuting	Pant [29]
R-weakly commuting type	Pathak, Cho & Kang [37]
co-ordinatewise weakly commuting	Gairola, Singh & Whitfield [12]
weakly compatible	Jungck & Rhoades [23]
partially commuting	Sastry & Krishna Murthy [43]
coincidentally commuting	Dhage [9]
compatible type (N)	Shrivastava, Bawa & Singh [47]
f-compatible	Pathak & Khan [39]
intimate maps	Sahu, Dhagat & Srivastava [42]

Some of these forms have been compared in a recent work by Murthy [27] and Pathak & Khan [39]. Such maps play a vital role in metric fixed point theory. In Section 2 we give a historical development and comparison of these maps. Section 3 is devoted to some general coincidence and fixed point theorems under tight minimal conditions without using continuity or reciprocal continuity of maps and relaxing the requirement of completeness of the space.

2. Comparison of weaker forms of commuting maps

Let Y be a nonempty set and $f, g : Y \to Y$. Then f and g are commuting at a point $p \in Y$ if $fgp = gfp$. Maps f and g are commuting on Y if they are commuting at each $p \in Y$.

In the rest part of this section (X, d) is a metric space space and f, g are self-mps of X.

Definition 2.1 (Sessa [44], see also Fisher & Sessa [11]). Maps f and g are *weakly commuting at* a point $x \in X$ whenever $d(fgx, gfx) \leq d(fx, gx)$. The pair (f, g) is *weakly commuting on* X if they commute at each point $x \in X$.

Notice that commuting maps are weakly commuting and the reverse implication is not true (see Example 2.1 below).

Example 2.1. Let $X = [0, 1]$ be endowed with the usual metric. Let $f(x) = \frac{x}{6}$, $gx = \frac{x}{2}(1 + x)$. Then $fg \neq gf$. However, $|fgx - gfx| \leq |fx - gx|$ for all $x \in X$ and f and g are weakly commuting.

Definition 2.2 (Jungck [18]). Maps f and g are *compatible*[1)] if

$$\lim_n d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = z$ for some $z \in X$.

We remark that weakly commuting maps are compatible. However, a compatible pair of selfmaps need not be weakly commuting.

Example 2.2 (Jungck [18]). Let $X = [0, \infty)$ be endowed with the usual metric. Let $fx = x^3$ and $gx = 2x^3$. Then $fgx \neq gfx$. So, f and g are not commuting on X and $|fgx - gfx| > |fx - gx|$. Therefore, f and g are not weakly commuting on X as well. However, $\lim_{x\to 0} |fx - gx| = 0 \in X$ and it implies $\lim_{x\to 0} |fgx - gfx| = 0$. Therefore, f and g are compatible.

[1)] Also called *asymptotically commuting* by Gairola, Singh & Whitfield [12], Tivari & Singh [63].

We remark that for a pair of weakly commuting maps on a metric space X, there may not exist a sequence $\{x_n\}$ in X for which the conditions of compatibility are satisfied (see Pant [29]). This means that there are weakly commuting maps which may not be actually compatible but vacuously.

Example 2.3. Let $X = \mathbb{R}$ be endowed with the usual metric. Let $fx = x$ and $gx = 1 + x$. Maps f and g are weakly commuting, indeed commuting but there does not exist a sequence $\{x_n\}$ in X for which the condition of compatibility is satisfied. However $fgx = gfx = 1 + x$, that is, $d(fgx, gfx) = 0$. Hence f and g may be called *vacuously compatible.*

Definition 2.3 (Cho [4], Cho, Murthy & Jungck [6], Jungck, Murthy & Cho [21]). Maps f and g are said to be *compatible of type* (A)[2)] if $\lim_n d(fgx_n, ggx_n) = 0$ and $\lim_n d(gfx_n, ffx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = z$ for some $z \in X$.

Definition 2.4 (Pathak, Chang & Cho [36]). Maps f and g are said to be *compatible of type* (B)[3)] if $\lim_n d(ffx_n, ggx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = z$ for some $z \in X$.

If f and g are continuous then compatiblity, compatible type (A), compatible type (B) are equivalent (*cf.* Pathak, Cho, Kang & Lee [38, Proposition 2.6]).

Definition 2.5 (Pant [29]). Maps f and g are *R-weakly commuting* if there exists $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$ for each $x \in X$.

It is clear from the above definition that R- weakly commuting maps commute at their coincidence point.

Definition 2.6 (Pant [29]). Maps f and g are *pointwise R-weakly commuting on X* if given $x \in X$ there exists $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$.

Pant [30] also showed that f and g can fail to be point wise R-weakly commuting only if there is some $x \in X$ such that $Ax = Sx$ but $ASx \neq SAx$, that is, only if they possess a coincidence point at which they do not commute. Therefore, the notion

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2) Cho [4] called it compatible type (α) in an FM-space. FM-space stands for Fuzzy Metric space.

3) Also called type (β) by Cho, Pathak, Kang & Jung [7] in an FM-space and type (P) by Pathak, Cho, Kang & Lee [38].

of pointwise R-weak commutativity is equivalent to commutativity at coincidence points.

Obviously weak commutativity of a pair of self-maps implies their R-weak commutativity and the reverse implication is true only when $R \leq 1$ (see also Pant [29]).

Example 2.4 (Pant [29]). Let $X = [1, \infty)$ and d be the usual metric on X. Let $fx = 2x - 1, gx = x^2$ for all $x \in X$. Then for any

$$x \in X, d(fgx, gfx) = 2(x-1)^2, d(fx, gx) = (x-1)^2,$$

that is, $d(fgx, gfx) = 2d(fx, gx)$. Thus maps f and g are R-weakly commuting ($R = 2$) but are not weakly commuting.

Definition 2.7 (Definition 1.2 in Pathak, Cho & Kang [37]). Maps f and g are *R-weakly commuting of type* (A_f) if there exists $R > 0$ such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all $x \in X$.

Definition 2.8 (Definition 1.3 in Pathak, Cho & Kang [37]). Maps f and g are *R-weakly commuting of type* (A_g) if there exists $R > 0$ such that $d(gfx, ffx) \leq Rd(fx, gx)$ for all $x \in X$.

Notice that Definition 2.8 is obtained from Definition 2.7 by interchanging the role of f and g.

Example 2.4 shows that f and g are R-weakly commuting but not R-weakly commuting of type (A_f), since, $d(fgx, ggx) = (x^2 - 1)^2$ and $d(fgx, ggx) > Rd(fx, gx)$ for each $x > 1$ and some $R > 0$ (*e. g.*, take $R = 3$). Thus R-weakly commuting pair of selfmaps need not be are R-weakly commuting of type (A_f).

The following example shows that a pair of selfmaps may be R-weakly commuting and R-weakly of type (A_f).

Example 2.5 (Pant [32]). Let $X = [2, 20]$ endowed with the usual metric. Let

$$fx = \begin{cases} 2, & \text{if } x = 2 \\ 12, & \text{if } x < x \leq 5 \\ (x+1)/3, & \text{if } x > 5 \end{cases} \quad \text{and} \quad gx = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5 \\ 6, & \text{if } 2 < x \leq 5. \end{cases}$$

Pant [32] has shown that these maps are R-weakly commuting of type (A_f). Further it is easily verified that they are R-weakly commuting maps indeed.

Pathak, Cho & Kang [37, Example 2.1], have shown that R-weakly commuting maps of type (A_f) need not be R-weakly commuting. Thus, for a pair of selfmaps, the two concepts, viz. R-weakly commuting and R-weakly commuting of type (A_f) are independent.

Definition 2.9 (Jungck & Pathak [22]). Maps f and g are f-biased if and only if $\{x_n\}$ is a sequence in X and $\lim_n fx_n = \lim_n gx_n = z \in X$, then

$$\alpha d(fgx_n, fx_n) \leq \alpha d(gfx_n, gx_n), \text{ if } \alpha = \liminf \text{ or } \alpha = \limsup .$$

Jungck & Pathak [22, Remark 1.1] have shown that if the pair $\{f, g\}$ is compatible, then it is both f-biased and g-biased, but the converse is not true.

Example 2.6 (Jungck & Pathak [22]). Let $X = [0,1], fx = 1 - 2x, gx = 2x$ for $x \in [0, 1/2]$ and $fx = 0, gx = 1$ for $x \in (1/2, 1]$. Then $\{g, f\}$ are both g-biased and f-biased but not compatible.

Definition 2.10 (Jungck & Rhoades [23]). Maps f and g are *weakly compatible*[4)] if they commute at their coincidence points, that is, if $fgx = gfx$, whenever $fx = gx$ for $x \in X$.

However, Singh [51] in 1986 and Singh & Pant [52] in 1988 used this concept without giving any name while establishing common fixed point theorems for maps on noncomplete spaces. Subsequently, this concept has been widely used, among others, by Singh, Ha & Cho [54], Singh & Mishra [55, 56, 57, 58], Chadha [3], Dhage [9] and Talwar [62, pp. 19–33].

Definition 2.11 (Definition 3 in Pathak & Khan [39]). Maps f and g are f-*compatible* if

$$\lim_n d(fgx_n, ggx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = z$ for some $z \in X$.

Definition 2.12 (Definition 4 in Pathak & Khan [39]). Maps f and g are g-*compatible* if

$$\lim_n d(gfx_n, ffx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = z$ for some $z \in X$.

[4)] Also called *coincidentally commuting* by Dhage [9], partially commuting by Sastry & Murthy [43] and compatible type (N) by Shrivastava, Bawa & Singh [47].

Notice that appropriate change of role of f and g in Definition 2.11 yields Definition 2.12.

If f and g are continuous, f-compatible, g-compatible, compatibility of type (A) and compatibility of type (B) are equivalent (*cf.* Pathak & Khan [39, Proposition 3], see also, Pathak, Cho, Kang & Lee [38, Proposition 2.6]).

Definition 2.13 (Singh & Mishra [55]). If for $x_0 \in X$ there exists a sequence $\{x_n\}$ in X such that $fx_{n+1} = gx_n, n = 0,1,2,\ldots$, then $O(g,f;x_0) = \{fx_n : n = 0,1,2,\ldots\}$ is an orbit for g and f. Maps f and g are *weakly x_0-preorbitally commuting* if and only if there exist a positive integer N such that $d(fgx_n, gfx_n) \leq d(fx_n, gx_n)$ for every $x_n (n \geq N)$ occurring in $O(g,f;x_0)$.

Definition 2.14 (Sahu, Dhagat & Srivastava [42]). Maps f and g are *f-intimate* if and only if

$$\alpha d(fgx_n, fx_n) \leq \alpha d(ggx_n, gx_n),$$

where $\alpha = \limsup$ or $\liminf$ and $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = z$ for some $z \in X$.

Sahu, Dhagat & Srivastava [42] have shown by an example that intimate maps are more general than compatible maps.

Example 2.7 (Sahu, Dhagat & Srivastava [42]). Let $X = [0,1]$ be endowed with the usual metric. Let $fx = 2/(x+2)$ and $gx = 1/(1+x)$ for $x \in [0,1]$. Let $\{x_n\} = \{1/n\}, n \in N$ be a sequence in X. Then,

$$\lim_n fx_n = \lim_n gx_n = 1, \lim_n |fgx_n - fx_n| = 1/3, \lim_n |ggx_n - gx_n| = 1/2.$$

Thus, $\lim_n |fgx_n - fx_n| < \lim_n |ggx_n - gx_n|$, that is, $\{f,g\}$ is f-intimate. Also, $\lim_n |fgx_n - ggx_n| = 1/6$. Thus $\{f,g\}$ is not compatible type (A).

The weak commutativity of a pair of selfmaps on a metric space depends on the choice of the metric. This is true for compatibility, R-weak commutativity and other variants of commutativity of maps as well. We illustrate this important fact by an example.

Example 2.8 (Singh [52]). Let $X = [0,\infty)$ be endowed with the usual metric. Let $fx = 1 + x$ and $gx = 2 + x^2$. Then $|fgx - gfx| = 2x$ and $|fx - gx| = |x^2 - x + 1|$. So, f and g are commuting at $x = 0$ and are not weakly commuting on X with respect to the usual metric. But if X is endowed with the discrete metric d, then

$d(fgx, gfx) = 1 = d(fx, gx)$ for $x > 0$. So, f and g are weakly commuting on X when endowed with discrete metric.

Selfmaps f and g are always commuting at their common fixed points, and so are weakly commuting, asymptotically commuting or compatible and so its other weaker variants as well. We remark that if maps f and g have a common fixed point z (say), then they belong to the following classes at z:

- Pointwise R-weakly commuting.
- R-weakly commuting.
- R-weakly commuting (A_f) type.
- R-weakly commuting (A_g) type.
- Weakly compatible.
- Compatible maps of type (A).
- Compatible maps of type (B).
- f-compatible.
- g-compatible.

This observation is easily verified from the fact that $fz = z = gz$ implies that $gfz = fgz$.

If f and g are weakly commuting at a point $z \in X$ and if z is their coincidence point, that is, $fz = gz$, then $d(gfz, fgz) = 0$ and f, g are commuting at z. Thus at a coincidence point, compatibility and weak commutativity of a pair of selfmaps are equivalent to their commutativity (*cf.* Singh [52]). This observation applies to compatible maps of various types (*cf.* Pathak & Khan [39]) as well (see also Singh & Mishra [57]). However, a pair of selfmaps may commute at their coincidence point but may not be compatible on X.

Example 2.9 (Singh & Mishra [56]). Let $X = [0, \infty)$ be a metric space with the usual metric.

$$fx = \begin{cases} x, & \text{if } x \in [0,1) \\ 1, & \text{if } x \in [1,\infty) \end{cases} \quad \text{and} \quad gx = \frac{x}{(1+x)}, \quad \text{if } x \in X,$$

then f and g are not compatible on X but f and g are commuting at their coincidence point $x = 0$. Indeed f and g are weakly compatible.

The main purpose of these significantly weaker forms of commuting maps has been to relax the commutativity of maps to a smallest subset of the domain of maps.

Now we discuss the commutativity and compatibility of a pair of selfmaps at a point. We begin with the following example.

Example 2.10 (Tomar & Singh [64]). Let $X = [2, 20]$,

$$fx = \begin{cases} 2, & \text{if } x = 2 \\ 12 + x, & \text{if } 2 < x \leq 5 \\ x - 3, & \text{if } x > 5 \end{cases} \quad \text{and} \quad gx = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5 \\ 8, & \text{if } 2 < x \leq 5. \end{cases}$$

Here, $fg2 = gf2$. If we consider a decreasing sequence $\{x_n\}$ such that $\lim_n x_n = 5$, then $\lim_n fx_n = \lim_n (x_n - 3) = 2$ and

$$\lim_n gx_n = 2, fgx_n = f2 = 2, gfx_n = g(x_n - 3) = 8.$$

So $\lim_n d(fgx_n, gfx_n) = 6$, and f and g are not compatible on X. This is very exciting to see that f and g are commuting at $x = 2$ and $\lim_n fx_n = \lim_n gx_n = 2$, but f and g are not compatible on X.

Thus unless the definition of compatibility at a point is modified, commutativity at a point does not mean compatibility at the same point. But if one defines the compatibility at a point (say $x = z$) by only considering the sequence $x_n = z$, then commutativity and compatibility at this point ($x = z$) are equivalent.

If the maps f and g are not continuous, then weakly compatible maps in general need not be compatible of type (A) or compatible of type (B) or f-compatible or g-compatible or other weaker forms.

Example 2.11. Let $X = [0, 6]$ be a metric space endowed with the usual metric. Let

$$fx = \begin{cases} x, & \text{if } x \in [0, 3) \\ 5, & \text{if } x \in [3, 6] \end{cases} \quad \text{and} \quad gx = \begin{cases} 6 - x, & \text{if } x \in [0, 3) \\ 5, & \text{if } x \in [3, 6] \end{cases}.$$

Notice that f and g are not continuous at $x = 3$. Let $\{x_n\}$ be a sequence such that $\lim_n x_n = 3, x_n < 3$. Then $\lim_n gx_n = \lim_n 6 - x_n = 3$ and $\lim_n fx_n = \lim_n x_n = 3$. Moreover,

$$fgx_n = f(6 - x_n) = 5, gfx_n = g(x_n) = 6 - x_n, ffx_n = f(x_n) = x_n$$

and $ggx_n = g(6 - x_n) = 5$ and so, f and g are not compatible since $\lim_n d(fgx_n, gfx_n) = \lim_n |5 - 6 + x_n| = 2$. Further

$$\lim_n d(ffx_n, ggx_n) = \lim_n |x_n - 5| = 2$$

and, f and g are not compatible type (B). However, we see that

$$fg3 = f5 = 5, gf3 = g5 = 5$$

and f and g are weakly compatible but not compatible or compatible type (B).

Example 2.12. Let $X = R$ be the metric space endowed with usual metric

$$fx = \begin{cases} \frac{1}{x^3}, & \text{if } x \neq 0 \\ 3, & \text{if } x = 0 \end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{1}{x^2}, & \text{if } x \neq 0 \\ 4, & \text{if } x = 0. \end{cases}$$

Let $\{x_n\}$ be a sequence in X such that $x_n = x^2$. Then

$$\lim_n fx_n = \lim_n \frac{1}{n^6} = 0, \lim_n gx_n = \lim_n \frac{1}{n^4} = 0,$$

Since

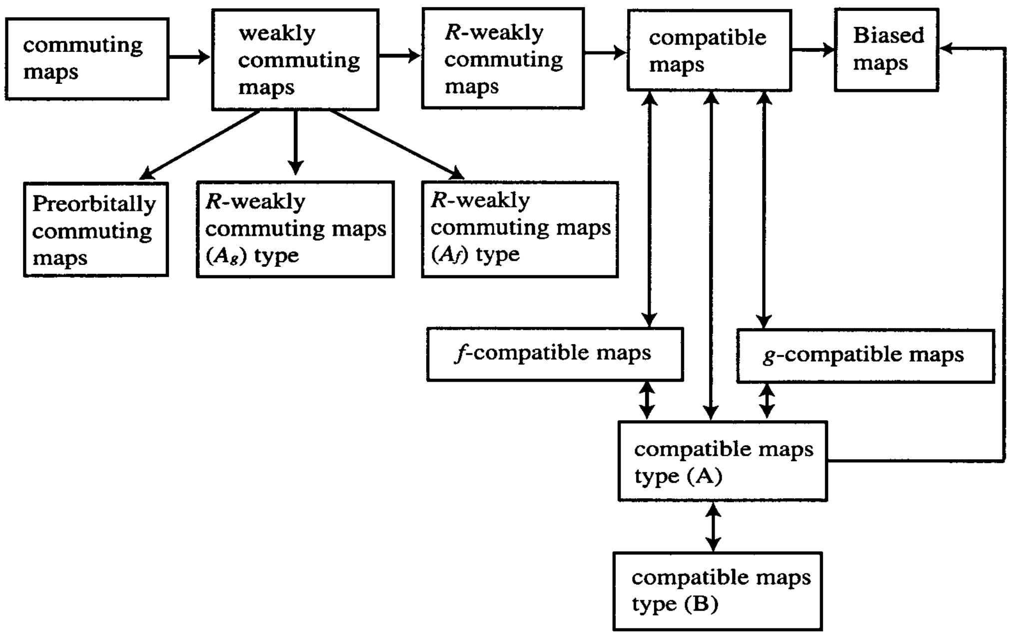
$$fgx_n = f\left(\frac{1}{n^4}\right) = n^{12}, \quad gfx_n = g\left(\frac{1}{n^6}\right) = n^{12},$$

$$ffx_n = f\left(\frac{1}{n^6}\right) = n^{18}, \quad ggx_n = g\left(\frac{1}{n^4}\right) = n^8,$$

we have

$$\lim_n d(fgx_n, ggx_n) = \infty \quad \text{and} \quad \lim_n d(gfx_n, ffx_n) = \infty.$$

Thus f and g are neither f-compatible, nor g-compatible. Moreover f and g are not compatible type (A). Further, $\lim_n d(ffx_n, ggx_n) = \infty$ shows that f and g are not compatible type (B). One may notice that f and g are compatible.



This shows that compatible maps f and g in general need not be f-compatible or g-compatible or compatible type (A) or compatible type (B).

If maps are continuous, some of the definitions are related to each other as shown above, wherein f and g are selfmaps of a metric space.

3. Coincidences And Fixed Points

A pair of selfmaps $f, g : X \to X$ is reciprocally continuous if $\lim_n fgx_n = ft$ and $\lim_n gfx_n = gt$ whenever $\{x_n\}$ is a sequence such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$ Pant [31]. Pant [31] has shown that reciprocally continuous maps need not be continuous.

The intent of this section is to drop reciprocal continuity and relax compatibility to weak compatibility from the theorem of Pant [31]. Further the requirement of completeness is also relaxed significantly. Let $\{A_i\}, i = 1, 2, \ldots, S$ and T are selfmaps of a metric space (X, d), and

$$M_{1i}(x,y) := \max\left\{d(Sx,Ty), d(A_1x,Sx), d(A_iy,Ty), \frac{[d(A_1x,Ty)+d(A_iy,Sx)]}{2}\right\}.$$

Also let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ (where $\mathbb{R}_+$ is the set of non-negative real numbers) be upper semicontinuous such that $\phi(t) < t$ for each $t > 0$.

Theorem 3.1. *Let $\{A_i\}, i = 1, 2, \ldots,$ S and T be selfmaps of a metric space (X, d). If one of SX, TX or A_iX is a complete subspace of X such that*

(i) *$A_1X \subseteq TX$ and $A_iX \subseteq SX$ when $i > 1$,*
(ii) *$d(A_1x, A_2y) \leq \phi(M_{12}(x,y))$, whenever $M_{12}(x,y) > 0$,*
(iii) *$d(A_1x, A_iy) < M_{1i}(x,y)$, whenever $M_{1i}(x,y) \neq 0$.*

Then:

(I) *A_1 and S have a coincidence,*
(II) *A_i and T have a coincidence,*
(III) *A_1 and S have a common fixed point provided that they are weakly compatible,*
(IV) *A_i and T have a common fixed point provided that they are weakly compatible,*
(V) *A_i, S and T, for each i, have a unique common fixed point provided that* (III) *and* (IV) *both are true.*

Proof. Pick $x_0 \in X$ and construct sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner.

$$y_{2n} = A_1 x_{2n} = T x_{2n+1} \text{ and } y_{2n+1} = A_2 x_{2n+1} = S x_{2n+2}, n = 0, 1, 2, \ldots.$$

Then it comes from Pant's theorem (*cf.* Pant [31]) that $\{y_n\}$ is a Cauchy sequence.

Now let SX be complete. The sequence $\{y_{2n+1}\}$ is contained in SX, and has a limit in SX. Let it be u. Let $v \in S^{-1}u$. Then $Sv = u$. The subsequence $\{y_{2n}\}$ also converges to u.

Now, we show that $A_1 v = u$. If $A_1 v \neq u$,

$$\begin{aligned} d(A_1 v, y_{2n+1}) &= d(A_1 v, A_2 x_{2n+1}) \\ &\leq \phi(M_{12}(v, x_{2n+1})), \text{ by (ii)} \\ &= \phi\Big(\max\Big\{d(Sv, Tx_{2n+1}), d(A_1 v, Sv), d(A_2 x_{2n+1}, Tx_{2n+1}), \\ &\qquad \frac{[d(A_1 v, Tx_{2n+1}) + d(A_2 x_{2n+1}, Sv)]}{2}\Big\}\Big). \end{aligned}$$

As $n \to \infty$,

$$\begin{aligned} d(A_1 v, u) &\leq \phi\Big(\max\Big\{d(Sv, u), d(A_1 v, u), d(u, u), \frac{[d(A_1 v, u) + d(u, Sv)]}{2}\Big\}\Big) \\ &= \phi(d(A_1 v, u)) \\ &< d(A_1 v, u), \end{aligned}$$

a contradiction. Therefore, $A_1 v = u$. Thus $Sv = A_1 v = u$, that is, A_1 and S have a coincidence. This proves (I).

Since $A_1 X \subseteq TX, A_1 v = u$ implies $u \in TX$. Let $w \in T^{-1}u$. Then $Tw = u$. Now we show that $A_i w = u, i > 1$. If $A_i w \neq u$,

$$d(u, A_i w) = d(A_1 v, A_i w) < M_{1i}(v, w) = d(A_i w, u),$$

a contradiction, Therefore $A_i w = u$. Thus $A_i w = Tw = u$, that is, A_i and T have a coincidence for each $i > 1$. This proves (II).

If we assume TX is complete, then argument analogous to the previous completeness argument establishes (I) and (II). If $A_1 X$ is complete then $u \in A_1 X \subseteq TX$. Similarly if $A_i X$ is complete then $u \in A_i X \subseteq SX$. Thus (I) and (II) are completely established.

Since A_1, S and A_i, T are weakly compatible at v and w respectively, and $A_1 u = A_1 Sv = SA_1 v = Su, A_i u = A_i Tw = TA_i w = Tu, i > 1$. This proves (III) and (IV).

Now

$$\begin{aligned} d(u, A_i u) = d(A_1 v, A_i u) &< M_{1i}(v,u) \\ &= \max\{d(Sv,Tu), d(A_1 v, Sv), d(A_i u, Tu), [d(A_1 v, Tu) + d(A_i u, Sv)]/2\} \\ &= \max\{d(u, A_i u), d(u,u), d(A_i u, A_i u), [d(u, A_i u) + d(A_i u, u)]/2\} \\ &= d(u, A_i u), \end{aligned}$$

a contradiction, therefore $A_i u = u$. Similarly $A_1 u = u$. This proves (III) and (IV). So, $u = A_i u = A_1 u = Su = Tu$, that is u is a common fixed point of A_i, S and T. The uniqueness of the common fixed point follows easily. □

The following result is a slightly more interesting when the above theorem is considered for three maps.

Theorem 3.2. *Let A_1, A_2 and T be selfmaps of a metric space (X,d). If one of A_1X, A_2X or TX is a complete subspace of X such that*

(i) $A_1X \cup A_2X \subseteq TX$,

(ii) $d(A_1x, A_2y) \le \phi(M_{12}(x,y))$, *whenever* $M_{12}(x,y) > 0$,

where $M_{12}(x,y) = \max\{d(Tx,Ty), d(A_1x,Tx), d(A_2y,Ty), [d(A_1x,Ty)+d(A_2y,Tx)]/2\}$. *Then:*

(i) *A_1, A_2 and T have coincidences.*

(ii) *A_1, A_2 and T have a unique common fixed point if T is weakly compatible with each of A_1 and A_2.*

Proof. Take $S = T$ in Theorem 3.1 and $\{A_1, A_2\} = \{A_i\}, i = 1, 2, \ldots$ □

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References

1. Ravi P. Agarwal, Maria Meehan & Donald O'Regan: Fixed Point Theory and Applications, Cambridge Tracts in Mathematics, vol. 141. *Cambridge University Press*, Cambridge, 2001. MR **2002c**:47122

2. Veena Chadha: *A Study in Fixed Point Theory and Stability Problems.* Ph. D. thesis (supervised by Prof. S. L. Singh), Gurukula Kangri Univ., Hardwar, India, 1994.
3. Shih Sen Chang: A common fixed point theorem for commuting mappings. *Proc. Amer. Math. Soc.* **83** (1981), no. 3, 645–652. MR **82j**:54091
4. Y. J. Cho: Fixed points for compatible mappings of type (A). *Math. Japon.* **38** (1993), no. 3, 497–508. MR **94d**:54093
5. ———: Fixed points in fuzzy metric spaces. *J. Fuzzy Math.* **5** (1997), no. 4, 949–962. MR **98h**:54011
6. Y. J. Cho, P. P. Murthy & G. Jungck: A common fixed point theorem of Meir and Keeler type. *Internat. J. Math. Math. Sci.* **16** (1993), no. 4, 669–674. CMP 1 234 811
7. Y. J. Cho, H. K. Pathak, S. M. Kang & J. S. Jung: Common fixed points of compatible maps of type (β) on fuzzy metric spaces. *Fuzzy Sets and Systems* **93** (1998), no. 1, 99–111. CMP 1 600 404
8. K. M. Das & K. Viswanatha Naik: Common fixed point theorems for commuting maps on metric spaces. *Proc. Amer. Math. Soc.* **77** (1979), no. 3, 369–373. MR **80k**:54082
9. B. C. Dhage: On common fixed point of coincidentally commuting mappings in D-metric space. *Indian J. Pure Appl. Math.* **30** (1999), no. 4, 395–406. CMP 1 695 692
10. B. Fisher: Mappings with a common fixed point. *Math. Sem. Notes Kobe Univ.* **7** (1979), no. 1, 81–84. MR **80j**:54038
11. B. Fisher & S. Sessa: Two common fixed point theorems for weakly commuting mappings. *Period. Math. Hungar* **20** (1989), no. 3, 207–218. MR **91c**:54055
12. U. C. Gairola, S. L. Singh & J. H. M. Whitfield: Fixed point theorems on product of compact spaces. *Demonstratio Math.* **28** (1995), no. 3, 541–548. CMP 1 362 184
13. K. Goebel: A coincidence theorem. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **16** (1968), 733–735. MR **44**#878
14. Olga Hadžić: Common fixed point theorems for family of mappings in complete metric spaces. *Math. Japon.* **29** (1984), no. 1, 127–134. MR **85f**:54098
15. G. S. Jeong & B. E. Rhoades: Some remarks for improving fixed point theorems for more than two maps. *Indian J. Pure Appl. Math.* **28** (1997), no. 9, 1177–1196. MR **98j**:54080
16. G. Jungck: Commuting mappings and fixed points. *Amer. Math. Monthly* **83** (1976), no. 4, 261–263. MR **53**#4031
17. ———: Periodic and fixed points, and commuting mappings. *Proc. Amer. Math. Soc.* **76** (1979), no. 2, 333–338. MR **80e**:54057
18. ———: Compatible mappings and common fixed points. *Internat. J. Math. Math. Sci.* **9** (1986), no. 4, 771–779. MR **87m**:54122
19. ———: Common fixed points for commuting and compatible maps on compacta. *Proc. Amer. Math. Soc.* **103** (1988), no. 3, 977–983. MR **89h**:54030

20. ______: Observations on a variant of compatibility. *Internat. J. Math. Math. Sci.* **21** (1998), no. 2, 261–268. CMP 1 609 759
21. G. Jungck, P. P. Murthy & Y. J. Cho: Compatible mappings of type (A) and Common fixed points. *Math. Japon.* **38** (1993), no. 2, 381–390. MR **94c:**54078
22. G. Jungck & H. K. Pathak: Fixed points via "biased maps". *Proc. Amer. Math. Soc.* **123** (1995), no. 7, 2049–2060. MR **95i:**54055
23. G. Jungck & B. E. Rhoades: Fixed points for set valued functions without continuity. *Indian J. Pure Appl. Math.* **29** (1998), no. 3, 227–238. CMP 1 617 919
24. T. Kubiak: Common fixed point theorems of pairwise commuting maps. *Math. Nachr.* **118** (1984), 123–127. MR **86f:**54076
25. R. Machuca: A Coincidence theorem. *Amer. Math. Monthly* **74** (1967), 569–572.
26. S. N. Mishra: Common fixed points of compatible mappings in PM-spaces. *Math. Japon.* **36** (1991), no. 2, 283–289. CMP 1 095 742
27. P. P. Murthy: Important tools and possible applications of metric fixed point theory. *Nonlinear Analysis* **47** (2001), 3479–3490.
28. Slobodan Č. Nešić: Results on fixed points of asymptotically regular mappings. *Indian J. Pure Appl. Math.* **30** (1999), no. 5, 491–494. CMP 1 694 694
29. R. P. Pant: Common fixed points of noncommuting mappings. *J. Math. Anal. Appl.* **188** (1994), 436–440. MR **95j:**54037
30. ______: Common fixed points of four mappings. *Bull. Calcutta Math. Soc.* **90** (1998), no. 4, 281–286. CMP 1 697 238
31. ______: A common fixed point theorem under a new condition. *Indian J. Pure Appl. Math.* **30** (1999), no. 2, 147–152. CMP 1 681 592
32. ______: Discontinuity and fixed points. *J. Math. Anal. Appl.* **240** (1999), no. 1, 284–289. MR **2000j:**54047
33. H. K. Pathak: Weak* commuting maps and fixed points. *Indian J. Pure Appl. Math.* **17** (1986), no. 2, 201–211. MR **87c:**54059
34. ______: A Meir-Keeler type fixed point theorem for weakly uniformly contraction maps. *Bull. Malaysian Math. Soc.(2)* **13** (1990), no. 1, 21–29. MR **91k:**54078
35. ______: Weak** commuting mappings and fixed points. *J. Indian Acad. Math.* **14** (1992), no. 2, 94–98. CMP 1 261 770
36. H. K. Pathak, S. S. Chang & Y. J. Cho: Fixed point theorems for compatible mappings of type (P). *Indian J. Math.* **36** (1994), no. 2, 151–166. CMP 1 345 267
37. H. K. Pathak, Y. J. Cho & S. M. Kang: Remarks on R-weakly commuting mappings and common fixed point theorems. *Bull. Korean Math. Soc.* **34** (1997), no. 2, 247–257. CMP 1 455 445
38. H. K. Pathak, Y. J. Cho, S. M. Kang & B. S. Lee: Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming. *Matematiche (Catania)* **50** (1995), no. 1, 15–33. MR **97i:**47123

39. H. K. Pathak & M. S. Khan: A comparison of various types of compatible maps and common fixed points. *Indian J. Pure Appl. Math.* **28** (1997), no. 4, 477–485. MR **97m:**54069

40. B. E. Rhoades & S. Sessa: Common fixed point theorems for three mappings under a weak commutativity condition. *Indian J. Pure Appl. Math.* **17** (1986), no. 1, 47–57. MR **88b:**54026

41. B. E. Rhoades, S. Sessa, M. S. Khan & M. Swaleh: On fixed points of asymptotically regular mappings. *J. Austral. Math. Soc. Ser. A* **43** (1987), no. 3, 328–346. MR **88h:**54064

42. D. R. Sahu, Vinita Ben Dhagat & Madhu Srivastava: Fixed points with intimate mappings (I). *Bull. Calcutta Math. Soc.* **93** (2001), no. 2, 107–114. CMP 1 890 862

43. K. P. R. Sastry & I. S. R. Krishna Murthy: Common fixed points of two partially commuting tangential selfmaps on a metric space. *J. Math. Anal. and Appl.* **250** (2000), no. 2, 731–734. CMP 1 786 095

44. S. Sessa: On a weak commutativity condition of mappings in fixed point considerations. *Publ. Inst. Math. (Beograd) (N.S.)* **32(46)** (1982), 149–153. MR **85f:**54107

45. S. Sessa, B. E. Rhoades & M. S. Khan: On common fixed points of compatible mappings in metric and Banach Spaces. *Internat. J. Math. Math. Sci.* **11** (1988), no. 2, 375–392. MR **89k:**54111

46. S. Sharma & B. Deshpande: Common fixed points of compatible maps of type (β) on fuzzy metric spaces. *Demonstratio Math.* **35** (2002), no. 1, 165–174. MR **2002j:**54045

47. P. K. Shrivastava, N. P. S. Bawa & Pankaj Singh: Coincidence theorems for hybrid contraction II. *Soochow J. Math.* **26** (2000), no. 4, 411–421. MR **2001i:**54051

48. S. L. Singh: On common fixed points of commuting mappings. *Math. Sem. Notes Kobe Univ.* **5** (1977), no. 2, 131–134. MR **56#**16607

49. ______: A note on the convergence of a pair of sequence of mappings. *Arch. Math. (Brno)* **15** (1979), no. 1, 47–51. MR **81a:**54050

50. ______: A note on recent fixed point theorem for commuting mappings. *Vijnana Parishad Anusandhan Patrika* **26** (1983), no. 3, 259–261. MR **84m:**54050

51. ______: Coincidence theorems, fixed point theorems and convergence of the sequences of coincidence values. *Punjab Univ. J. Math. (Lahore)* **19** (1986), 83–97. MR **88f:**54088

52. ______: Commuting maps & their weaker forms (in Hindi). *Vijnana Garima Sindu* **13** (1991), 64–66.

53. S. L. Singh, V. Chadha & S. N. Mishra: Remarks on recent fixed point theorems for compatible maps. *Internat. J. Math. Math. Sci.* **19** (1996), no. 4, 801–804. CMP 1 397 849

54. S. L. Singh, K. S. Ha & Y. J. Cho: Coincidence and fixed points of nonlinear hybrid contractions. *Internat. J. Math. Math. Sci.* **12** (1989), no. 2, 247–256. MR **90g:**54042

55. S. L. Singh & S. N. Mishra: Coincidence points, hybrid fixed and stationary points of orbitally weakly dissipative maps. *Math. Japon.* **39** (1994), no. 3, 451–459. MR **95b**:47078
56. ______: Remarks on Jachymski's fixed point theorems for compatible maps. *Indian J. Pure Appl. Math.* **28** (1997), no. 5, 611–615. MR **98c**:54036
57. ______: Coincidences and fixed points of nonself hybrid contractions. *J. Math. Anal. Appl.* **256** (2001), no. 2, 486–497. CMP 1 821 752
58. ______: Remarks on recent fixed point theorems and applications to integral equations. *Demonstratio Math.* **34** (2001), no. 4, 847–857. MR **2002g**:47127
59. S. L. Singh & B. D. Pant: Coincidence and fixed point theorems for a family of mappings on Menger spaces and extension to uniform spaces. *Math. Japon.* **33** (1988), no. 6, 957–973. MR **90d**:54095
60. S. L. Singh & D. D. Sharma: *Solutions Of Coincidence And Fixed Point Equations On 2-metric and 2-normed Spaces.* Commission On Scientific & Tech. Terms, Dep. of Education, Human Resource Development, Govt. of India, New Delhi, 1999.
61. S. L. Singh & Rekha Talwar: Fixed point theorems for expansions. *Proc. Nat. Acad. Sci. India Sect. A* **62** (1992), no. 2, 269–273. CMP 1 270 881
62. Rekha Talwar: *Fixed point theorems in probabilistic analysis and uniform spaces.* Ph. D. thesis (supervised by Prof. S. L. Singh), Gurukula Kangri Univ., Hardwar, India, 1991.
63. B. M. L. Tivari & S. L. Singh: A note on recent generalizations of Jungck contraction principle. *J. Uttar Pradesh Gov. Colleges Acad. Soc.* **3** (1986), no. 1, 13–18. MR **88k**:54081
64. Anita Tomar & S. L. Singh: *A development of weaker forms of commuting maps.* A part presented in National Conference on History of Mathematics, held at Kumaon Univ., Nainital, Oct. 13–16, 2000. [Also a part presented in 66th Annual Conf. of Indian Math. Soc., held at Aurangabad, Dec. 19–22, 2000.]
65. J. S. Ume & T. H. Kim: Common fixed point theorems for weak compatible mappings. *Indian J. Pure Appl. Math.* **32** (2001), no. 4, 565–571. CMP 1 838 832
66. R. Vasuki: Common fixed points for R-weakly commuting maps in fuzzy metric spaces. *Indian J. Pure Appl. Math.* **30** (1999), no. 4, 419–423. CMP 1 695 690

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